FORCED TORSIONAL VIBRATIONS OF A SEMI-INFINITE PIEZOELECTRIC MEDIUM OF (622) CLASS

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Torsional vibrations of a piezoelectric half-space have been investigated when twisting is accomplished by the rotatory vibration of a rigid circular disc. The dual integral equations derived from the mixed boundary conditions are reduced to a non-homogeneous Fredholm's integral equation by Noble's method. Numerical results are presented for applied torque.

INTRODUCTION

WAVE propagation in piezoelectric medium has a wide range of physical applications (Paul & Sarma 1977). Reissner and Sagoci (1944) have introduced oblate spheroidal coordinates in order to solve exactly the problem of torsional oscillations in a semi-infinite homogeneous isotropic medium produced by a rigid circular disc attached to its surface. Robertson (1967), Paul (1968) and Gladwell (1968) have employed the Fourior-bessel method which reduces the same kind of problem to the solution of a pair of dual integral equations. In the present investigations, we consider the torsional wave propagation in a piezoelectric half-space of (622) class caused by the forced rotatory oscillations of a rigid circular disc. Ice and β-quartz which are available plenty in nature belong to this class. The mixed boundary conditions give rise to the dual integral equations. These are reduced to a non-homogeneous Fredholm's integral equation by using Noble's technique. Following Robertson (1967), the kernel is evaluated by the method of contour integration. The choice of the branches admits the outgoing wave at infinity. Numerical results for the solution of the integral equation and for the applied torque are presented.

BASIC EQUATIONS

We refer to the cylindrical polar coordinate system such that the free surface is along the z-plane while the positive direction of the z-axis passes through the piezoelectric half-space. Let v be the non-vanishing component of the displacement along the cross-radial direction and ϕ be the electric potential function. The piezoelectric relations for the solid of the (622) class executing torsional vibrations are (Paul & Rao 1969).

$$T_{\theta z} = c_{44} v_{,z} + e_{14}\phi_{,r} \; ; \; T_{r\theta} = c_{66}(v_{,r} - r^{-1} v) \; ;$$

$$D_r = e_{14} v_{,z} - \epsilon_{11}\phi_{,r} \; ; \; D_z = - \epsilon_{33}\phi_{,z} \; ; \qquad \ldots(1)$$

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where $T_{\theta z}$ and $T_{r\theta}$ are the components of the stress, D_r and D_z are the components of the electric displacement, c_{44} and c_{66} are the elastic, e_{14} is the piezoelectric and ϵ_{11} and ϵ_{33} are the dielectric constants. The subscripts following a comma denote the partial derivatives with respect to them.

In the absence of a body force the equation of motion and the charge equation of electrostatics are

$$c_{44}v_{,zz} + c_{66}\left(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} - r^{-2}\right)v + e_{14}\phi_{,rz} = sv_{,tt}, \qquad \ldots(2)$$

and

$$e_{14}(v_{,r} + r^{-1}v)_{,z} - \epsilon_{11}(\phi_{,rr} + r^{-1}\phi_{,r}) - \epsilon_{33}\phi_{,zz} = 0,$$

where s is the density of the material.

We try to seek the solution in the form

$$v = \exp(iwt)\int_0^\infty V(\alpha, z)\, J_1(r\alpha)\, d\alpha \;;$$

$$\phi = \exp(iwt)\int_0^\infty \phi(\alpha, z)\, J_0(r\alpha)\, d\alpha, \qquad \ldots(3)$$

where $J_n(r)$ denotes the bessel function of order n.
From the eqns. (2) and (3), we obtain

$$[c_{44}D^2 + (sw^2 - c_{66}\,\alpha^2)]\,V - e_{14}\,\alpha\,D\,\phi = 0,$$

and

$$e_{14}\alpha DV + (\alpha^2\epsilon_{11} - \epsilon_{33}D^2)\,\phi = 0, \qquad \ldots(4)$$

where $D = d/dz$. This system of equations may be expressed as

$$(D^4 - PD^2 + Q)\,(V, \phi) = 0, \qquad \ldots(5)$$

where

$$\left.\begin{aligned}
P &= (K_{14}^2 + \bar{\epsilon}_{11} + \bar{c}_{66})\,\alpha^2 - c_1^2\,;\\
Q &= \bar{\epsilon}_{11}\bar{c}_{66}\alpha^2(\alpha^2 - c_2^2)\,;\\
K_{14}^2 &= e_{14}^2/(\epsilon_{33}c_{44})\,;\\
\bar{c}_{66} &= c_{66}/c_{44}\,;\\
\bar{\epsilon}_{11} &= \epsilon_{11}/\epsilon_{33}\,;\\
c_1^2 &= sw^2/c_{44} \text{ and } c_2^2 = sw^2/c_{66}.
\end{aligned}\right\} \qquad \ldots(6)$$

Let x_1^2 and x_2^2 be the roots of the auxiliary equation

$$x^4 - Px^2 + Q = 0.$$

Hence we have

$$x_i = [(P \pm (P^2 - 4Q)^{1/2})/2]^{1/2} \; ; i = 1, 2, \qquad ...(7)$$

where we choose a + sign for $i = 1$ and a − sign for $i = 2$. We also have

$$x_1 x_2 = (\bar{\epsilon}_{11} \bar{c}_{66})^{1/2} \, \alpha\beta, \qquad ...(8)$$

where we define (Gladwell 1968)

$$\left.\begin{aligned} \beta &= (\alpha^2 - c_2^2)^{1/2} \quad \text{if } \alpha > c_2 \\ &= i(c_2^2 - \alpha^2)^{1/2} \quad \text{if } - c_2 < \alpha < c_2 \\ &= -(\alpha^2 - c_2^2)^{1/2} \quad \text{if } \alpha < - c_2 \end{aligned}\right\} \qquad ...(9)$$

We obtain the solution of the system of eqn. (4) as

$$\left.\begin{aligned} \phi &= A_1 \exp(-x_1 z) + A_2 \exp(-x_2 z), \\ V &= (e_{14}/c_{44}) \, [K_1 A_1 \exp(-x_1 z) + K_2 A_2 \exp(-x_2 z)], \end{aligned}\right\} \qquad ...(10)$$

where A_1 and A_2 are arbitrary constants.

From the eqns. (10) and (4) we find that

$$K_i = (\alpha^2 \bar{\epsilon}_{11} - x_i^2)/(\alpha \, x_i \, K_{14}^2) \; ; \; i = 1, 2. \qquad ...(11)$$

With the help of the eqns. (1) and (3) the shearing stress is given by

$$T_{\theta z} = - e_{14} \exp(iwt) \int_0^\infty \left(\sum_{i=1}^{2} h_i A_i \exp(-x_i z) \right) J_1(r\alpha) \, d\alpha, \qquad ...(12)$$

where

$$h_i = \alpha + K_i x_i \; ; \; i = 1, 2. \qquad ...(13)$$

Formulation of Dual-Integral Equations

If the surface $z = 0$ is coated with electrodes that are shorted, then the electric boundary condition is

$$\phi(r, o, t) = 0. \qquad ...(14)$$

We consider that a rigid circular disc of unit radius, attached to the surface $z = 0$, executes forced rotatory oscillations. The rest of the surface is kept traction-free. Hence, we obtain the mixed boundary conditions

$$\left.\begin{aligned} & T_{\theta z}(r, o, t) = 0 \text{ for } r > 1, \\ \text{and} \quad & v(r, o, t) = (e_{14}/c_{44}) \exp(iwt) \, rd \text{ for } r \leqslant 1, \end{aligned}\right\} \qquad ...(15)$$

where d is a constant.

The boundary conditions, with the help of the eqns. (10) and (3), yield

$$A_1 + A_2 = 0 \; ; \; \int_0^\infty (K_1 A_1 + K_2 A_2) J_1(r\alpha) \, d\alpha = rd \text{ for } r \leqslant 1. \quad \ldots(16)$$

We write

$$h_1 A_1 + h_2 A_2 = f(\alpha), \quad \ldots(17)$$

where $f(\alpha)$ is an unknown function.

From the eqns. (8), (11), (12), (13), (15), (16) and (17), the unknown $f(\alpha)$ is determined by the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty f(\alpha) J_1(r\alpha) \, d\alpha = 0 \text{ for } r > 1, \\ &\int_0^\infty \frac{\alpha(\bar{\epsilon}_{11})^{1/2} + (\bar{c}_{66})^{1/2} \beta}{(\bar{c}_{66})^{1/2} \beta(x_1 + x_2)} f(\alpha) J_1(r\alpha) \, d\alpha = rd : \text{ for } r \leqslant 1. \end{aligned}\right\} \ldots(18)$$

For convenience, we put

$$\alpha \left[\frac{\alpha(\bar{\epsilon}_{11})^{1/2} + (\bar{c}_{66})^{1/2} \beta}{(\bar{c}_{66})^{1/2} \beta(x_1 + x_2)} \right] = L \, [1 + H(\alpha)], \quad \ldots(19)$$

where

$$L = [(\bar{\epsilon}_{11})^{1/2} + (\bar{c}_{66})^{1/2}] / [\bar{c}_{66} \{K_{14}^2 + ((\bar{\epsilon}_{11})^{1/2} + (\bar{c}_{66})^{1/2})^2\}]^{1/2}. \quad \ldots(20)$$

We observe that the function $H(\alpha)$ defined by the eqn. (19) is such that $H(\alpha) \to 0$ as $\alpha \to \infty$. Hence the mixed boundary conditions (15), by virtue of the eqns. (18) and (19), give rise to the following dual integral equations in the standard form

$$\int_0^\infty f(\alpha) J_1(r\alpha) \, d\alpha = 0 \quad \text{for } r > 1 \quad \ldots(21)$$

and

$$\int_0^\infty \alpha^{-1} [1 + H(\alpha)] \, f(\alpha) J_1(r\alpha) \, d\alpha = rd/L : \quad \text{for } r \leqslant 1. \quad \ldots(22)$$

We follow the method of Noble and derive the Fredholm's integral equation of the second kind from the above dual integral equations. The details are omitted as they may be found in the reference (Gladwell 1968).

Eqn. (21) is automatically satisfied, if we put

$$f(\alpha) = (2\alpha/\pi) \int_0^1 \theta(x) \sin(\alpha x) \, dx. \quad \ldots(23)$$

Substitution of the result in eqn. (23) into (22) leads to the non-homogeneous Fredholm's integral equation

$$\theta(x) + \pi^{-1} \int_0^1 M(x, y)\, \theta(y)\, \mathrm{d}y = 2\mathrm{d}x/L, \qquad \ldots(24)$$

where the kernel is given by

$$M(x, y) = 2 \int_0^\infty H(\alpha) \sin(\alpha x) \sin(\alpha y)\, \mathrm{d}\alpha. \qquad \ldots(25)$$

Following Robertson (1967), we adopt the counter integration technique to evaluate the kernel. We select that branch which admits the outgoing wave at infinity. From the eqns. (19) and (25), we infer that the integrand has no pole but has only branch points given by $\beta = 0$. We also observe that the external radical of x_2, given by eqn. (7), gives rise to the same branch points. The internal radical that occurs in x_1 and x_2 has no relevant branch point for the integrand. The sign convention for either β or x_2 is that already defined in eqn. (9). Hence we obtain

$$M(x, y) = 2(\bar{\epsilon}_{11}/\bar{c}_{66})^{1/2} \int_0^{c_2} \frac{\alpha x_1 \exp(i\alpha x) \sin(\alpha y)\, \mathrm{d}\alpha}{(c_2^2 - a^2)^{1/2}[x_1^2 + i\alpha(\bar{\epsilon}_{11}\bar{c}_{66}(c_2^2 - \alpha^2))^{1/2}]} \qquad \ldots(26)$$

The applied torque produced by the rigid disc is given by

$$T = -\,2\pi \int_0^1 r^2 T_{\theta z}(r, o, t)\, \mathrm{d}r. \qquad \ldots(27)$$

As in the reference (Gladwell 1968) using the results of the eqns. (12), (17) and (23), we finally obtain

$$T = 8e_{14} \exp(iwt) \int_0^1 t\theta(t)\, \mathrm{d}t. \qquad \ldots(28)$$

Numerical Results

For the numerical computations, we consider the piezoelectric half-space of β-quartz. The following non-dimensional quantities are evaluated from the physical constants given in the reference (Paul and Rao 1969)

$$\bar{c}_{66} = 1.400552;\ \bar{\epsilon}_{11} = 0.955642;\ K_{14}^2 = 0.002933.$$

For convenience the constant of proportionality d appearing in eqn. (15), is assumed to be such that $2d/L$ is unity. We obtain the kernel from eqn. (26), on replacing α by the non-dimensional variable α/c_1, as

$$M(x, y) = 2(\bar{\epsilon}_{11}/\bar{c}_{66})^{1/2} \int_0^a \frac{\alpha y_1 \exp(ic_1\alpha x) \sin(c_1\alpha y)}{(a^2-\alpha^2)^{1/2}[y_1^2 + i\alpha(\bar{\epsilon}_{11}\bar{c}_{66}(a^2-\alpha^2))^{1/2}]} d\alpha,$$

where

$$a = c_2/c_1,\ y_1 = [(P_1 + (P_1^2 - 4Q_1)^{1/2})/2]^{1/2},$$

$$P_1 = (K_{14}^2 + \bar{\epsilon}_{11} + \bar{c}_{66})\,\alpha^2 - 1 \text{ and } Q_1 = \bar{\epsilon}_{11}\bar{c}_{66}\alpha^2(\alpha^2 - a^2).$$

TABLE I

$\theta(x)$ *when* $c_1 = 0.5$

x	Real part of $\theta(x)$	Imaginary part of $\theta(x)$
0	– 0.0407	0 0232
0.1	0.0586	0.0217
0.2	0.1579	0.0201
0.3	0.2573	0.0185
0.4	0 3568	0.0169
0.5	0.4563	0.0153
0.6	0.5559	0.0137
0.7	0.6555	0.0121
0.8	0.7552	0.0104
0.9	0.8550	0.0087
1.0	0.9548	0.0071

TABLE II

Values of $\int_0^1 t\theta(t)\,dt$

c_1	Real part	Imaginary	Modulus
0.1	0.6721	0.0041	0.6721
0.2	0.6640	0.0071	0.6640
0.3	0 6559	0.0090	0.6559
0.4	0.6479	0.0099	0.6480
0.5	0.6403	0.0098	0.6404
1.0	0.6088	– 0.0011	0.6088
1.5	0.5918	0.0223	0.5923

Numerical calculations are obtained by programming on IBM 370/155 digital computer. For $c_1 = 0.5$ the real and imaginary parts of the solution $\theta(x)$ of the Fredholm integral eqn. (24) are shown in Table I. The argument x ranges from 0 to 1 at steps of 0.1. From eqn. (29), we find that the applied torque T, produced by the rigid disc, is adequately determined by knowing $\int_0^1 t\theta(t)\, dt$. For seven different values of c_1, this integral is found in Table II. We infer from Table I, that the real part of $\theta(x)$ gradually increases while the imaginary part decreases. Table II shows that the real part of $\int_0^1 t\theta(t)\, dt$ decreases while the imaginary part increases as c_1 ranges from .1 to .4. Both the real and imaginary parts decrease when c_1 ranges from .5 to 1.5. From the last column of Table II, we observe that the modulus of the applied torque decreases as c_1 ranges from .1 to 1.5.

References

Gladwell, G. M. L. (1968). Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid. *Int. J. Engng. Sci.*, **6**, 591–607.

Paul, H. S. (1968). Rotatary vibration of a rigid circular die on a semi-infinite elastic solid. *J. acoust. Soc. Am.*, **44**. 1232–1236.

Paul, H. S., and Rao, B. S. (1969). Propagation of disturbance in an infinite solid of piezoelectric β-quartz. *Proc. Camb. phil. Soc. Math. phys. Sci.*, **66**, 205–213.

Paul, H. S., and Sarma, K. V. (1977). Torsional wave propagation in a finite piezoelectric cylindrical shell. *Proc. Indian natn. Sci. Acad.*, A, **43**, 169–182.

Reissner, E., and Sagoci, H. F. (1944). Forced torsional oscillation of an elastic half-space. *J. appl. Phys.*, **15**, 652–654.

Robertson, I. A. (1967). On a proposed determination of the shear modulus of an isotropic elastic half-space by the forced torsional oscillations of a circular disc. *Appl. Sci. Res.*, **17**, 305–312.