ON A GENERAL CLASS OF POLYNOMIALS RELEVANT TO QUANTUM MECHANICS

by T. R. PRABHAKAR *and* SUDHA JAIN, *Faculty of Mathematics, Delhi University, Delhi-110 007*

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This is a study of a very general class of polynomials $R_n^{\mu}(x)$, which include squares and products of many of the standard polynomials and are of interest in the problems of quantum mechanics and mathematical physics. The polynomials $R_n^{\mu}(x)$ also contain as special cases a large number of well-known polynomials. We obtain several integral formulae, series relations and recurrence relations for these polynomials. On specialization of parameters they yield a host of results on squares and products of various known polynomials.

INTRODUCTION

In problems of quantum mechanics and related fields there often occur not only various polynomials such as those of Gegenbauer, Legendre, but also the products or squares of these polynomials (cf. Cohen & Landsbury 1967 ; Cohen 1968-69 ; and Landsberg *et al.* 1964). Cohen 1968-69 introduced a general polynomial set Θ_n, which included several polynomials as also squares of these polynomials as particular cases; he defined Θ_n by the generating relation

$$(1 - z)^{-2\nu} {}_{p+1}F_{q+1}\left(\begin{matrix} \nu, a_1, \ldots, a_p \\ \\ b_0, b_1, \ldots, b_q \end{matrix} ; -\frac{4xz}{(1-z)^2}\right) = \sum_{n=0}^{\infty} z^n \Theta_n \quad \ldots(1)$$

and stated recurrence relations for the polynomial. He also gave some simple integrals involving Θ_n. One of the aims of this paper is to evaluate some very general integrals involving Θ_n and to sum up certain infinite series of these polynomials. We, however, prove all the results for the polynomial $R_n^{\mu}(x)$, which is much more general than Θ_n and is defined by

$$(1 - t)^{-c} {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ \\ b_1, \ldots, b_q \end{matrix} ; -\frac{\lambda xt}{(1-t)^{\mu}}\right) = \sum_{n=0}^{\infty} \frac{t^n (c)_n}{n!} R_n^{\mu}(x), \quad \ldots(2)$$

where μ is any positive integer. Evidently $R_n^{\mu}(x)$ includes as a particular case the polynomial Θ_n, in addition to many other important polynomials such as those of

Laguerre, Jacobi, Sister Celine, Konhauser etc. It also includes as special cases the squares and products of some polynomials such as those of Laguerre polynomials, which are not contained in Θ_n. Indeed on specializing parameters, one can obtain a number of results believed to be new for Θ_n and for many other polynomials as also for the products and squares of the polynomials.

Lastly we remark that though several polynomial sets with generating functions of the form

$$(1 - t)^{-c} \psi \left[\frac{-4xt}{(1-t)^2} \right]$$

have been studied by various authors (cf. Rainville 1960), no attempt seems to have been made so far to investigate a polynomial set which has a generating function of the form

$$(1 - t)^{-c} \psi \left[\frac{-\lambda xt}{(1-t)^\mu} \right],$$

where μ is any positive integer. The present study may also be regarded as an attempt in this direction.

SPECIAL CASES OF $R_n^\mu(x)$

From (2), it follows that the polynomial $R_n^\mu(x)$ has the following representation as a generalized hypergeometric function :

$$R_n^\mu(x) \equiv R_n^\mu(c\ ;\ a_1, \ldots, a_p\ ;\ b_1, \ldots, b_q\ ;\ \delta x)$$

$$= {}_{p+\mu}F_{q+\mu}\left(\begin{matrix} -n, a_1, \ldots, a_p, \triangle(\mu - 1, c + n) \\ b_1, \ldots, b_q, \triangle(\mu, c) \end{matrix} ;\ \delta x \right), \qquad \ldots(3)$$

where $\triangle(r, \alpha)$ denotes set of r parameters $\frac{\alpha}{r}, \frac{\alpha+1}{r}, \ldots, \frac{\alpha+r-1}{r}$, r being a positive integer. For $r = 0$, $\triangle(r, \alpha)$ is to be taken as empty and

$$\delta = \left(1 - \frac{1}{\mu}\right)^{\mu-1} \frac{\lambda}{\mu}, \qquad \mu > 1\ ;$$

evidently

$$\delta = \lambda, \qquad \text{when } \mu = 1.$$

We next list some of the particular cases of $R_n^\mu(x)$.

$$R_n^1\left[1 + \alpha\ ;\ \alpha + n + 1\ ;\ \frac{1+\alpha}{2}, 1 + \frac{\alpha}{2}\ ;\ x \right]$$

$$= \left(\frac{n!}{(1+\alpha)_n} \right)^2 L_n^{(\alpha)}(2\sqrt{x})\, L_n^{(\alpha)}(-2\sqrt{x}). \qquad \ldots(4)$$

[Product of Laguerre]

$$R_n^2\,(2\nu\,;\,\nu,\,\nu\,;\,2\nu\,;\,x) = \left(\frac{n!}{(2\nu)_n}\right)^2 \{C_n^\nu\,(1-x)^{1/2}\}^2. \qquad \ldots(4)$$

[Square of Gegenbauer]

$$R_n^2\,(2\nu\,;\,\nu,\,\nu\,;\,2\nu\,;\,x) = \{\Gamma(\tfrac{1}{2}+\nu)\}^2\left(-\frac{x}{4}\right)^{(1/2)-\nu}\{P^{1/2-\nu}_{n+\nu-1/2}\,(1-x)^{1/2}\}^2. \qquad \ldots(5)$$

[Square of associated Legendre]

$$R_n^2\,(1\,;\,\tfrac{1}{2},\,\tfrac{1}{2}\,;\,1\,;\,x) = \{P_n(1-x)^{1/2}\}^2 \qquad \ldots(6)$$

[Square of Legendre]

$$R_n^2\,(1\,;\,\tfrac{1}{2},\,\tfrac{1}{2},\,1\,;\,1+g,\,1-g\,;\,x)$$

$$= \frac{(-1)^n\,(n!)^2}{(1+g)_n\,(1-g)_n}\;P_n^{(g,-g)}\,\{(1-x)^{1/2}\}\;P_n^{(g,-g)}\,\{-(1-x)^{1/2}\}. \qquad \ldots(7)$$

[Product of Jacobi]

$$R_n^2\,(2\nu\,;\,\nu,\,a_2,\,\ldots,\,a_p\,;\,b_1,\,\ldots,\,b_q\,;\,x)$$

$$= \frac{n!}{(2\nu)_n}\,\Theta_n(\nu\,;\,a_2\,\ldots\,a_p\,;\,b_1\,\ldots\,b_q\,;\,x) \qquad \ldots(8)$$

[Cohen]

$$R_n^2\,(1+\alpha+\beta;\,\tfrac{1}{2}(\alpha+\beta+1),\,\tfrac{1}{2}(\alpha+\beta+2),\,a_3,\ldots,a_p;\,1+\alpha,\,\tfrac{1}{2},\,b_3,\ldots,b_q;\,x)$$

$$= \frac{n!}{(1+\alpha)_n}\,f_n^{(\alpha,\,\beta)}\left(\begin{matrix} a_3,\,\ldots,\,a_p \\ \\ b_3,\,\ldots,\,b_q\end{matrix}\;;\,x\right) \qquad \ldots(9)$$

Here $f_n^{(\alpha,\,\beta)}\left(\begin{matrix} a_3,\,\ldots,\,a_p \\ \\ b_3,\,\ldots,\,b_q\end{matrix}\;;\,x\right)$ is generalized Sister Celine's polynomial (Shah 1967) and reduces to Sister Celine's polynomial (Rainville 1960) when $\alpha = \beta = 0$.

$$R_n^2\,(1+\alpha+\beta\,;\,\tfrac{1}{2}(1+\alpha+\beta),\,\tfrac{1}{2}(2+\alpha+\beta),\,\xi\,;\,1+\alpha,\,p\,;\,x)$$

$$= \frac{n!}{(1+\alpha)_n}\,H_n^{(\alpha,\,\beta)}\,(\xi,\,p,\,x), \qquad \ldots(10)$$

where $H_n^{(\alpha,\,\beta)}\,(\xi,\,p,\,x)$ is generalized Rice polynomial (Khandekar 1964), which reduces to Rice polynomial $H_n(\xi,\,p,\,x)$ when $\alpha = \beta = 0$, (Rainville 1960).

$$R_n^2\,(1+\alpha+\beta\,;\,\tfrac{1}{2}(1+\alpha+\beta),\,\tfrac{1}{2}(2+\alpha+\beta)\,;\,1+\alpha\,;\,x)$$

$$=\frac{n!}{(1+\alpha)_n}\,P_n^{(\alpha,\,\beta)}\,(1-2x) \qquad ...(11)$$

[Jacobi]

$$R_n^2\,(2\nu\,;\,\nu\,:\,-\,;\,x)=\Gamma(\nu+\tfrac{1}{2})\,\{x(x-1)\}^{1/2-\nu/2}\,P_{n+\nu-1/2}^{1/2-\nu}\,(1-2x), \qquad ...(12)$$

[Associated Legendre]

and

$$R_n^2\left(a-1\,;\,\tfrac{1}{2}(a-1),\,\frac{a}{2};\,-\,:\,-\frac{x}{b}\right)=y_n(x,\,a,\,b), \qquad ...(13)$$

where $y_n(x, a, b)$ is generalized Bessel polynomial which reduces to simple Bessel polynomial $y_n(x)$ when $a = b = 2$, (Rainville 1960).

$$R_n^1\left[1\,;\,1\,:\,\frac{\alpha+1}{k},\,...,\,\frac{\alpha+k}{k}\,;\,\frac{x^k}{k^k}\right]=\frac{n!}{\Gamma(kn+\alpha+1)}\,Z_n^{\alpha}\,(x\,;\,k). \qquad ...(14)$$

Here $Z_n^{\alpha}(x\,;k)$ is one of the polynomials in the biorthogonal pair studied recently by Konhauser (1967) and Prabhakar (1970).

$$R_n^1\,[1+\alpha\,;\,-\,:\,-\,;\,x]=\frac{n!}{(1+\alpha)_n}\,L_n^{\alpha}\,(x) \qquad ...(15)$$

[Generalized Laguerre]

$$R_n^2\,[1\,;\,\tfrac{1}{2}\,;\,1\,;\,x]=Z_n(x) \qquad ...(16)$$

[Bateman's polynomial]

INTEGRALS

In this section some integrals involving the polynomial $R_n^{\mu}(x)$ have been evaluated. We prove that

$$\int_0^{\pi/2}\cos 2u\theta(\sin\theta)^{\nu}\,[R_n^{\mu}\,(c\,;\,a_1,\,...,\,a_p\,;\,b_1,\,...,\,b_q\,;\,\delta x(\sin\theta)^{2h}]\,d\theta$$

$$=\frac{\Gamma(\tfrac{1}{2}+u)\,\Gamma(\tfrac{1}{2}-u)\,\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u+1)\,\Gamma(\nu/2-u+1)}\,R_n^{\mu}(c\,;\,a_1,\,...,\,a_p,$$

$$\triangle(2h,\,\nu+1)\,;\,b_1,\,...,\,b_q,\,\triangle(h,\,\nu/2+u+1),$$

$$\triangle(h,\,\nu/2-u+1)\,;\,\delta x) \qquad ...(17)$$

and

$$\int_0^{\pi/2} \cos u\theta (\cos\theta)^{\nu} \, [R_n^{\mu}(c\,;a_1,\ldots,a_p\,;b_1,\ldots,b_q\,;\delta x(\cos\theta)^{2h})]\,d\theta$$

$$= \frac{\pi\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u/2+1)\,\Gamma(\nu/2-u/2+1)}\; R_n^{\mu}(c\,;a_1,\ldots a_p,$$

$$\triangle(2h,\nu+1) : b_1,\ldots,b_q, \triangle(h,\nu/2+u/2+1),$$

$$\triangle(h,\nu/2-u/2+1)\,;\,\delta x), \qquad \ldots(18)$$

where $u = 0, 1, \ldots$, $Re\ \nu > 0$, h is positive integer and $0 < \theta < \frac{\pi}{2}$.

Proof :

To prove (17), we use (3) in the integrand and change the order of integration and summation which is easily justified. The left-hand member can, then, be written as

$$\sum_{r=0}^{\infty} \frac{\prod_{i=0}^{\mu-2}\left(\frac{c+n+i}{\mu-1}\right)_r (a_1)_r \ldots (a_p)_r (-n)_r\,\delta^r x^r}{\prod_{i=0}^{\mu-1}\left(\frac{c+i}{\mu}\right)_r (b_1)_r \ldots (b_q)_r\, r!} \times$$

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^{\nu+2hr}\,d\theta.$$

Now evaluating the integral with the help of the result [3, 1.5.1 (30)] :

$$\times\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^{\nu}\,d\theta = \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{\nu+1}\,\Gamma(\nu/2+u+1)\,\Gamma(\nu/2-u+1)},$$

where $u = 0, 1, \ldots$ and $Re\ \nu \geqslant 0$, and using the relations

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\ (\alpha)_{nk} = k^{nk}\prod_{i=0}^{k-1}\left(\frac{\alpha+i}{k}\right)_n, \qquad \ldots(19)$$

we can write the above series as

$$\frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{\nu+1}\,\Gamma(\nu/2+u+1)\,\Gamma(\nu/2-u+1)}\times$$

$$\sum_{r=0}^{\infty}\frac{\prod_{i=0}^{\mu-2}\left(\frac{c+n+i}{\mu-1}\right)_r (a_1)_r \ldots (a_p)_r\,(-n)_r \prod_{i=0}^{2h-1}\left(\frac{\nu+1+i}{2h}\right)_r \delta^r x^r}{\prod_{i=0}^{\mu-1}\left(\frac{c+i}{\mu}\right)_r (b_1)_r \ldots (b_q)_r \prod_{i=0}^{h-1}\left(\frac{\nu/2+u+1+i}{h}\right)_r \prod_{i=0}^{h-1}\left(\frac{\nu/2-u+1+i}{h}\right)_r r!}$$

which leads to (17).

The integral (18) may be evaluated on applying the same procedure as above and using the known results [3, 1.5.1, (30)]

$$\int_0^{\pi/2}\cos m\theta\,\cos^n\theta\,d\theta=\frac{\pi\Gamma(n+1)}{2^{n+1}\,\Gamma(n/2+m/2+1)\,\Gamma(n/2-m/2+1)}$$

and (19).

Some Series Relations

In this section, we use the results of the previous section to establish some series relations involving $R_n^{\mu}(x)$. We prove that :

$$(\sin\theta)^{\nu}\,[R_n^{\mu}(c\,;a_1,\ldots,a_p\,;b_1,\ldots,b_q\,;\delta x(\sin\theta)^{2h})]$$

$$=\frac{\Gamma(\nu+1)}{\pi\,2^{\nu-1}}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}+r)\,\Gamma(\frac{1}{2}-r)}{\Gamma(\nu/2+r+1)\,\Gamma(\nu/2-r+1)}\,R_n^{\mu}(c;a_1,\ldots,a_p,$$

$$\triangle(2h,\nu+1)\,;b_1,\ldots,b_q,\triangle(h,\nu/2+r+1),$$

$$\triangle(h,\nu/2-r+1)\,;\delta x)\cos 2r\theta \qquad \ldots(20)$$

and

$$(\cos\theta)^{\nu}\,R_n^{\mu}(c\,;a_1,\ldots,a_p\,;b_1,\ldots,b_q\,;\delta x(\sin\theta)^{2h})]$$

$$=\frac{\Gamma(\nu+1)}{2^{\nu-1}}\sum_{r=0}^{\infty}\frac{1}{\Gamma(\nu/2+r/2+1)\,\Gamma(\nu/2-r/2+1)}\,R_n^{\mu}(c;a_1,\ldots a_p,$$

$$\triangle(2h,\nu+1)\,;b_1,\ldots,b_q,\triangle(h,\nu/2+r/2+1),$$

$$\triangle(h,\nu/2-r/2+1)\,;\delta x)\cos r\theta, \qquad \ldots(21)$$

where $0<\theta<\pi/2$, $Re\,(\nu)\geqslant 0$ and h is a positive integer.

Proof :

Let

$$f(\theta) = (\sin\theta)^{\nu} R_n^{\mu}(c\ ;\ a_1, \ldots, a_p\ ;\ b_1, \ldots, b_q\ ;\ \delta x(\sin\theta)^{2h})$$

$$= \sum_{r=0}^{\infty} A_r \cos 2r\theta, \qquad 0 < \theta < \pi/2\ ;$$

then since $f(\theta)$ is continuous and of bounded variation in $(0, \pi/2)$, the co-efficients A_r can be computed by the help of (17), we have that

$$A_r = \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}+r)\,\Gamma(\frac{1}{2}-r)}{\pi\, 2^{\nu-1}\,\Gamma(\nu/2+r+1)\,\Gamma(\nu/2-r+1)} R_n^{\mu}(c\ ;\ a_1, \ldots, a_p,$$

$$\triangle(2h, \nu+1);\ b_1, \ldots, b_q,\ \triangle(h, \nu/2+r-1),\ \triangle(h, \nu/2-r+1)\ ;\ \delta x).$$

The series in (21) is derived in a similar manner on using (18).

APPLICATIONS

We now apply the results of the preceding sections to evaluate several integrals and series involving various standard polynomials as also integrals and series involving the products and squares of these polynomials.

(*i*) If we take $c = 1+\alpha$, $\mu = \lambda = 1$, $a_1 = (1+\alpha+n)$, $b_1 = \frac{1+\alpha}{2}$, $b_2 = 1+\alpha/2$ and $p = 2$, $q = 2$, we get the following results for products of Laguerre polynomial $L_n^{\alpha}(x)$ valid for $Re\ \nu \geqslant 0$, $u = 0, 1, \ldots$ and $0 < \theta < \pi/2$.

$$\int_0^{\pi/2} \cos 2u\theta(\sin\theta)^{\nu} L_n^{\alpha}(2(\sin\theta)^h\sqrt{x})\, L_n^{\alpha}(-2(\sin\theta)^h\sqrt{x})\, d\theta$$

$$= \left(\frac{(1+\alpha)_n}{n!}\right)^2 \frac{\Gamma(1/2+u)\,\Gamma(1/2-u)\,\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u+1)\,\Gamma(\nu/2-u+1)} \times$$

$$ {}_{2+2h}F_{3+2h}\left(\begin{matrix} -n,\ \alpha+n+1,\ \triangle(2h, \nu+1) \\ 1+\alpha,\ \frac{1+\alpha}{2},\ 1+\alpha/2,\ \triangle(h, \nu/2+u+1),\ \triangle(h, \nu/2-u+1) \end{matrix}\ ; x\right) \qquad \ldots(22)$$

$$(\sin\theta)^{\nu}\ L_n^{\alpha}(2(\sin\theta)^h\sqrt{x})\, L_n^{\alpha}(-2(\sin\theta)^h\sqrt{x})$$

$$= \left(\frac{(1+\alpha)_n}{n!}\right)^2 \frac{\Gamma(\nu+1)}{\pi\, 2^{\nu-1}} \sum_{r=0}^{\infty} \frac{\Gamma(1/2+r)\,\Gamma(1/2-r)}{\Gamma(\nu/2+r+1)\,\Gamma(\nu/2-r+1)} \times$$

$$ {}_{2+2h}F_{3+2h}\left(\begin{matrix} -n, \alpha+n+1, \triangle(2h, \nu+1) \\ 1+\alpha, \frac{1+\alpha}{2}, 1+\alpha/2, \triangle(h, \nu/2+r+1), \triangle(h, \nu/2-r+1) \end{matrix}; x\right) $$

$$ \times \cos 2r\theta \qquad \ldots(23) $$

$$ \int_0^{\pi/2} \cos 2u\theta (\cos\theta)^{\nu} \; L_n^{\alpha}(2(\cos\theta)^h \sqrt{x}) \; L_n^{\alpha}(-2(\cos\theta)^h \sqrt{x}) \, d\theta $$

$$ = \left(\frac{(1+\alpha)_n}{n!}\right)^2 \frac{\pi\,\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u/2+1)\,\Gamma(\nu/2-u/2+1)} \times $$

$$ {}_{2+2h}F_{3+2h}\left(\begin{matrix} -n, \alpha+n+1, (2h, \nu+1) \\ 1+\alpha, \frac{(1+\alpha)}{2}, 1+\alpha/2, \triangle(h, \nu/2+u/2+1), \triangle(h, \nu/2-u/2+1) \end{matrix}; x\right). \qquad \ldots(24) $$

$$ (\cos\theta)^{\nu} \; L_n^{\alpha}(2(\cos\theta)^h \sqrt{x}) \; L_n^{\alpha}(-2(\cos\theta)^h \sqrt{x}) $$

$$ = \left(\frac{(1+\alpha)_n}{n!}\right)^2 \frac{\Gamma(\nu+1)}{2^{\nu-1}} \sum_{r=0}^{\infty} \frac{1}{\Gamma(\nu/2+r/2+1)\,\Gamma(\nu/2-r/2+1)} \times $$

$$ {}_{2+2h}F_{3+2h}\left(\begin{matrix} -n, \alpha+n+1, \triangle(2h, \nu+1) \\ 1+\alpha, 1+\alpha/2, \frac{1+\alpha}{2}, \triangle(h, \nu/2+r/2+1), \triangle(h, \nu/2-r/2+1) \end{matrix}; x\right) $$

$$ \times \cos r\theta \qquad \ldots(25) $$

(*ii*) If $c = 2\alpha$, $\lambda = 4$, $\mu = 2$ and $a_1 = \alpha$, we obtain for the polynomial $\Theta_n(x)$:

$$ \int_0^{\pi/2} \cos 2u\theta (\sin\theta)^{\nu}\, \Theta_n(\alpha\,;\, a_2, \ldots, a_p\,;\, b_1, \ldots b_q\,;\, x(\sin\theta)^{2h})\, d\theta $$

$$ = \frac{(2\alpha)_n\,\Gamma(1/2+u)\,\Gamma(1/2-u)\,\Gamma(\nu+1)}{n!\,2^{\nu+1}\,\Gamma(\nu/2+u+1)\,\Gamma(\nu/2-u+1)}\, \Theta_n(\alpha\,;\, a_2, \ldots, a_p, \triangle(2h, \nu+1)\,; $$

$$ b_1, \ldots, b_q, \triangle(h, \nu/2+u+1), \triangle(h, \nu/2-u+1)\,;\, x), \qquad \ldots(26) $$

$$ (\sin\theta)^{\nu}\, \Theta_n(\nu\,;\, a_2, \ldots, a_p\,;\, b_1, \ldots, b_q\,;\, x(\sin\theta)^{2h}) $$

$$ = \frac{(2\nu)_n}{n!} \frac{\Gamma(\nu+1)}{\pi\,2^{\nu-1}} \sum_{r=0}^{\infty} \frac{\Gamma(1/2+r)\,\Gamma(1/2-r)}{\Gamma(\nu/2+r+1)\,\Gamma(\nu/2-r+1)}\, \Theta_n(\nu\,;\, a_2, \ldots, a_p, $$

$\triangle(2h, \nu + 1) ; b_1, \ldots, b_q, \triangle(h, \nu/2 + r + 1), \triangle(h, \nu/2 - r + 1) ; x) \times$
$\cos 2r\theta$ ...(27)

and

$$\int_0^{\pi/2} \cos 2u\theta(\cos\theta)^\nu (\Theta_n(\alpha ; a_2, \ldots, a_p : b_1, \ldots, b_q ; x(\cos\theta)^{2h}) \, d\theta$$

$$= \frac{(2\alpha)_n}{n!} \frac{\pi\,\Gamma(\nu+1)}{\Gamma(\nu/2 + u/2 + 1)\,\Gamma(\nu/2 - u/2 + 1)} \Theta_n(\alpha ; a_2, \ldots a_p, \triangle(2h, \nu+1) :$$
$$b_1, \ldots, b_q, \triangle(h, \nu/2 + u/2 + 1), \triangle(h, \nu/2 - u/2 + 1) ; x), \qquad \ldots(28)$$

$(\cos\theta)^\nu\,\Theta_n(\alpha : a_2, \ldots, a_p : b_1, \ldots, b_q ; x(\sin\theta)^{2h})$

$$= \frac{(2\alpha)_n}{n!} \frac{\Gamma(\nu+1)}{2^{\nu-1}} \sum_{r=0}^{\infty} \frac{1}{\Gamma(\nu/2 + r/2 + 1)\,\Gamma(\nu/2 - r/2 + 1)} \times$$
$$\Theta_n(\alpha ; a_2, \ldots, a_p, \triangle(2h, \nu+1) : b_1, \ldots, b_q, \triangle(h, \nu/2 + r/2 + 1),$$
$$\triangle(h, \nu/2 - r/2 + 1) ; x) \cos r\theta \qquad \ldots(29)$$

valid for $0 < \theta < \pi/2$, $Re\ \nu \geqslant 0$ and h is positive integer, where Θ_n is polynomial introduced by Cohen.

(*iii*) Putting $c = 1$, $\lambda = 4$, $\mu = 2$, $a_1 = 1/2$, $a_2 = 1/2$, $b_1 = 1$ and $q = 1$, $p = 2$, we get for Legendre polynomials $P_n(x)$:

$$\int_0^{\pi/2} \cos 2u\theta(\sin\theta)^\nu \{P_n(1 - x(\sin\theta)^{2h})^{1/2}\}^2 \, d\theta$$

$$= \frac{\Gamma(1/2 + u)\,\Gamma(1/2 - u)\,\Gamma(\nu+1)}{2^{\nu-1}\,\Gamma(\nu/2 + u + 1)\,\Gamma(\nu/2 - u + 1)} \times$$
$$ {}_{3+2h}F_{2+2h}\left(\begin{matrix} -n, n+1, 1/2, \triangle(2h, \nu+1) \\ 1, 1, \triangle(h, \nu/2 + u + 1), \triangle(h, \nu/2 - u + 1) \end{matrix} ; x \right), \qquad \ldots(30)$$

$(\sin\theta)^\nu \{P_n(1 - x(\sin\theta)^{2h})^{1/2}\}^2$

$$= \frac{\Gamma(\nu+1)}{\pi\, 2^{\nu-1}} \sum_{r=0}^{\infty} \frac{\Gamma(1/2 + r)\,\Gamma(1/2 - r)}{\Gamma(\nu/2 + r + 1)\,\Gamma(\nu/2 - r + 1)} \times$$
$$ {}_{3+2h}F_{2+2h}\left[\begin{matrix} -n, n+1, 1/2, \triangle(2h, \nu+1) \\ 1, 1, \triangle(h, \nu/2 + r + 1), \triangle(h, \nu/2 - r + 1) \end{matrix} ; x \right] \cos 2r\,\theta$$
$$\ldots(31)$$

and

$$\int_0^{\pi/2} \cos 2u\theta (\cos\theta)^\nu \{P_n(1-x(\cos\theta)^{2h})^{1/2}\}^2 \, d\theta$$

$$= \frac{\pi\,\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u/2+1)\;\Gamma(\nu/2-u/2+1)} \times$$

$${}_{3+2h}F_{2+2h}\left(\begin{matrix} -n,\ n+1,\ 1/2,\ \triangle(2h,\ \nu+1) \\ 1,\ 1,\ \triangle(h,\ \nu/2+u/2+1),\ \triangle(h,\ \nu/2-u/2+1) \end{matrix};\ x\right) \quad \ldots(32)$$

and

$(\cos\theta)^\nu\ \{P_n(1-x(\cos\theta)^{2h})^{1/2}\}^2$

$$= \frac{\Gamma(\nu+1)}{2^{\nu-1}} \sum_{r=0}^{\infty} \frac{1}{\Gamma(\nu/2+r/2+1)\,\Gamma(\nu/2-r/2+1)} \times$$

$${}_{3+2h}F_{2+2h}\left(\begin{matrix} -n,\ n+1,\ 1/2,\ \triangle(2h,\ \nu+1) \\ 1,\ 1,\ \triangle(h,\ \nu/2+r/2+1),\ \triangle(h,\ \nu/2-r/2+1) \end{matrix};\ x\right)\cos r\theta, \quad \ldots(33)$$

where $Re\ \nu \geqslant 0$ and $0 < \theta < \pi/2$, $u = 0, 1, \ldots$ and $P_n(x)$ is Legendre polynomial.

(*iv*) Putting $c = 2\alpha$, $\mu = 2$, $\lambda = 4$, $a_1 = \alpha$, $a_2 = \alpha$, $b_1 = 2\alpha$ and $p = 2$, $q = 1$ we have

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^\nu \{C_n^{\alpha}(1-x(\sin\theta)^{2h})^{1/2}\}^2 \, d\theta$$

$$= \left(\frac{(2\alpha)_n}{n!}\right)^2 \frac{\Gamma(1/2+u)\,\Gamma(1/2-u)\,\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u+1)\,\Gamma(\nu/2-u+1)} \times$$

$${}_{3+2h}F_{2+2h}\left(\begin{matrix} -n,\ n+2\alpha,\ \alpha,\ \triangle(2h,\ \nu+1) \\ 2\alpha,\ \alpha+1/2,\ \triangle(h,\ \nu/2+u+1),\ \ \triangle(h,\ \nu/2-u+1) \end{matrix};\ x\right) \quad \ldots(34)$$

$(\sin\theta)^\nu\ \{C_n^{\alpha}(1-x(\sin\theta)^{2h})^{1/2}\}^2$

$$= \left(\frac{(2\alpha)_n}{n!}\right)^2 \frac{\Gamma(\nu+1)}{\pi\,2^{\nu-1}} \sum_{r=0}^{\infty} \frac{\Gamma(1/2+r)\,\Gamma(1/2-r)}{\Gamma(\nu/2+r+1)\,\Gamma(\nu/2-r+1)} \times$$

$$ {}_{3+2h}F_{2+2h}\left(\begin{matrix} -n,\ n+2\alpha,\ \alpha,\ \triangle(2h,\ \nu+1) \\ 2\alpha,\ \alpha+1/2,\ \triangle(h,\ \nu/2+r+1),\ \triangle(h,\ \nu/2-r+1) \end{matrix}\ ;\ x\right) \times \cos 2r\theta, \quad \ldots(35) $$

and

$$ \int_0^{\pi/2} \cos u\theta\ (\cos\theta)^{\nu}\ \{C_n^{\alpha}(1-x(\cos\theta)^{2h})^{1/2}\}^2\ d\theta. $$

$$ = \left(\frac{(2\alpha)_n}{n!}\right)^2 \frac{\pi\ \Gamma(\nu+1)}{2^{\nu+1}\ \Gamma(\nu/2+u/2+1)\ \Gamma(\nu/2-u/2+1)} \times $$

$$ {}_{3+2h}F_{2+2h}\left[\begin{matrix} -n,\ n+2\alpha,\ \alpha,\ \triangle(2h,\ \nu+1) \\ 2\alpha,\ \alpha+1/2,\ \triangle(h,\ \nu/2+u/2+1),\ \triangle(h,\ \nu/2-u/2+1) \end{matrix}\ ;\ x\right], \quad \ldots(36) $$

$$ (\cos\theta)^{\nu}\ \{C_n^{\alpha}(1-x(\cos\theta)^{2h})^{1/2}\}^2 $$

$$ = \frac{\Gamma(\nu+1}{2^{\nu-1}} \left(\frac{(2\alpha)_n}{n!}\right)^2 \sum_{r=0}^{\infty} \frac{1}{\Gamma(\nu/2+r/2+1)\ \Gamma(\nu/2-r/2+1)} $$

$$ {}_{3+2h}F_{2+2h}\left(\begin{matrix} -n,\ n+2\alpha,\ \alpha,\ \triangle(2h,\ \nu+1) \\ 2\alpha,\ \alpha+1/2,\ \triangle(h,\ \nu/2+r/2+1),\ \triangle(h,\ \nu/2-r/2+1) \end{matrix}\ ;\ x\right) \times \cos r\theta \quad \ldots(36) $$

valid for $Re\ \nu \geqslant 0$, $u = 0, 1, \ldots$ and $0 < \theta < \pi/2$, where $C_n^{\alpha}(x)$ is Gegenbauer polynomial.

(v) Put $\lambda = \mu = c = 1$, $b_1 = 1 + \alpha$ and $p = 0$, $q = 1$, we get

$$ \int_0^{\pi/2} \cos 2u\theta (\sin\theta)^{\nu}\ L_n^{\alpha}(x(\sin\theta)^{2h})\ d\theta $$

$$ = \frac{(1+\alpha)_n}{n!} \frac{\Gamma(1/2+u)\ \Gamma(1/2-u)\ \Gamma(\nu+1)}{2^{\nu+1}\ \Gamma(\nu/2+u+1)\ \Gamma(\nu/2-u+1)} \times $$

$$ {}_{1+2h}F_{1+2h}\left(\begin{matrix} -n,\ \triangle(2h,\ \nu+1) \\ 1+\alpha,\ \triangle(h,\ \nu/2+u+1),\ \triangle(h,\ \nu/2-u+1) \end{matrix}\ ;\ x\right) \ldots(37) $$

$$(\sin\theta)^{\nu}\ L_n^{\alpha}\,(x(\sin\theta)^{2h})$$

$$= \frac{\Gamma(\nu+1)}{\pi\, 2^{\nu-1}} \sum_{r=0}^{\infty} \frac{\Gamma(1/2+r)\,\Gamma(1/2-r)}{\Gamma(\nu/2+r+1)\,\Gamma(\nu/2-r+1)} \times$$

$$ {}_{1+2h}F_{1+2h}\left[\begin{matrix} -n,\ \triangle(2h,\ \nu+1) \\ 1+\alpha,\ \triangle(h,\ \nu/2+r+1),\ \triangle(h,\ \nu/2-r+1) \end{matrix};\ x\right] \times \cos 2r\theta \qquad \ldots(38)$$

and

$$\int_0^{\pi/2} \cos u\theta(\cos\theta)^{\nu}\ L_n^{\alpha}\,(x(\cos\theta)^{2h})\, d\theta$$

$$= \frac{(1+\alpha)_n}{n!} \frac{\pi\,\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\nu/2+u/2+1)\,\Gamma(\nu/2-u/2+1)} \times$$

$$ {}_{1+2h}F_{1+2h}\left(\begin{matrix} -n,\ \triangle(2h,\ \nu+1) \\ 1+\alpha,\ \triangle(h,\ \nu/2+u/2+1),\ \ \triangle(h,\ \nu/2-u/2+1) \end{matrix};\ x\right) \qquad \ldots(39)$$

$$(\cos\theta)^{\nu}\ L_n^{\alpha}\ (x(\cos\theta)^{2h})$$

$$= \frac{\Gamma(\nu+1)}{2^{\nu-1}} \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{\infty} \frac{1}{\Gamma(\nu/2+r/2+1)\,\Gamma(\nu/2-r/2+1)} \times$$

$$ {}_{1+2h}F_{1+2h}\left(\begin{matrix} -n,\ \triangle(2h,\ \nu+1) \\ 1+\alpha,\ \triangle(h,\ \nu/2+r/2+1),\ \triangle(h,\ \nu/2-r/2+1) \end{matrix};\ x\right) \times \cos r\theta \qquad \ldots(40)$$

valid for $0 < \theta < \pi/2$, $Re\ \nu \geqslant 0$ and $u = 0, 1, \ldots,$ where $L_n^{\alpha}(x)$ is generalized Laguerre polynomial.

On specializations of parameters many more results involving other polynomials mentioned earlier can be obtained. We have selected only a few.

Recurrence Relations and More Integral Properties of $R_n^{\mu}(x)$

Using (2), it is not difficult to see that a number of recurrence relations can be obtained; some of them are listed below. We also list some integral formulae which can be proved by using known techniques.

(*i*) $x[(c+n-1)\ DR_n^{\mu}(x) - n(1-\mu)\ DR_n^{\mu}(x)]$

$$= (c+n-1)\ [nR_n^{\mu}(x) - nR_{n-1}^{\mu}(x)]\ ;\ \text{for } n \geqslant 1$$

(*ii*) $xD\ R_n^{\mu}(x) - nR_n^{\mu}(x)$

$$= -\frac{n!}{(c)_n}\left[\sum_{k=0}^{n-1}\frac{(c)_k}{k!}\ \{R_k^{\mu}(x) + \mu x DR_k^{\mu}(x)\}\right];\ \text{for } n \geqslant 1$$

(*iii*) $xDR_n^{\mu}(x) - nR_n^{\mu}(x)$

$$= \frac{n!}{(c)_n}\left[\sum_{k=0}^{n-1}(-1)^k(\mu-1)^{n-k}\frac{(c)_k}{k!}\ \{(c+\mu k)\ R_k^{\mu}(x)\right];$$

for $n \geqslant 1$

(*iv*) $$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\ R_n^{\mu}(c\ ;\ a_1, \ldots, a_p\ ;\ b_1, \ldots, b_q\ ;\ \delta x^k(t-x)^s)\ dx$$

$$= \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\alpha+\beta)}\ t^{\alpha+\beta-1}\ R_n^{\mu}\cdot\Big(c\ ;\ a_1, \ldots, a_p, \alpha/k, \frac{\alpha+1}{k}, \ldots, \frac{\alpha+k-1}{k}$$

$$\frac{\beta}{s}, \frac{\beta+1}{s}, \ldots, \frac{\beta+s-1}{s}\ ;\ b_1, \ldots, b_q, \frac{\alpha+\beta}{k+s}, \ldots, \frac{\alpha+\beta+k+s-1}{k+s}\ ;$$

$$\frac{k^k s^s\ \delta t^{k+s}}{(k+s)^{k+s}}\Big)\ ;$$

where k, s are non-negative integers both not equal to zero and $Re\ (\alpha) > 0$, $Re\ (\beta) > 0$.

(*v*) $$\int_0^{\infty} y^{\alpha-1}\ e^{-sy}\ R_n^{\mu}\ (c\ ;\ a_1, \ldots, a_p\ ;\ b_1, \ldots, b_q\ ;\ \delta y^k\ x^k)\ dy$$

$$= \frac{\Gamma(\alpha)}{s^{\alpha}}\ R_n^{\mu}\left(c\ ;\ a_1, \ldots, a_p, \alpha/k, \ldots, \frac{\alpha+k-1}{k}\ :\ b_1, \ldots, b_q\ ;\ \delta\left(\frac{x^k k}{s}\right)^k\right)$$

with $Re\ \alpha > 0$ and $Re\ s > 0$, (cf. Rice 1960).

(*vi*) The general integral for the interval $(-1, +1)$ is

$$\int_{-1}^{+1}(1-x)^{\alpha-1}(1+x)^{\beta-1}\ R_n^{\mu}\ (c\ ;\ a_1, \ldots, a_p\ ;\ b_1, \ldots, b_q\ :$$

$$\delta(1-x)^k(1+x)^s)\ dx$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, 2^{\alpha+\beta-1}\, R_n^{\mu}\Bigg[c\,;\, a_1,\, \ldots,\, a_p,\, \alpha/k,\, \frac{\alpha+1}{k},\, \ldots,\, \frac{\alpha+k-1}{k},$$

$$\frac{\beta}{s},\, \ldots,\, \beta+s-1/k\,;\, b_1,\, \ldots,\, b_q,\, \frac{\alpha+\beta}{k+s},\, \ldots,\, \frac{\alpha+\beta+k+s-1}{k+s}\,;$$

$$\frac{\delta s^s\, k^k\, 2^{k+s}}{(k+s)^{k+s}}\Bigg].$$

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