MAGNETOHYDRODYNAMIC ROTATING FLUID PAST A DISK IN A CYLINDRICAL TUBE

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In this paper, we study the effects of a uniform transverse magnetic field on a homogeneous rotating fluid confined in a cylindrical tube past a circular disk, the magnetic field being in the direction parallel to the axis of the cylinder. The medium being assumed to be slightly conducting and the magnetic Reynolds number being very small, the induced magnetic field is neglected. The governing equations in terms of Stoke's stream function and its general solutions are obtained assuming that the disturbance die down far upstream and the waves propagate only in the downstream side of the obstacle. By imposing the matching conditions for ψ and its first two derivatives along the axial direction on the plane of the disk, and using the boundary condition on the disk we obtain Dual-Bessel series which is solved by the method of J. C. Cooke. It can be observed that in contrast to the cases of homogeneous or stratified rotating fluids the oscillations N in number in the downstream side are no longer persistent and they decay at far distance in the order of

$$\exp\left[-\frac{K(2R^{-2}-\alpha_n^2)}{2(R^{-2}-\alpha_n^2)}Z\right].$$

Apart from it the presence of a uniform magnetic field induces a finite number N of additional decaying modes in the upstream side and an infinite number of similar modes in the downstream side of the disk.

Introduction

A rotating fluid in an unbounded or partially bounded medium exhibits remarkably different features from that of a non-rotating fluid. In particular, the problem of the flow pattern generated, when a homogeneous rotating fluid confined inside a pipe goes past an obstacle (body or source), has been attempted by Long (1953), Frankle (1956), Trustrum (1964), Stewartson (1968), and Krishna and Sarma (1969). Long (1956) obtained the governing equations of an anisymmetric steady flow of an inviscid homogeneous rotating fluid in terms of Stoke's stream function.

In this paper, we study the effects of a uniform transverse magnetic field on a homogeneous rotating fluid confined in a cylindrical tube past a circular disk, the magnetic field being in the direction parallel to the axis of the cylinder. The medium being assumed to be slightly conducting and the magnetic Reynolds number is very small, the induced magnetic field is neglected (Sparrow & Cess 1962). The governing equations in terms of Stoke's stream function and its general solutions are obtained assuming that the disturbance die down far upstream and the waves propagate only in the downstream side of the obstacle. By imposing the matching conditions for ψ and its first two derivatives along the axial direction on the plane of the disk, and using the boundary condition on the disk we obtain Dual-Bessel series which is solved by the method of Cooke and Trantev (1959). It can be observed that in contrast to the cases of homogeneous or stratified rotating fluids the oscillations N in number in the downstream side are no longer persistent and they decay at far distances in the order of

$$\exp\left[\frac{K(2R^{-2}-\alpha_n^2)}{2(R^{-2}-\alpha_n^2)}Z\right].$$

Apart from it the presence of a uniform magnetic field induces a finite number N of additional decaying modes in the upstream side and an infinite number of similar modes in the downstream side of the disk.

Governing Equations and Solution

Choose the cartesian frame of reference (r', θ, z'). We consider an incompressible inviscid fluid confined within a circular cylinder $r' = b$ and rotating with a constant angular velocity Ω about the z' axis, past a circular disk $r' \leqslant a$ situated on the plane $z' = 0$ with a uniform velocity U far upstream, in the presence of a uniform magnetic field H_0 along the z' axis. The governing equations of motion of the magnetohydrodynamic flow with respect to a frame rotating with the same angular velocity of the fluid in the vector form are

$$\rho' \frac{d\mathbf{q}'}{dt'} = -\nabla p' + \mu_e \mathbf{J} \times H - 2\rho\,\Omega\,\mathbf{k} \times \mathbf{q}' - \rho\,\Omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{r}) \quad \text{...(1)}$$

$$\nabla \cdot \mathbf{q}' = 0 \quad \text{...(2)}$$

$$\mathbf{J} = \sigma\mu_e \mathbf{q}' \times H \quad \text{...(3)}$$

where $\mathbf{k}$ is the unit vector along z' axis, $\mathbf{q}'$ the velocity vector, ρ' the density, p' the pressure, $\mathbf{J}$ the current density and H is the magnetic field. μ_e is the magnetic permeability and σ is the electrical conductivity of the fluid. The undisturbed state consists of velocity $(0, 0, U)$ with respect to the rotating frame, along (r', θ, z') directions respectively and a uniform magnetic field H_0 in the z' direction. Let $(u, v, w + U)$ be the velocity in the perturbed state and $(H_r, H_\theta, H_z + H_0)$ the magnetic field where H_r, H_θ and H_z are the components of the induced magnetic field. Let $p_0 + p'$ be the corresponding pressure, p_0 being the undisturbed pressure. Substituting these in the governing equations and neglecting the induced magnetic field in comparison with the applied magnetic field, the Oseen-type linearized equations of motion for the axisymmetric steady flow in the absence of any input electric field, in cylindrical polars are

$$\rho'\left(U\frac{\partial u'}{\partial z'} - 2\Omega v'\right) = -\frac{\partial p'}{\partial r'} - \sigma\mu_e^2 H_0^2 u' \qquad \ldots(4)$$

$$\rho'\left(U\frac{\partial v'}{\partial z'} + 2\Omega u'\right) = -\sigma\mu_e^2 H_0^2 v' \qquad \ldots(5)$$

$$\rho'\frac{\partial w'}{\partial z'} = -\frac{\partial p'}{\partial z'} \qquad \ldots(6)$$

$$\frac{\partial u'}{\partial r'} + \frac{u'}{r'} + \frac{\partial w'}{\partial z'} = 0 \qquad \ldots(7)$$

Introducing the non-dimensional variables u, v, w, r, z and p as

$$u' = Uu;\; v' = Uv;\; w' = Uw$$

$$r' = ar;\; z' = az;\; p' = \rho' U^2 p$$

the above equations become

$$\frac{\partial u}{\partial z} - \frac{v}{R} = -\frac{\partial p}{\partial r} - Ku \qquad \ldots(8)$$

$$\frac{\partial v}{\partial z} + \frac{u}{R} = -Kv \qquad \ldots(9)$$

$$\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} \qquad \ldots(10)$$

and

$$\frac{\partial u}{\partial z} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \qquad \ldots(11)$$

where $R = \dfrac{U}{2\Omega a}$ is the Rossby number and

$$K = \frac{\sigma\mu_e^2 H_0^2 a}{\rho' U} \text{ is the magnetic parameter.}$$

Using the equation of continuity (11), we can define the stream function ψ such that

$$u = \frac{1}{r}\frac{\partial\psi}{\partial z} \text{ and } w = -\frac{1}{r}\frac{\partial\psi}{\partial r}. \qquad \ldots(12)$$

Eliminating p from eqns. (8) to (11) and using the condition that the flow is uniform far upstream, the governing equation in ψ is

$$\left(\frac{\partial}{\partial z} + K\right) D^2\psi + K^2\frac{\partial^2\psi}{\partial z^2} + (K^2 + R^{-2})\frac{\partial\psi}{\partial z} = 0, \qquad \ldots(13)$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The boundary conditions to be satisfied are on the cylinder

$$r = \frac{b}{a}, \ \psi = 0 \qquad \text{...(14)}$$

and the relative normal velocity on the disk is zero.

The general solution for ψ which satisfies the boundary condition on the cylinder $r = b/a$ can be taken as

$$\psi = \sum_{n=1}^{\infty} A_n r J_1(\alpha_n r) \exp(\alpha_n z), \qquad \text{...(15)}$$

where α_n's are roots of

$$m^3 + 2Km^2 + (K^2 + R^{-2} - \alpha_n^2)\, m - \alpha_n^2 K = 0 \qquad \text{...(16)}$$

and

$$J_1(\alpha_n \lambda) = 0,$$

where

$$\lambda = \frac{b}{a}.$$

Assuming K to be small, the approximate roots of the above equation are

$$-\left(K + \sqrt{\alpha_n^2 - R^{-2}}\right) + \frac{\alpha_n^2 K}{6K\sqrt{\alpha_n^2 - R^{-2}} + 2(\alpha_n^2 - R^{-2})},$$

$$\left(-K + \sqrt{\alpha_n^2 - R^{-2}}\right) - \frac{\alpha_n^2 K}{6K\sqrt{\alpha_n^2 - R^{-2}} - 2(\alpha_n^2 - R^{-2})} \qquad \text{...(17)}$$

and

$$\frac{\alpha_n^2 K}{R^{-2} - \alpha_n^2}$$

Let $\alpha_n > R^{-1}$ for all $n > N$

$\alpha_n < R^{-1}$ for all $n \leqslant N$

$(\alpha_N < R^{-1} < \alpha_{N+1})$

(Also we assume that there exists no root of $J_1(\alpha_n\lambda) = 0$ between R^{-2} and $R^{-2} + K^2$, K being small). The conditions at infinity are $\psi \to 0$ as $z \to -\infty$ and ψ is bounded as $z \to +\infty$. Hence, separating the regions $z < 0$ and $z > 0$, the solutions corresponding to the upstream and downstream regions satisfying the above conditions are

$$\psi_- = \sum_{n=1}^{N} A_n r J_1(\alpha_n r) \exp(p_n z) + \sum_{n=N+1}^{\infty} D_n r J_1(\alpha_n r) \exp(q_n z) \text{ for } (z < 0) \qquad \ldots(18)$$

$$\psi_+ = \sum_{n=N+1}^{\infty} B_n r J_1(\alpha_n r) \exp(-p_n z) + \sum_{n=N+1}^{\infty} E_n r J_1(\alpha_n r) \exp(r_n z) + \sum_{n=1}^{N} [G_n \cos s_n z + H_n \sin s_n z] \times r \exp(t_n z) J_1(\alpha_n r) \text{ for } (z > 0), \qquad \ldots(19)$$

where

$$p_n = \frac{\alpha_n^2 K}{R^{-2} - \alpha_n^2}$$

$$q_n = \left(-K + \sqrt{\alpha_n^2 - R^{-2}}\right) - \frac{\alpha_n^2 K}{6K\sqrt{\alpha_n^2 - R^{-2}} - 2(\alpha_n^2 - R^{-2})}$$

$$r_n = \left(-K - \sqrt{\alpha_n^2 - R^{-2}}\right) + \frac{\alpha_n^2 K}{6K\sqrt{\alpha_n^2 - R^{-2}} + 2(\alpha_n^2 - R^{-2})}$$

$$s_n = (R^{-2} - \alpha_n^2)^{1/2}$$

$$t_n = -\frac{K(2R^{-2} - \alpha_n^2)}{2(R^{-2} - \alpha_n^2)}$$

Since the region $z > 0$ is an analytic continuation of $z < 0$ when $1 < r < b/a$, we require that the function ψ and its derivatives with respect to z should be continuous at $z = 0$ i.e.,

$$\left.\begin{aligned} &\psi_- = \psi_+ ; \frac{\partial \psi_-}{\partial z} = \frac{\partial \psi_+}{\partial z} \quad \text{on } z = 0 \\ \text{and} \quad &\frac{\partial^2 \psi_-}{\partial z^2} = \frac{\partial^2 \psi_+}{\partial z^2} \quad 1 < r < \frac{b}{a} \end{aligned}\right\} \quad \ldots(20)$$

The boundary condition on the disk ($z = 0$, $0 < r < 1$) gives

$$\psi_- = \psi_+ = \frac{r^2}{2} \quad \ldots(21)$$

Using these conditions we get

$$\sum_{n=1}^{N} (A_n - G_n)\, rJ_1(\alpha_n r) + \sum_{n=N+1}^{\infty} (D_n - B_n - E_n)\, rJ_1(\alpha_n r) = 0 \quad \text{for all } r, \quad \ldots(22)$$

$$\sum_{n=1}^{N} (p_n A_n - p_n G_n - s_n H_n)\, rJ_1(\alpha_n r) + \sum_{n=N+1}^{\infty} (q_n Dn + p_n B_n - r_n E_n)\, rJ_1(\alpha_n r) = 0 \text{ for } 1 < r < \frac{b}{a} \quad \ldots(23)$$

and

$$\sum_{n=1}^{N} (p_n^2 A_n - p_n^2 G_n + s_n^2 G_n - 2p_n s_n H_n)\, rJ_1(\alpha_n r) + \sum_{n=N+1}^{\infty} (q_n^2 D_n - p_n^2 B_n - r_n^2 E_n)\, rJ_1(\alpha_n r) = 0 \quad \text{for } 1 < r < \frac{b}{a} \quad \ldots(24)$$

Eqn. (22) together with the condition (21) implies

$$A_n = G_n \text{ and } D_n = B_n + E_n \quad \ldots(25)$$

On using (25), eqns. (23) and (24) can be represented in the form

$$\sum_{n=1}^{\infty} X_n r J_1(\alpha_n r) = 0 \quad \left(1 < r < \frac{b}{a}\right) \quad \ldots(26)$$

where X_n's are connected to A_n, B_n, D_n, E_n, G_n and H_n by the relations

$$A_n = G_n = \left(\frac{1-2p_n}{s_n^2}\right) X_n; \; H_n = -\frac{1}{s_n} X_n$$

$$B_n = \frac{[r_n(r_n-1) - q_n(q_n-1)]\, X_n}{q_n r_n(r_n-q_n) - p_n q_n(p_n+q_n) + p_n r_n(p_n+r_n)} \qquad \ldots(27)$$

$$D_n = \frac{[r_n(r_n-1) - p_n(p_n+1)]\, X_n}{q_n r_n(r_n-q_n) - p_n q_n(p_n+q_n) + p_n r_n(p_n+r_n)}$$

$$E_n = \frac{(q_n^2 - p_n^2 - q_n - p_n)\, X_n}{q_n r_n(r_n-q_n) - p_n q_n(p_n+q_n) + p_n r_n(p_n+r_n)}.$$

The boundary condition (21) gives

$$\sum_{n=1}^{N} A_n J_1(\alpha_n r) + \sum_{n=N+1}^{\infty} D_n J_1(\alpha_n r) = \frac{r}{2} \qquad (0 < r < 1) \quad \ldots(28)$$

(26) and (28) give rise to two series connecting the constants X_n as follows:

$$\sum_{n=1}^{N} \frac{(1-2p_n)}{s_n^2} X_n J_1(\alpha_n r) + \sum_{n=N+1}^{\infty} \theta_n X_n J_1(\alpha_n r) = \frac{r}{2} (0 < r < 1), \ldots(29)$$

where

$$\theta_n = \frac{r_n(r_n-1) - p_n(p_n+1)}{q_n r_n(r_n-q_n) - p_n q_n(p_n+q_n) + p_n r_n(p_n+r_n)}$$

$$\sum_{n=1}^{\infty} X_n r J_1(\alpha_n r) = 0 \qquad \left(1 < r < \frac{b}{a}\right) \quad \ldots(30)$$

(29) and (30) constitute Dual-Bessel series which have been solved by Cooke and Trantev (1959) for a particular type of coefficients. He considers a series of the type

$$\text{(A)} \quad \sum_{n=1}^{\infty} \alpha_n^{p} a_n J_\nu(\alpha_n r) = F(r) \qquad (0 < r < 1)$$

$$\text{(B)} \quad \sum_{n=1}^{\infty} a_n J_\nu(\alpha_n r) = 0 \qquad (1 < r < a)$$

given $(-1 \leqslant p \leqslant 1)$ and $J_\nu(\alpha_n a) = 0$ to determine a_n's. By virtue of the identity

$$\sum_{m=1}^{\infty} \frac{J_{\nu+2m+1+p/2}(\alpha_n)\, J_\nu(\alpha_n r)}{\alpha_n^{1+p/2}\, J_{\nu+1}^2(\alpha_n a)} = 0 \qquad (1 < r < a),$$

where $m \geqslant 0$ and $\nu > -1$.

Taking

$$a_n = \frac{\sum_{m=0}^{\infty} b_m J_{\nu+2m+1+(p/2)}(\alpha_n)}{\alpha_n^{1+(p/2)} J_{\nu+1}^2 (\alpha_n a)}$$ equation (B) is automatically satisfied.

Using the result

$$\frac{J_{\nu+2s+1+(p/2)}(\alpha_n)}{\alpha_n^{1+(p/2)}} = \frac{\Gamma(\nu + s + 1)}{2^{p/2}\Gamma(\nu + 1)\Gamma\left(s + 1 + \frac{p}{2}\right)} \times$$

$$\int_0^1 r^{\nu+1}(1 - r^2)^{p/2} \mathcal{F}_s\left(1 + \frac{p}{2} + \nu, \nu + 1, r^2\right) J_\nu(\alpha_n r)\, dr,$$

where

$$\mathcal{F}_s\left(1 + \frac{p}{2} + \nu, \nu + 1, r^2\right) = {}_2F_1\left(- s, 1 + \frac{p}{2} + \nu + s, \nu + 1, r^2\right)$$

is Jacobi's polynominal, in equation (A) he gets

$$\sum_{m=0}^{\infty} b_m \sum_{n=1}^{\infty} \frac{J_{\nu+2m+1+(p/2)}(\alpha_n)\, J_{\nu+2s+1+(p/2)}(\alpha_n)}{\alpha_n^2 J_{\nu+1}^2 (\alpha_n a)} = E(\nu, s, p)$$

where

$$E(\nu, s, p) = \frac{\Gamma(s + \nu + 1)}{2^{p/2}\,\Gamma(\nu + 1)\,\Gamma\left(s + 1 + \frac{p}{2}\right)} \times$$

$$\int_0^1 r^{\nu+1}(1 - r^2)^{p/2} \mathcal{F}_s\left(1 + \frac{p}{2} + \nu, \nu + 1, r^2\right) F(r)\, dr.$$

The above equation with $s = 0, 1, 2, \ldots$ provides a set of algebraic equations to determine the coefficients b_m and hence a_n. For this simple case, an iterative process for obtaining the solution has also been indicated, wherein the summation of left hand side infinite series of Bessel functions can be converted into an infinite integral and then approximating the integral for small parameters.

Following this procedure for solving the Dual-Bessel series (29) and (30), we choose X_n's as

$$X_n = \frac{1}{\alpha_n J_2^2 (\lambda\alpha_n)} \sum_{m=0}^{\infty} b_m J_{2m+2}(\alpha_n), \qquad \ldots(31)$$

which satisfy (30) automatically by virtue of Tranter's identity (1959). The constants b_m in (31) are to be determined from the eqn. (29). Substituting (31) into (29) and interchanging the order of summation we get

$$\sum_{m=0}^{\infty} b_m \left[\sum_{n=1}^{N} \frac{(1-2p_n)\, J_{2n+2}(\alpha_n)\, J_1(\alpha_n r)}{s_n^2\, \alpha_n J_2^2\,(\lambda\alpha_n)} + \sum_{n=N+1}^{\infty} \frac{\theta_n J_{2n+2}(\alpha_n)\, J_1(\alpha_n r)}{\alpha_n J_2^2\,(\lambda\alpha_n)} \right] = \frac{r}{2}\ (0 < r < 1) \quad \ldots(32)$$

Also if s is a positive integer or zero

$$\frac{J_{2s+2}(\lambda_m)}{\lambda_m} = (s+1)\int_0^1 r^2 \mathcal{F}_s(2, 2, r^2)\, \mathcal{F}_1(\lambda_m r)\, dr,$$

where

$$\mathcal{F}_s(2, 2, r^2) = {}_2F_1\,(-s, s+2; 2; r^2).$$

Multiplying both sides of (32) by $r^2\mathcal{F}_s(2, 2, r^2)\, dr$ and integrating between 0 to 1, we get

$$\sum_{m=0}^{\infty} b_m \left[\sum_{n=1}^{N} \frac{(1-2p_n)\, J_{2n+2}(\alpha_n)}{s_n^2\, \alpha_n J_2^2\,(\lambda\alpha_n)} \times \int_0^1 r^2\mathcal{F}_s(2, 2, r^2)\, J_1(\alpha_n r)\, dr + \sum_{n=N+1}^{\infty} \frac{\theta_n J_{2n+2}(\alpha_n)}{\alpha_n J_2^2(\lambda\alpha_n)} \int_0^1 r^2\mathcal{F}_s(2, 2, r^2)\, J_1(\alpha_n r)\, dr \right]$$

$$= \tfrac{1}{2}\int_0^1 r^3\mathcal{F}_s(2, 2, r^2)\, dr$$

i.e.,

$$\sum_{m=0}^{\infty} b_m \left[\sum_{n=1}^{N} \frac{(1-2p_n)\, J_{2n+2}(\alpha_n)}{s_n^2\, \alpha_n J_2^2\,(\lambda\alpha_n)}\, \frac{J_{2s+2}(\alpha_n)}{\alpha_n} + \sum_{n=N+1}^{\infty} \frac{\theta_n J_{2n+2}(\alpha_n)\, J_{2s+2}(\alpha_n)}{\alpha_n J_2^2\,(\lambda\alpha_n)} \right]$$

$$= \frac{s+1}{2} \int_0^1 r^3 \mathcal{F}_s(2, 2, r^2)\, dr$$

$$= 1/8 \quad \text{for } s = 0$$

$$= 0 \quad \text{for } s = 1, 2, 3, \ldots$$

In evaluating the integral on the right hand side above we have used the following orthogonality property of the Jacobi's polynomials, (Magnus 1949).

$$\int_0^1 X^{r-1}(1-X)^{\alpha-r} \mathcal{F}_m(X)\, \mathcal{F}_n(X)\, dX$$

$$= 0 \qquad m \neq n$$

$$= \frac{\Gamma(r)\,\Gamma(\alpha+1-r)\,(\alpha+1-r)_n \lfloor n}{\Gamma(\alpha)\,(a)_n (r)_n (\alpha+2n)}$$

for $m = n$ and $\mathrm{Re}\,(r) > 0$, $\mathrm{Re}\,(\alpha - r) > -1$.

These are a set of algebraic equations which determine the constants b_m's. Once b_m's are determined eqns. (31), (27), (18) and (19) determine the stream function ψ uniquely.

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