THREE BODY PROBLEM FOR GENERAL EVEN POWER POTENTIAL WITH APPLICATIONS

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Schrödinger equation for three body has been solved for a general even power potential using the technique formulated by Eyges (1961). Expression for the wave function of this general potential has then been utilised for evaluating the wave functions for harmonic oscillator, anharmonic oscillator and Gauss potentials as special cases.

Introduction

It is well known that simple three-body theories have been useful in explaining quite a few number of complex problems in nuclear physics provided these theories contain the essential features of the particular problem like correct born or driving term, and the correct thresholds and kinematics, unitarity etc. Truly speaking, the success of the simple nuclear physics calculations further led to the hope that a similar approach to other systems, like π-nucleon, may also prove successful. Numerical work in the three-body problem also gave an insight into the form of the theory and of reasonable approximation schemes in the study of final state interactions in three-body final states.

These days the standard formal starting point for the three-body problem is Faddeev formalism (1965). Exact three-body calculations using separable two-body interactions have been carried out for the neutron-deutron system at low and intermediate energies by Aaron *et al.* (1965). Amado (1966) has studied the exact bound state of the three-nucleon system with separable interactions between pairs. Bound state wave function from the scattering amplitude equation was obtained and this wave function was applied in the study of the proton- and neutron-body form factors of the triton. Mitra *et al.* (1965) obtained a set of equations of the Faddeev type with properly connected kernels from the four-particle Lippmann-Schwinger equation. These kernels contained only the two- and three particle scattering amplitudes that did not involve the potentials. Wong (1968) has applied a procedure based on the Faddeev equation to non-relativistic three-body problems with local interactions. Wong has discussed the examples of Yukawa potential, Coulomb potential and the phenomenological α-α potential to test the usefulness of his procedure. A number of relatively recent developments in atomic theory have evoked interest in explicitly correlated wave functions for three-body systems with Coulomb interaction. Ferrante *et al.* (1976) realising the crucial importance of these correlated wave functions in the calculation of several atomic and molecular properties and in obtaining meaningful results in many collision problems have derived wave function for three-particle Coulomb systems for the case of $\bar{H}$, $\bar{e}\overset{+}{e}\bar{e}$, $d\mu d$, $t\mu t$ and $p\mu d$.

Eyges (1961) treated the quantum mechanical problem of finding the ground-state energy and eigenfunction for three identical particles bound by attractive interparticle potentials. In this paper, the wave function was written in a special functional form for which an integral equation equivalent to Schrödinger equation was derived. This equation was then solved approximately for the case of exponential interparticle potentials. The results for the eigenvalue compared favourably with the variational calculations given by other authors. Eyges (1965) gave another method of solution to the integral equation and gave the possibility for the generalizability to the four-body and even to the N-body problem. Results obtained by this method were compared with exact results for the one-dimensional problem with δ-function potentials and it was found that approximate and exact functions differed by a maximum of 6.9 per cent. Further, Eyges (1966) extended his method of solution to repulsive potentials also. Raghuwanshi and Sharma (1977) discussed certain aspects of Eyges's method and have shown that this method does not give satisfactory solutions to hyperbolic type of potential functions but provides accurate solutions to more general exponential potentials.

The object of this paper is to find the expression for the wave function for a general even power potential using the results of Eyges. This has been done in the next Section.

In the last Section, the general expression obtained in the next Section, has been applied to the problem of harmonic oscillator, anharmonic oscillator and Gaussian potentials. For testing the accuracy of the results obtained by us, the ground state wave function for two body interparticle harmonic oscillator potential has been plotted against k and the shape of the curve is found to be in close agreement with that obtained by other accurate methods. Wave functions for the anharmonic oscillator with r^4 anharmonicity have also been evaluated numerically and plotted against k.

Solution of General Even Power Potential

The general even power potential considered by us in this paper has the form :

$$V(r) = -g^2 \sum_{i=0}^{\infty} N_{2i} r^{2i}, \qquad \ldots(1)$$

where g^2, the coupling constant being both real and positive. The coefficients N_{2i} of the potential (1) are chosen such that

$$V(r) = 0 \ \text{ for } r > r_0.$$

Eyges (1961) has proved that for forces of finite range for which the potential is essentially zero when r is greater than some radius r_0, the zeroth and first order iterates of the s-wave part of momentum space wave functions are given as:

$$\phi_0^{(0)} = \frac{1}{k^2 + \frac{3}{4}\kappa^2 + K^2} \qquad \ldots(2)$$

and

$$\phi_0^{(1)}(k, \kappa) = \frac{1}{k^2 + \frac{3}{4}\kappa^2 + K^2}\left[f(k, \kappa) + \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(2l+1)F_{lm}(k, \kappa)\right] \qquad \ldots(3)$$

respectively, where

$$f(k, \kappa) = \int_0^{\infty} \frac{\omega_0(k, k')\, k'^2\, dk'}{k'^2 + \frac{3}{4}\kappa^2 + K^2}, \qquad \ldots(4)$$

and

$$F_{lm}(k, \kappa) = \int_0^{\infty} \frac{k'^2\, W_{0ll}(k, k', \kappa)\,(3k'\kappa)^m\, dk'}{(k'^2 + 3\kappa^2 + K^2)^{m+1}} \int_{-1}^{+1} P_l(y)\, y^m dy. \qquad \ldots(5)$$

Functions $\omega_0(k, k')$ and $W_{0ll}(k, k', \kappa)$ are the integrals defined in eqns. (23) and (28) of Eyges's paper.

As the contributions of the higher F_{lm} to the series for $\phi_0^{(1)}(k, \kappa)$ fall off rapidly with l and m, thus we shall make the approximation of truncating the series, keeping only the term F_{00}. With the above mentioned approximation, eqn. (3) can be written as:

$$\phi_0^{(1)}(k, \kappa) = \frac{1}{k^2 + \frac{3}{4}\kappa^2 + K^2}\left[f(k, \kappa) + F_{00}(k, \kappa)\right] \qquad \ldots(6)$$

For evaluating the value of $\phi_0^{(1)}(k, \kappa)$ we now calculate $f(k, \kappa)$ and $F_{00}(k, \kappa)$ for the potential (1). The function $f(k, \kappa)$, on substituting the values of integral $\omega_0(k, k')$ from Eyges's paper [eqn. 23] and the potential (1) takes the form :

$$f(k, \kappa) = -4\pi g^2 \int_0^{\infty} \frac{\sum_{i=0}^{\infty} \int_0^{\infty} N_{2i} r^{2i+2} j_0(kr)\, j_0(k'r)\, k'^2\, dk'\, dr}{k'^2 + \frac{3}{4}\kappa^2 + K^2}. \qquad \ldots(7)$$

Eqn. (7) can be further written as

$$f(k, \boldsymbol{\kappa}) = -\,4\pi g^2 \sum_{i=0}^{\infty} \int_0^{\infty} N_{2i}\, r^{2i+2}\, j_0(kr)\, \mathrm{d}r \int_0^{\infty} \frac{j_0(kr)\, k'^2\, \mathrm{d}k'}{\beta^2 + k'^2}, \qquad \ldots(8)$$

where

$$\beta = (\tfrac{3}{4}\,\boldsymbol{\kappa}^2 + K^2)^{1/2}.$$

The k' integration gives $\left(\frac{\pi}{2r}\right) e^{-r\beta}$, thus $f(k, \boldsymbol{\kappa})$ assumes the form

$$f(k, \boldsymbol{\kappa}) = -\,2\pi^2 g^2 \sum_{i=0}^{\infty} \int_0^{\infty} N_{2i}\, r^{2i+1}\, e^{-r\beta}\, j_0(kr)\, \mathrm{d}r. \qquad \ldots(9)$$

The r integration in eqn. (9) can then be solved with the help of Gradshteyn and Ryzhik (1965), which finally yields the following for $f(k, \boldsymbol{\kappa})$:

$$f(k, \boldsymbol{\kappa}) = -\,2\pi g^2 \sum_{i=0}^{\infty} N_{2i}(-\,1)^{2i+1} \frac{\mathrm{d}^{2i+1}}{\mathrm{d}\beta^{2i+1}} \{(\beta^2 + k^2)^{-1/2}\}. \qquad \ldots(10)$$

The value of $F_{00}(k, \boldsymbol{\kappa})$ can be similarly evaluated with the help of Gradshteyn and Ryzhik (1965) and gives:

$$F_{00}(k, \boldsymbol{\kappa}) = -\,2\pi^2 g^2 \sum_{i=0}^{\infty} N_{2i}\, \alpha^{-2i-2} \sum_{m=0}^{\infty} \frac{\sqrt{2(m+i+1)}}{\sqrt{m}\,\sqrt{m+1}} \times F\left(-\,m, -m; 1; \frac{9K^2}{4k^2}\right) \times \left(-\,\frac{k^2}{4\alpha^2}\right)^m, \qquad \ldots(11)$$

where

$$\alpha = (3k^2 + K^2)^{1/2}, \quad m = 2i.$$

and $F\left(-\,m, -\,m; 1; \frac{9K^2}{4k^2}\right)$ is the hypergeometric function.

Eqns. (10) and (11) finally give the following for the momentum space wave function for general even power potential:

$$\phi_0^{(1)}(k, \boldsymbol{\kappa}) = \frac{-\,2\pi^2 g^2}{(k^2 + \frac{3}{4}\boldsymbol{\kappa}^2 + K^2)} \left[\sum_{i=0}^{\infty} \left\{ N_{2i}(-\,1)^{2i+1} \frac{\mathrm{d}^{2i+1}}{\mathrm{d}\beta^{2i+1}} (\beta^2 + k^2)^{-1/2} \right\} + \sum_{i=0}^{\infty} N_{2i}\, \alpha^{-2i-2} \sum_{m=0}^{\infty} \frac{\sqrt{2(m+i+1)}}{\sqrt{m}\,\sqrt{m+1}} \times F\left(-\,m, -\,m; 1; \frac{9K^2}{4k^2}\right) \left(-\,\frac{k^2}{4\alpha^2}\right)^m \right] \qquad \ldots(12)$$

The radial wave function can then be determined by transforming eqn. (12), utilizing the following Fourier transform:

$$\psi(r, \rho) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \int_0^\infty \phi_0^{(1)}(k, \kappa)\, e^{[ikr + i\kappa\cdot\rho]} . \mathrm{d}k . \mathrm{d}\kappa \qquad ...(13)$$

Applications of the General Formula

We now utilize the general expression (12) to three well-known problems of harmonic oscillator, anharmonic oscillator and Gauss potentials.

Harmonic Oscillator : In eqn. (1), taking $N_{2i} = 0$ for $i \geqslant 2$ and $N_0 = 0$, gives the harmonic oscillator potential

$$V(r) = - g^2 N_2 r^2. \qquad ...(14)$$

With the above substitutions, wave function for 3-body harmonic oscillator problem assumes the form:

$$\phi_0^{(1)}(k, \kappa) = \frac{1}{(k^2 + \frac{3}{4}\kappa^2 + K^2)} \Bigg[- 2\pi^2 g^2 N_2 (-1)^3 \frac{\mathrm{d}^3}{\mathrm{d}\beta^3} \{(\beta^2 + k^2)^{-1/2}\} - 2\pi^2 g^2 N_2 \alpha^{-4} \sum_{m=0}^{\infty} \frac{\sqrt{2(m+2)}}{\sqrt{m}\sqrt{(m+1)}} \times F\left(-m, -m; 1; \frac{9K^2}{4k^2}\right) \left(\frac{-k^2}{4\alpha^2}\right)^m \Bigg] \qquad ...(15)$$

For reducing the above expression to two body problem, set $\kappa = 0$ in eqn. (15) which yields after simplification, the following for the momentum space wave function.

$$\phi_0^{(1)}(k, 0) = \frac{1}{(k^2 + K^2)} g^2 N_2 \frac{\mathrm{d}^3}{\mathrm{d}K^3} \{(k^2 + k^2)^{-1/2}\} + 0\,(1/k^4). \qquad ...(16)$$

We now show below that eqn. (16) can also be obtained directly by utilizing the following equation of Eyges paper (eqn. 9)

$$[\phi_0^{(1)}(k, 0) \equiv \phi_0^{(1)}(k)]$$

$$\phi_0^{(1)}(k) = - \frac{1}{2\pi^2(k^2 + K^2)} \int_0^\infty \phi_0(k')\, \omega_0(k, k')\, k'^2\, \mathrm{d}k', \qquad ...(17)$$

where $\omega_0(k, k')$ has been defined in eqn. (23) of Eyges's paper. For harmonic oscillator potential (14), (17) can be written as :

$$\phi_0^{(1)}(k) = \frac{2g^2N_2}{\pi(k^2+K^2)} \int_0^\infty \int_0^\infty \phi_0^{(0)}(k')\, r^4 j_0(kr)\, j_0(k'r)\, k'^2 \mathrm{d}r\, \mathrm{d}k', \qquad \text{...(18)}$$

where $\phi_0^{(0)}(k')$ is the zeroth order iterate.

On solving the double integral in eqn. (18), one gets:

$$\phi_0^{(1)}(k) = \frac{g^2N_2}{(k^2+K^2)} \frac{\mathrm{d}^3}{\mathrm{d}K^3} \{(k^2+K^2)^{-1/2}\} \qquad \text{...(19)}$$

this expression resembles with eqn. (16) for large values of K.

Expression (16) can also be simplified to the following forms :

$$\phi_0^{(1)}(k) = g^2N_2\,[9K(k^2+K^2)^{-7/2} - 15K^3(k^2+K^2)^{-9/2}] \qquad \text{...(20)}$$

and

$$\phi_0^{(1)}(k) = 6g^2N_2K^{-6}\left[1 - \frac{6k^2}{K^2}\right] + O\left(\frac{1}{K^{10}}\right). \qquad \text{...(21)}$$

For higher energies terms in the $O\left(\frac{1}{K^{10}}\right)$ may be neglected.

In order to evaluate the radial wave function $\phi_0^{(1)}(r)$ of the harmonic oscillator, eqn. (21) can be transformed using Fourier transform of $\phi_0^{(1)}(k)$ as,

$$\phi_0^{(1)}(r) = \frac{6g^2N_2K^{-6}}{(2\pi)^{3/2}} \int \left(1 - \frac{6k^2}{K^2}\right) e^{ikr}\, \mathrm{d}k \qquad \text{...(22)}$$

On solving eqn. (22) with the help of Ryzhik and Gradshteyn (1965), the final expression for $\phi_0^{(1)}(r)$ is

$$\phi_0^{(1)}(r) = \frac{3g^2N_2K^{-15/2}}{\pi} \left(\frac{\sqrt{6}}{r}\right)^{3/2} \left[J_{3/2}\left(\frac{\kappa r}{\sqrt{6}}\right) - iH^{(1)}_{-3/2}\left(\frac{\kappa r}{\sqrt{6}}\right)\right], \qquad \text{...(23)}$$

where

$$J_{3/2}\left(\frac{\kappa r}{\sqrt{6}}\right) \text{ and } H^{(1)}_{-3/2}\left(\frac{\kappa r}{\sqrt{6}}\right)$$

are the Bessel function of first kind and Hankel functions respectively.

Further, using (20) and for $K = 1$, numerical values of $\phi_0^{(1)}(k)/\eta$ have been obtained for different values of k. These values have been tabulated in Table I.

Anharmonic Oscillator : In eqn. (1), taking $N_{2i} = 0$ for $i \geqslant 3$ and $N_0 = 0$, gives the following anharmonic oscillator potential :

$$V(r) = -(g^2N_2)\, r^2 - (g^2N_4)\, r^4 \qquad \text{...(24)}$$

TABLE I

The values of the first iterate wave function $\phi_0^{(1)}(k)/\eta$ for the harmonic oscillator in momentum space for different values of k.

($\eta = g^2N_2$)

k	$\phi_0^{(1)}(k)/\eta$	k	$\phi_0^{(1)}(k)/\eta$
0.	− 6	0	− 6
0.1	− 5.651465	− 0.1	− 5.651465
0.2	− 4.727592	− 0.2	− 4.727592
0.3	− 4.184091	− 0.3	− 4.184091
0.4	− 2.3387329	− 0.4	− 2.3387329
0.5	− 1.3739545	− 0.5	− 1.3739545
0.6	− 0.6918076	− 0.6	− 0.6918076
0.7	− 0.2642767	− 0.7	− 0.2642767
0.8	− 0.0174029	− 0.8	− 0.0174029
0.9	+ 0.1325825	− 0.9	+ 0.1325825
1.0	+ 0.1379835	− 1.0	+ 0.1378935
1.1	+ 0.0214662	− 1.1	+ 0.0214662

TABLE II

The values of the first iterate wave function $\phi_{0\,per}^{(1)}(k)/\eta$ for the anharmonic oscillator in momentum space for different values of k.

($K = 1$, $\xi = \eta/225$, $\eta = g^2N_2$, $\xi = g^2N_4$)

0	− 7.1321	0	− 7.1321
0.1	− 5.1756416	− 0.1	− 5.1756416
0.2	− 4.394189	− 0.2	− 4.394189
0.3	− 3.9800516	0.3	− 3.9800516
0.4	− 2.3387329	− 0.4	− 2.3387329
0.5	− 1.3900169	− 0.5	− 1.3900169
0.6	− 0.7316500	− 0.6	− 0.7316500
0.7	− 0.3039486	− 0.6	− 0.3039486
0.8	− 0.0378601	− 0.8	− 0.0378601
0.9	+ 0.115716	− 0.9	+ 0.1195716
1.0	+ 0.129955	− 1.0	+ 0.129955

Following the procedure adopted in harmonic oscillator problem, expression for the two body wave function of anharmonic oscillator in momentum space can be written as :

$$\phi_{0\,per}^{(1)}(k) = +\eta[9K(K^2+k^2)^{-7/2} - 15K^3(K^2+k^2)^{-9/2}] + \xi\,[225\,K(K^2+k^2)^{-9/2} - 1050\,K^3(K^2+k^2)^{-11/2} + 945\,K^5(K^2+k^2)^{-13/2}], \quad \ldots(25)$$

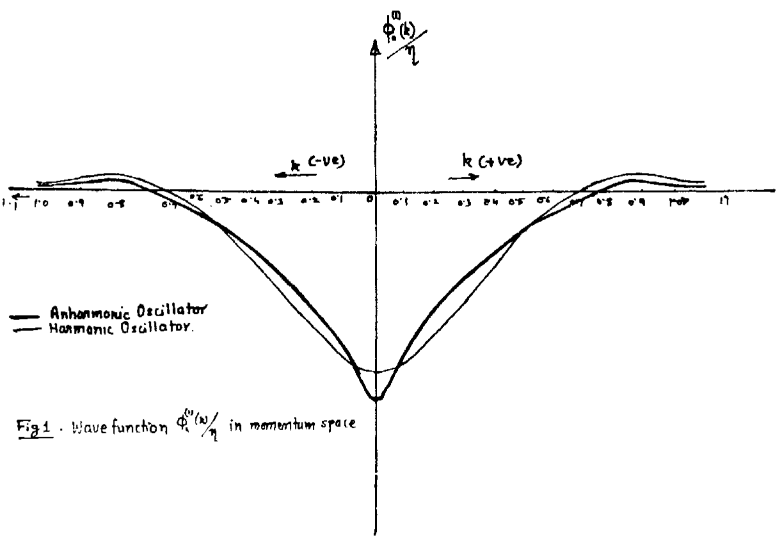

FIG. 1. Variation of wave functions $\phi^{(1)}_{0\,per}(k)/\eta$ and $\phi^{(1)}_{0}(k)/\eta$ with k.

where

$$\eta = (g^2 N_2), \quad \xi = (g^2 N_4),$$

and $\phi^{(1)}_{0\,per}(k)$ is the first iterate of momentum space wave function for perturbed harmonic oscillator. For $K = 1$ and $\xi = \eta/225$, the numerial values of $\phi^{(1)}_{0\,per}(k)/\eta$ for different values of k have been evaluated and given in Table II.

The variation with k of wave functions $\phi^{(1)}_{0\,per}(k)/\eta$ for the perturbed harmonic oscillator with ξx^4 perturbation has been shown in Fig. 1. The wave functions $\phi^{(1)}_{0}(k)/\eta$ of an unperturbed harmonic oscillator are also shown for comparison.

Gauss Potential : The Gauss potential

$$V(r) = -\, g^2\, e^{-\alpha_1^2 r^2} \qquad \ldots(26)$$

can be obtained from (1) by taking

$$N_{2i} = (-\,1)^i\, [\alpha_1^{2i}/i!] \qquad \ldots(27)$$

Thus three-body wave function $\phi^{(1)}_{0}(k, \kappa)$ for this potential can be written as:

$$\phi^{(1)}_{0}(k, \kappa) = \frac{1}{2\pi^2(k^2 + \frac{3}{4}\kappa^2 + K^2)}\Big[-\, 2\pi^2 g^2 \sum_{i=0}^{\infty}\Big\{(-\,1)^i \frac{\alpha_1^{2i}}{i!}(-\,1)^{2i+1} \times$$

(equation continued)

$$\frac{d^{2i+1}}{d\beta^{2i+1}} (\beta^2 + k^2)^{-1/2} \Big\} - 2\pi^2 g^2 \times$$

$$\sum_{i=0}^{\infty} (-1)^i \frac{\alpha_1^{2i} \alpha^{-2i-2}}{i!} \sum_{m=0}^{\infty} \frac{\sqrt{2(m+i+1)}}{\sqrt{m}\sqrt{m+1}} \times$$

$$F\left(-m, -m; 1; \frac{9K^2}{4k^2}\right) \times \left(\frac{-k^2}{4\alpha^2}\right)^m \Big]. \qquad ...(28)$$

The two body wave function $\phi_0^{(1)}(k)$ for this potential can be obtained by taking $\kappa = 0$, as before.

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