SCATTERING OF ELECTROMAGNETIC WAVES BY INHOMOGENEOUS SPHERICAL PARTICLES

by S. B. MAL *and* HARI L. BHATNAGAR, *Department of Applied Sciences, Regional Engineering College, Kurukshetra, Haryana*

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Vector wave equations obtained by introducing radially variable dielectric constant of the form $\varepsilon(r) = \varepsilon_0 \exp.(-\mu^2 r^2)$, $0 \leqslant r \leqslant a$, under the restriction $\mu^2 a^2 << 1$, have been solved exactly. A method for obtaining the radial equations has been suggested and successfully tried in the present case. The scalar wave functions have been obtained in terms of confluent hypergeometric functions. These functions have been used to calculate the scattering coefficients with the help of usual boundary conditions. For the particular case of dielectric profile of the spherical particles under the restriction $\mu^2 a^2 << 1$, the results are general and any restrictions on the size or the ratio of the refractive indices of the particle core to that of the surrounding medium are unnecessary.

INTRODUCTION

The solutions of Maxwell's equations for the case of a sphere possessing continuously or discontinuously variable dielectric constant embedded in a homogeneous medium has been attempted in recent years. (Aden & Kerker 1951; Arnush 1964; Garbacz 1962; Gould & Burmann 1964; Herman & Batten 1961; Kerker *et al.* 1966; Murphy 1965; Lynch 1963; Margulies & Scarf 1964; Negi 1962; Nomura & Takaku 1956; Rheinstein 1964; Scharfman 1954; Tai 1958; and Wyatt 1962, 1964). The cases considered differ only in their dependence on the power of the radial distance from the centre of the sphere. Kerker had adequately reviewed such cases and has described the mathematical formulism developed. It is found that explicit analytical solutions exist only for certain functional forms of $\epsilon(r)$, the dielectric constant of the sphere. For rest of the cases, high speed computerised programming has been used. This approach has been over-emphasised by Wyatt (1962, 1964) and Garbacz (1962) while attempting to obtain numerical solutions of certain practical problems. This approach need not be regarded as solution to all the problems arising out of continuously variable dielectric constant. In general, it is desirable to obtain analytical or series solutions with known coefficients to be exact.

In the present communication, the latter approach of obtaining analytical solutions for vector wave equations arising out of the function :

$$\epsilon(r) = \epsilon_0 e^{-\mu^2 r^2}, 0 \leqslant r \leqslant a$$

where $\mu^2 a^2 << 1$, have been given.

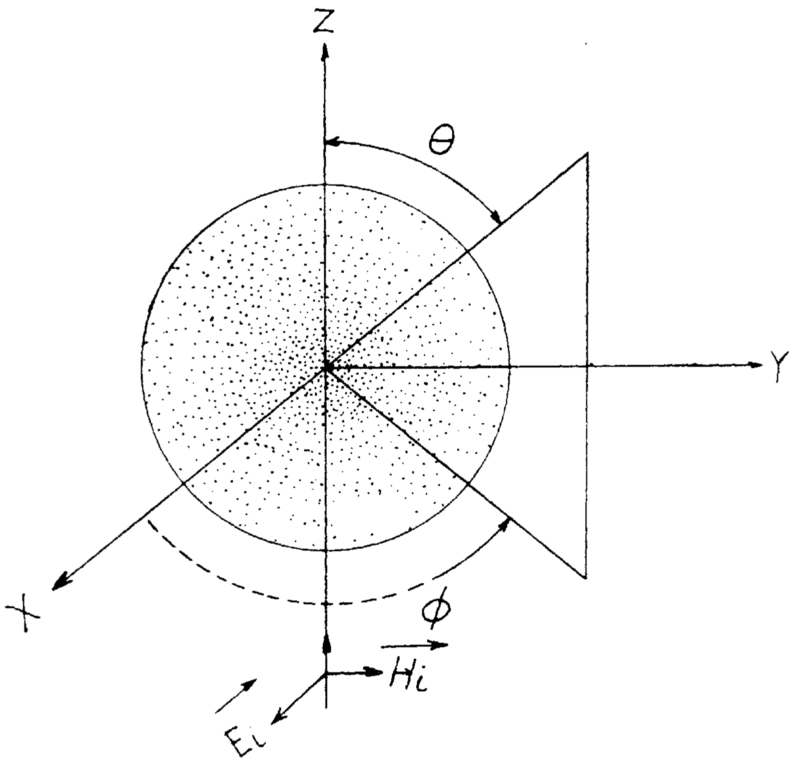

FIG. 1. Coordinate system showing E.M. Scattering from a non-homogeneous sphere.

MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a plane electromagnetic wave, whose electric vector is linearly polarised in the x-direction and which is propagated in the positive z-direction, interacting with a non-homogeneous sphere as shown in Fig. 1. The centre of the sphere is taken as the origin.

The spherical partical has been assumed to be non-conducting of constant permeability and possessing a radially varying dielectric constant, except for its outermost surface which is discontinuous with respect to the surrounding medium. The surrounding medium has been assumed to be uniform as regards its dielectric constant.

In this case, if one assumes the time dependence of the form $e^{-i\omega t}$, Maxwell's equations take the simple form :

$$\nabla \times \vec{H} = - i\omega \epsilon^{v} \epsilon_1(r) . \vec{E} \quad \ldots(1)$$

$$\nabla \times \vec{E} = i\omega K_m^{v} K_{m_1} \vec{H} \quad \ldots(2)$$

$$\nabla \cdot (\epsilon_1(r) \vec{E}) = 0 \quad \ldots(3)$$

and

$$\nabla \cdot \vec{H} = 0, \qquad \text{...(4)}$$

where, $\epsilon_1(r)$ is the relative permittivity with respect to ϵ^v (permittivity of free space) and K_{m_1} is the relative permeability with respect to K_m^v (permeability of free space). $\epsilon_1(r)$ has been assumed to be of the form :

$$\left.\begin{aligned} \epsilon_1(r) &= \epsilon_0 e^{-\mu^2 r^2}, \; 0 \leqslant r \leqslant a \\ &= \epsilon_2 \qquad\quad , \; r \geqslant a, \end{aligned}\right\} \qquad \text{...(5)}$$

where μ is an adjustable constant.

The corresponding vector wave equations are as follows:

$$\nabla^2 \vec{E} + k_0^2 \, \epsilon_1(r) \, . \vec{E} = - \nabla \left(\frac{\vec{E}}{\epsilon_1(r)} \cdot \nabla \epsilon_1(r) \right), \; 0 \leqslant r \leqslant a, \qquad \text{...(6)}$$

$$\nabla^2 \vec{H} + k_0^2 \, \epsilon_1(r) \, \vec{H} = \frac{1}{\epsilon_1} \left(\nabla \times \vec{H} \times \nabla \epsilon_1(r) \right), \; 0 \leqslant r \leqslant a, \qquad \text{...(7)}$$

and

$$\nabla^2 \vec{E} + k_0'^2 \, \epsilon_2 \vec{E} = 0, \;\; r \geqslant a, \qquad \text{...(8)}$$

$$\nabla^2 \vec{H} + k_0'^2 \, \epsilon_2 \vec{H} = 0, \;\; r \geqslant a, \qquad \text{...(9)}$$

where,

$$\left.\begin{aligned} k_0^2 &= \omega^2 K_m^v \, k_{m_1} \, . \epsilon^v = k_v^2 \, k_{m_1}, \\ k_0'^2 &= \omega^2 K_m^v \, . k_{m_2} \epsilon^v = k_v^2 \, . k_{m_2}, \end{aligned}\right\} \qquad \text{...(10)}$$

The incident and scattered fields in spherical co-ordinate system have been assumed to be the same as given by Stratton (1941). For the transmitted field, the functions $\vec{m}^{(t)}_{\substack{e\\0}1l}$ and $\vec{n}^{(t)}_{\substack{e\\0}1l}$ can be expressed as follows:

$$\vec{m}^{(t)}_{\substack{e\\0}1l} = \frac{1}{k_1} \cdot \nabla \times (\alpha \, \psi_{\substack{e\\0}1l} \cdot \vec{i}_1), \qquad \text{...(11)}$$

and

$$\vec{n}^{(t)}_{\substack{e\\0}1l} = \frac{1}{k_1} \; \frac{1/\epsilon_1(r) \cdot \nabla \times \vec{m}^{(t)}_{\substack{e\\0}1l}}{1}, \qquad \text{...(12)}$$

where

$$\left.\begin{aligned} \psi_{\substack{e\\o}1l} &= \frac{R_l^{(1)}(\alpha)}{R_l^{(3)}(\alpha)} \cdot P_l^1(\cos\theta) \cdot \frac{\cos}{\sin}\phi, \\ \alpha &= k_1 r \text{ and } \epsilon(r) = e^{-\mu^2 r^2}. \end{aligned}\right\} \quad \ldots(13)$$

Two different expressions are obtained due to the unsymmetrical nature of non-homogeneous vector Helmholtz equations.

Thus the problem of solving the non-homogeneous vector Helmholtz equations, arising out of radial dependence of the dielectric constant of a spherical particle, is reduced to obtaining two suitable wave functions $\psi_{\substack{e\\o}1l}$.

To obtain $\psi_{\substack{e\\o}1l}$, the following operations may be performed without any loss of generality:

(*i*) Take the scalar product of the Eqns. (6, 7) with $\vec{l_2}$

(*ii*) Integrate both sides with respect to θ within the limits $\theta = 0$ to π, after multiplying with $P_l^1(\cos\theta) \cdot d\theta$

These processes are represented by the following equations:

$$\int_0^\pi \vec{l_2} \cdot \left[\left\{\nabla^2 + k_0^2\,\epsilon_1(r)\right\}\vec{E^t} + \nabla\left\{\vec{E^t} \cdot \frac{\nabla\,\epsilon_1(r)}{\epsilon_1(r)}\right\}\right] P_l^1(\cos\theta) \cdot d\theta = 0 \quad \ldots(14)$$

and

$$\int_0^\pi \vec{l_2} \cdot \left[\left\{\nabla^2 + k_0^2\,\epsilon_1(r)\right\}\vec{H^t} + \frac{\nabla\,\epsilon_1(r)}{\epsilon_1(r)} \times \nabla \times \vec{H^t}\right] P_l^1(\cos\theta) \cdot d\theta = 0 \quad \ldots(15)$$

The integrations are performed subject to the usual divergence conditions $\nabla \cdot \left\{\epsilon_1(r)\,\vec{E^t}\right\} = 0$ and $\nabla \cdot \vec{H^t} = 0$. The transmitted field can be expressed as:

$$\vec{E^t} = E_0 e^{-i\omega t} \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} \left[a_l^t\,\vec{m}_{o1l}^{(t)} - i\,b_l^t\,\vec{n}_{e1l}^{(t)}\right] \quad \ldots(16)$$

$$\vec{H^t} = -\frac{k_1}{\omega\,k_m^v \cdot k_{m_1}} \cdot E_0\,e^{-i\omega t} \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} \left[b_l^t\,\vec{m}_{e1l}^{(t)} + i\,a_l^t\,\vec{n}_{o1l}^{(t)}\right], \quad \ldots(17)$$

where $k_1^2 = k_0^2\,\epsilon_0$.

Using the operations (14) and (15), one gets the following radial equations for $R_l^{(1)}(\alpha)$ and $R_l^{(3)}(\alpha)$:

$$\frac{d^2 R_l^{(1)}(\alpha)}{d\alpha^2} + \left[\frac{2}{\alpha} - \frac{1}{\epsilon_1(r)}\frac{\partial \epsilon_1}{\partial \alpha}\right]\frac{d R_l^{(1)}(\alpha)}{d\alpha} + \left[\frac{k_0^2\,\epsilon_1(r)}{k_1^2} - \frac{1}{\epsilon_1(r)}\cdot\frac{1}{\alpha}\cdot\frac{\partial \epsilon_1}{\partial \alpha} - \frac{l(l+1)}{\alpha^2}\right] R_l^{(1)}(\alpha) = 0 \quad \ldots(18)$$

and

$$\frac{d^2}{d\alpha^2} R_l^{(3)}(\alpha) + \frac{2}{\alpha}\frac{d R_l^{(3)}(\alpha)}{d\alpha} + \left[\frac{k_0^2\,\epsilon_1(r)}{k_1^2} - \frac{l(l+1)}{\alpha^2}\right] R_l^{(3)}(\alpha) = 0, \quad \ldots(19)$$

where $\alpha = k_1 r$.

Case I:

$$\begin{aligned}\epsilon_1(r) &= \epsilon_0 \cdot e^{-\mu^2 r^2}\\ &\simeq \epsilon_0(1 - \mu^2 r^2)\,, \quad 0 \leqslant r \leqslant a,\\ &= \epsilon_2 \qquad\qquad\quad , \quad r \geqslant a,\end{aligned}$$

neglecting higher powers.

The equations are:

$$\alpha^2 \frac{d^2 R_l^{(1)}(\alpha)}{d\alpha^2} + \alpha(2 + 2a_1\alpha^2)\frac{d R_l^{(1)}(\alpha)}{d\alpha} + [- l(l+1) + (2a_1 + 1)\,\alpha^2 - a_1\alpha^4]\, R_l^{(1)}(\alpha) = 0 \quad \ldots(20)$$

$$\alpha^2 \frac{d^2 R_l^{(3)}(\alpha)}{d\alpha^2} + 2\alpha \frac{d R_l^{(3)}(\alpha)}{d\alpha} + [- l(l+1) + \alpha^2 - a_1\alpha^4]\, R_l^{(3)}(\alpha) = 0 \quad \ldots(21)$$

where, $a_1 = \mu^2/k_1^2$.

The solutions of the Eqns. (20) and (21) which are regular at the origin are:

$$R_l^{(1)}(\alpha) = \alpha^l \times e^{-1/2(a_1)^{1/2}\cdot\alpha^2 + 1/2(a_1^2 + a_1)^{1/2}\alpha^2} \times {}_1F_1\left[\frac{(2l+3)(a_1^2 + a_1)^{1/2} - a_1 + 1}{4(a_1^2 + a_1)^{1/2}}, \tfrac{3}{2} + l; (-)(a_1^2 + a_1)^{1/2}\alpha^2\right] \quad \ldots(22)$$

and

$$R_l^{(3)}(\alpha) = \alpha^l \cdot e^{1/2\, a_1^{1/2} \cdot \alpha^2} \times$$

$$_1F_1\left[\frac{3}{4} + \frac{l}{2} + \frac{1}{4\sqrt{a_1}}, \frac{3}{2} + l; -(a_1)^{1/2} \cdot \alpha^2\right], \quad \ldots(23)$$

where $_1F_1$ is the confluent Hypergeometric function.

The usual tangential boundary conditions lead to two pairs of inhomogeneous equations for the expansion coefficients. They further yield

$$a_l^r = -\frac{k_{m_1} \cdot R_l^{(3)}(N\rho) \cdot [\rho j_l(\rho)]' - k_{m_2} j_l(\rho)[N\rho \cdot R_l^{(3)}(N\rho)]'}{k_{m_1} \cdot R_l^{(3)}(N\rho) \cdot [\rho h_l^{(1)}(\rho)]' - k_{m_2} h_l^{(1)}(\rho) \cdot [N\rho \cdot R_l^{(3)}(N\rho)]'} \quad \ldots(24)$$

and

$$b_l^r = -\frac{k_{m_1} j_l(\rho)[N\rho \cdot R_l^{(1)}(N\rho)]' - k_{m_2} \cdot N^2 \cdot e^{-a_1\alpha^2} \cdot R_l^{(1)}(N\rho) \cdot [\rho j_l(\rho)]'}{k_{m_1} h_l^{(1)}(\rho) \cdot [N\rho R_l^{(1)}(N\rho)]' - k_{m_2} N^2 \cdot e^{-a_1\alpha^2} \cdot R_l^{(1)}(N\rho)[\rho h_l^{(1)}(\rho)]'}, \quad \ldots(25)$$

where

$$N\rho = k_1 a$$

$$\rho = k_2 a$$

$$N = k_1/k_2.$$

and primes denote differentiation with respect to the respective arguments.

Scattering Cross-Section

The scattering cross-section of the sphere is defined as the ratio of the total scattered energy per second to the energy density of the incident wave. Thus,

$$Q_s = \frac{2\pi}{k_2^2} \sum_{l=1}^{\infty} (2l+1)[\,|a_l^r|^2 + |b_l^r|^2] \cdot m^2. \quad \ldots(26)$$

To evaluate (26) numerically very involved calculations have to be carried out. For the present, only two limiting cases have been discussed with a view to comparing these results with those obtained by Mie.

Limiting cases :

Case I : $|\rho| << 1$,

The radius of the sphere is much less than the wavelength of the incident wave.

Case II : $|\rho| >> 1$

The radius of the sphere is much greater than the wavelength of the incident wave.

Case III : For $|\rho| << 1$, the following approximation can be made :

$$j_l(\rho) \simeq 2^l \cdot \frac{l!}{(2l+1)!} \cdot \rho^l \left\{1 - \frac{\rho^2}{4l+6}\right\}, \qquad \ldots(27)$$

$$h_l^{(1)}(\rho) \simeq -\frac{i}{2^l} \cdot \frac{(2l)!}{l!} \frac{1}{\rho^{l+1}}, \qquad \ldots(28)$$

$$\chi_1 = [N\rho \,.\, R_l^{(3)}(N\rho)]' / R_l^{(3)}(N\rho) \simeq l + 1 - (N\rho)^2 \cdot \frac{1}{2l+3} \qquad \ldots(29)$$

$$\chi_2 = [N\rho\, R_l^{(1)}(N\rho)]' / R_l^{(1)}(N\rho) \simeq l + 1 - (N\rho)^2 \cdot \left[\frac{2a_1 l + 2a_1 + 1}{2l+3}\right] \qquad \ldots(30)$$

With the help of eqns. (27) to (30), eqns. (24) and (25) are reduced to the following :

$$a_l^r = -iD$$

$$\frac{k_{m_2}\left\{4l + 6 - \rho^2\right\}\left\{l + 1 - \frac{N^2\rho^2}{2l+3}\right\} - k_{m_1}\left\{(l+1)(4l+6) - (l+3)\rho^2\right\}}{k_{m_2}\left\{(l+1)(4l+6) - 2(N\rho)^2\right\} + k_{m_1} \,.\, l(4l+6)} \qquad \ldots(31)$$

$$b_l^r = -iD$$

$$\frac{k_{m_1}\left\{1 - \frac{\rho^2}{4l+6}\right\}\left\{l + 1 - (N\rho)^2 \,.\, \frac{2a_1 l + 2a_1 + 1}{2l+3}\right\} - k_{m_2} N^2 \left\{1 - a_1 N^2\rho^2\right\}\left\{l + 1 - \frac{(l+3)\rho^2}{4l+6}\right\}}{k_{m_1}\left\{l + 1 - (N\rho)^2 \frac{2a_1 l + 2a_1 + 1}{2l+3}\right\} + k_{m_2} N^2 l \left\{1 - a_1 \,.\, N^2\rho^2\right\}}. \qquad \ldots(32)$$

where

$$D = 2^{2l} \cdot \frac{(l!)^2 \,.\, \rho^{2l+1}}{(2l)!\,(2l+1)!}.$$

If one assumes ρ to be very small, so that all powers higher than ρ^5 can be neglected, only a_1^r, b_1^r and b_2^r are of any consequence. Thus,

$$a_1^r \simeq -i\frac{\rho^3}{3}\left[\frac{10k_{m_2} - 10k_{m_1} - \rho^2\{k_{m_2} + N^2 k_{m_2} - 2k_{m_1}\}}{10k_{m_2} + 5k_{m_1} - k_{m_2} N^2 \rho^2}\right]. \qquad \ldots(33)$$

$$a_2^r \simeq -i\frac{\rho^5}{15}\left[\frac{k_{m_2} - k_{m_1}}{3k_{m_2} + 2k_{m_1}}\right], \qquad \ldots(34)$$

$$b_1^r \simeq -i\frac{2\rho^3}{3}\left[\frac{k_{m_1}-N^2k_{m_2}}{2k_{m_1}+N^2k_{m_2}}\right]-i\frac{\rho^5}{15(2k_{m_1}+N^2k_{m_2})^2}\times$$
$$\left[N^4\{18a_1k_{m_1}k_{m_2}-3k_{m_1}k_{m_2}+2k^2_{m_2}\}+3N^2k_{m_1}.k_{m_2}-2k^2_{m_1}\right] \quad ...(35)$$

and

$$b_2^r \simeq -i\frac{\rho^5}{15}\left[\frac{k_{m_1}-N^2k_{m_2}}{3k_{m_1}+2N^2k_{m_2}}\right]. \quad ...(36)$$

Case IV : For $|\rho| >> 1$, the following asymptotic values can be used :

$$j_l(\rho) \simeq \frac{1}{\rho}.\cos\left\{\rho-(l+1)\frac{\pi}{2}\right\}, \quad ...(37)$$

$$h_l^{(1)}(\rho) \simeq \frac{1}{\rho}.e^{i\rho}(-i)^{l+1}, \quad ..(38)$$

$$[N\rho.R_l^{(3)}(N\rho)]'/R_l^{(3)}(N\rho) = \chi_1' \simeq -\tfrac{1}{2}(1+a_1^{-1/2})+a_1^{1/2}.N^2\rho^2 \quad ...(39)$$

and

$$[N\rho\, R_l^{(1)}(N\rho)]'/R_l^{(1)}(N\rho) = \chi_2' \simeq -\tfrac{1}{2}\times$$
$$\left(1+\frac{1-a_1}{(a_1^2+a_1)^{1/2}}\right)+(-a_1-\sqrt{a_1^2-a_1})N^2\rho^2 \quad ...(40)$$

With the help of eqns. (37) to (40), the eqns. (24) and (25) are reduced to the following equations respectively :

$$a_l^r \simeq (-i)^{-l}.e^{-i\rho}.\left[\frac{\rho.k_{m_1}.\sin y+k_{m_2}.\chi_1'\cos y}{\rho k_{m_1}-i\,k_{m_2}\chi_1'}\right]. \quad ...(41)$$

and

$$b_l^r \simeq (-i)^{-l}.e^{-i\rho}\left[\frac{k_{m_1}\chi_2'.\cos y+k_{m_2}.N^2\rho(1-a_1N^2\rho^2)\sin y}{i\,k_{m_1}.\chi_2'+k_{m_2}N^2\rho(1-a_1N^2\rho^2)}\right]. \quad ...(42)$$

where $Y=\rho-(l+1)\frac{\pi}{2}$. These results are significantly different from those of Mie.

Discussion and Conclusions

In the present communication, exact solutions have been obtained for the dielectric profile of the sphere given by the function $\epsilon_1(r)=\epsilon_0.e^{-\mu^2r^2}$ where $\mu^2a^2 << 1$.

Further equations co-relating the functions $\vec{m}^{(t)}$ and $\vec{n}^{(t)}$ for T.E. or T.M. waves have been obtained by expanding the non-homogeneous vector wave equations thereby proving that $\vec{m}^{(t)}$ and $\vec{n}^{(t)}$ so obtained are the solutions. It can be shown by lengthy but straight forward mathematical operations that these solutions are mathematically the only feasible solutions. The scattering coefficients for the cases studied are given by Eqns. (24) and (25) and are liable to simple computerised techniques.

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