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Review Article (Nuclear Scattering)

COMPLEX ANGULAR MOMENTUM APPROACH TO NUCLEAR SCATTERING : A REVIEW

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This review article describes the formulation and application of complex angular momentum method to nuclear scattering problems. Regge type representations for nuclear and Coulomb-nuclear problem are discussed for the scattering of spin zero, spin $\frac{1}{2}$, and spin 1 particles from spin zero targets. The nature of Regge trajectories constructed from the energy levels of compound nuclei are described for the light nuclear systems like $\alpha - \alpha$, $\alpha - C^{12}$, $\alpha - O^{16}$, $p - O^{16}$, $n - O^{16}$, $p - C^{12}$ and $n - C^{12}$. Different methods of using the Regge trajectory and pololoy in the analysis of the nuclear scattering are summarised including their role in the description of heavy ion scattering amplitude. The review brings out both the advantages and drawbacks that one finds in using the complex angular momentum method in the analysis of nuclear scattering.

Keywords : Complex Angular Momentum; Nuclear Scattering; Regge Trajectories.

INTRODUCTION

THE study of the analytic properties of the non-relativistic scattering amplitude and the S-matrix in the complex angular momentum plane initiated by Regge and coworkers (Regge, 1959; Bottino *et al.*, 1962; and de Alfaro & Regge, 1965) has been adopted to construct phenomenological models for describing collisions in high energy physics. Comparatively speaking, the application of complex angular momentum (CAM) approach in non-relativistic scattering problems, namely, nuclear and atomic collisions is rather limited. One would expect that since the CAM method is rigorously established only in potential scattering, its application will be numerous in nuclear and atomic physics problems. This is not so because of the fact that in non-relativistic scattering theory, one can in general solve the Schrödinger equation with the knowledge of the interaction and obtain all the physical quantities. In the absence of this approach in high energy physics, one adopts models constructed based on the results available in the non-relativistic

scattering theory. In spite of this, there have been some exercises in finding applications of the Regge method to nuclear and atomic scattering. Essentially these works are motivated by the advantages and elegance of the Regge method for particular set of problems in nuclear and atomic scattering. The purpose of this review is to analyse and summarise these applications.

Broadly speaking, the CAM method comes within the analytic S-matrix approach to scattering. In this approach, one attempts to understand and analyse scattering in terms of the analytic properties of the scattering amplitude and the S-matrix. The analytic properties of the scattering amplitude that are used may be as a function of energy or angles or the S-matrix as function of the energy, momentum, or angular momentum. For the study of scattering in the non-relativistic domain, the analytic properties of partial wave amplitude as a function of energy, momentum or angular momentum are of considerable interest. For example, Humblet and Rosenfeld (1961) have formulated the analytic S-matrix approach to nuclear reactions in terms of the poles of the S-matrix in the momentum or energy plane. But, in general, the analytic structures of the partial wave amplitude in the momentum and energy plane are complicated even for well-behaved potentials like Yukawa potential. For example, in the case of Yukawa type potentials, the S-matrix has cuts in addition to pole terms in the momentum (k) plane or energy (E) plane. The number of poles, in particular in the lower half k plane, are infinite for any partial wave. Therefore, in the analytic S-matrix approach based on the poles in the momentum plane, one has to analyse the S-matrix for each partial wave in terms of a large number of poles and background terms with cuts which are difficult to determine. Finally, the pole terms in the representation for the S-matrix in the momentum plane do not have well-known physical properties such as the threshold behaviour in momentum and asymptotic behaviour in angular momentum. Therefore, a pole representation of the S-matrix in the momentum plane, in general, will not generate the physical partial wave amplitudes except near sharp resonances. The structure of the scattering amplitude in terms of poles of the S-matrix in the angular momentum plane (Regge poles) is significantly different and provides an alternative description of the scattering amplitude.

In the complex angular momentum ($\lambda = l + \frac{1}{2}$) plane, the analytic properties of the S-matrix are simpler than those in the momentum plane (Newton, 1964). For example, in the case of Yukawa potential, the S-matrix is a meromorphic function in the λ-plane. and physically meaningful poles corresponding to bound states and resonances are always restricted to the right half plane and the number of poles is finite for a given positive energy. Secondly, in the complex angular momentum approach, the full scattering amplitude is usually represented in terms of a few poles in the right half plane and a background integral. This should be compared with the fact that in the momentum plane approach, the S-matrix for each partial wave is represented in terms of poles and a background term. In contrast to the momentum plane approach, the background integral in Regge pole theory represents essentially, the effect of singularities in the left half plane. In the modified Regge type representation, a part of the background effects are included. Another very satisfying feature of the complex angular momentum approach is that

it is possible to formulate Regge type representation for partial wave amplitudes such that in addition to keeping the background effect small, they exhibit the physical features of the S-matrix : namely, correct threshold behaviour in momentum, asymptotic behaviour in angular momentum and unitarity.

Another important aspect of the Regge pole approach is the following: One single Regge pole in general can partially generate larger number of partial waves in its representation to the scattering amplitude. For example, in the strong absorption models of nucleus-nucleus scattering, where a large number of partial waves for $l = 0$ to $l = L$ are more or less completely absorbed into the reaction channel, there appears to be a pronounced peak in the differential cross-section in the backward angle. A Regge pole which can be correlated with the quasi molecular states of the two ions seems to be able to explain this quite well in a number of cases.

From a survey of literature one finds that three broad approaches are being followed in applying the Regge pole theory to nuclear scattering. In one approach, the scattering amplitude is represented by Regge poles, and in this representation the poles and residues are treated as parameters to obtain a fit to the cross-section data. In this approach, one can choose a suitable background term too. In the second approach, one uses the states of the compound nucleus to identify various Regge poles and Regge trajectories and then uses this to obtain a fit to the cross-section based on Regge type representations. In these calculations, the Regge pole parameters are kept fixed, but the representations are modified to make the background integrals small. In the third type of calculation, Regge pole parameters are calculated from the corresponding optical potential and these parameters are used along with a suitable background term to analyse the scattering data. We shall make a survey of all these approaches in this review, and point out the significance of these methods in different physical problems.

The plan of this article is as follows: First, we give a brief outline of the Regge pole method in potential scattering and its extension to the Coulomb nuclear problem. Secondly, we summarise various Regge type representations. Thirdly, we have discussed the Regge trajectories of nuclear systems. Fourthly, we describe the method of analysing the nuclear scattering data and finally applications to ion-ion scattering are presented.

Regge Poles in Potential Scattering

In this section, we summarise the Regge pole formulation which is useful in analysing nuclear scattering cross-sections. One starts with the partial wave expansion of scattering amplitude $A(k, z)$ for the scattering of a zero spin projectile by a zero spin target:

$$A(k, z) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) [S_l(k) - 1] P_l(z). \qquad \ldots(1)$$

Here, k is the magnitude of the relative momentum, S_l is the S-matrix for lth partial wave and is related to the lth partial wave phase shift δ_l through the relation

$S_l = e^{2i\delta_l}$, $z = \cos\theta$, θ being the scattering angle in the centre of mass (c.m.) system. Using the relation $P_l(z) = (-1)^l P_l(-z)$ and assuming that $S_l(k)$ has analytic continuation in the complex λ plane, one can write,

$$A(k, z) = \frac{1}{2k} \int_C d\lambda \lambda \frac{[S(\lambda, k) - 1]}{\cos \pi\lambda} P_{\lambda-(1/2)}(-z) \qquad \ldots(2)$$

where C is the contour shown in Fig. 1 which crosses the real axis between 0 and $\frac{1}{2}$, and avoids all the poles of the $S(\lambda, k)$. $S(\lambda, k)$ is the analytically continued S-matrix in the complex angular momentum plane: $S_l(k) \equiv S(l + \frac{1}{2}, k)$. If the integrand in (2) is sufficiently well behaved in the right half plane including the imaginary axis, then the integral (2) can be shown to be equal to

$$A(k, z) = \frac{1}{2k} \int_{-\infty}^{\infty} d\lambda' \lambda' [S(i\lambda', k) - 1] P_{i\lambda'-(1/2)}(-z)/\cosh(\pi\lambda')$$

$$- \frac{i\pi}{k} \sum_n \lambda_n \beta_n P_{\lambda_n-(1/2)}(-z)/\cos(\pi\lambda_n) \qquad \ldots(3)$$

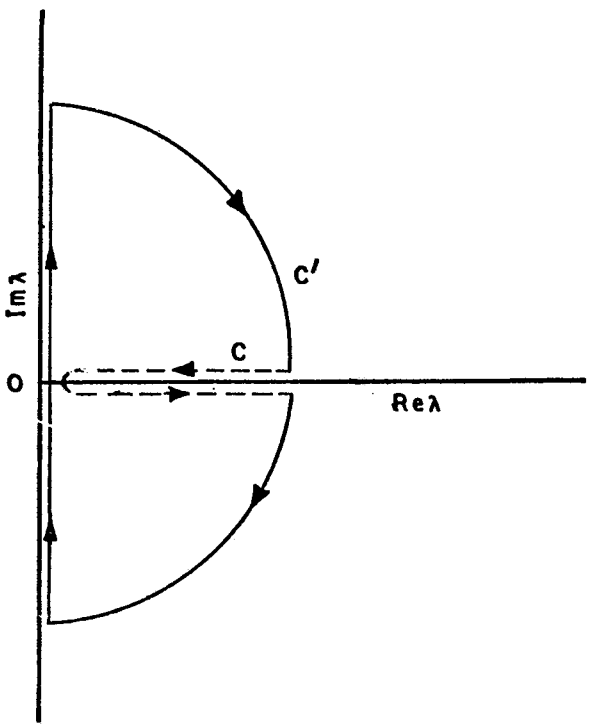


FIG. 1. Contour C of integration for the Sommerfeld-Watson Transform. C' indicates the modified contour to incorporate poles in λ-plane.

where λ_n are the positions of the poles (Regge poles) of $S(\lambda, k)$ in the right half λ-plane and β_n are the corresponding residues at these pole positions. Eqn. (3) provides the Regge representation for the scattering amplitude. The procedure with which one obtains the representation (3) from (1) is known as Sommerfeld-Watson transform. It has been established that for Yukawa type potentials defined by

$$V(r) = \int_{\mu_0}^{\infty} d\mu \sigma(\mu)\, e^{-\mu r}/r \qquad \ldots(4)$$

the function $[S(\lambda, k) - 1]$ approaches zero sufficiently rapidly as $|\lambda| \to \infty$ in the right half plane so that Sommerfeld-Watson transform (Sommerfeld, 1949) can be carried out. In fact for Yukawa potential $V(r) = G \exp(-\mu r)/r$, we have the result

$$|S(\lambda, k) - 1|_{\substack{|\lambda| \to \infty \\ \mathrm{Re}\,\lambda \geqslant 0}} = O(|\lambda|^{-1/2} \exp(-\mathrm{Re}\,\lambda\xi)) \qquad \ldots(5)$$

where

$$\cosh \xi = 1 + \mu^2/(2k^2). \qquad \ldots(6)$$

The asymptotic behaviour given by (5) also establishes the uniqueness of the analytic continuation of $S(\lambda, k)$ which can be used in a Sommerfeld-Watson transform. It can be shown that if there are more than one analytic continuation of $S(\lambda, k)$ in the half plane $\mathrm{Re}\,\lambda \geqslant 0$ behaving sufficiently well to carry out the Sommerfeld-Watson transform, they are identical to each other. This is proved using a theorem due to Carlson (Titchmarch, 1939) which states that if $f(z)$ is regular in the right half plane $\mathrm{Re}\, z > 0$ and if it is $O(e^{k|z|})$ as $|z| \to \infty$ with $k < \pi$ and $f(z) = 0$ for $z = 1, 2, \ldots$; then $f(z) \equiv 0$. It can also be shown that $S(\lambda, k)$ with the asymptotic behaviour given by (5) cannot have infinite number of poles in the right half plane, and hence the sum in eqn. (3) has only finite number of terms.

In order to apply the Regge representation or modified forms of it to the nuclear scattering problem like proton-nucleus or nucleus-nucleus scattering one has to consider the scattering potential which is a sum of nuclear part (which can be assumed to be of Yukawa type) and the Coulomb part (which has a long range r^{-1} behaviour even though within the nuclear dimension its r dependence gets modified by the nuclear charge distribution). The S-matrix $S(\lambda, k)$ for an ordinary Coulomb potential has the form

$$S_c(\lambda, k) = \frac{\Gamma(\lambda + \frac{1}{2} + i\eta)}{\Gamma(\lambda + \frac{1}{2} - i\eta)} = e^{2i\sigma_l} \qquad \ldots(7)$$

where $\eta = z_1 z_2 e^2 m/\hbar^2 k$ and z_1, z_2 are the charge numbers of the two interacting particles and m is the reduced mass. Obviously $[Sc(\lambda, k) - 1]$ does not have the asymptotic behaviour necessary for carrying out the Sommerfeld-Watson transform. However, in nuclear scattering problem involving Coulombic interaction it is advantageous to write the scattering amplitude as a sum of the Coulomb amplitude $A_c(k, z)$ and the "nuclear" amplitude $\bar{A}(k, z)$

$$A(k, z) = A_c(k, z) + \bar{A}(k, z) \qquad \ldots(8)$$

where

$$A_c(k, z) = \frac{-\eta}{2k \sin^2 \frac{\theta}{2}} \exp\left(-2i\eta \ln \sin \frac{\theta}{2} + 2i\sigma_0\right) \qquad \ldots(9)$$

and

$$\bar{A}(k, z) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)\, S_c(\lambda, k)\, [\bar{S}(\lambda, k) - 1]\, P_l(z). \qquad \ldots(10)$$

The full S-matrix $S(\lambda, k)$ is related to $\bar{S}(\lambda, k)$ through the relation

$$S(\lambda, k) = S_c(\lambda, k)\, \bar{S}(\lambda, k). \qquad \ldots(11)$$

It should, however, be noted that $\bar{S}(\lambda, k)$ has contribution from both nuclear and Coulomb potential.

In the Coulomb nuclear problem, it is not possible to carry out the Sommerfeld-Watson transform for the partial wave expansion of the full scattering amplitude. This is because $S(\lambda, k)$ does not have the desired asymptotic behaviour. However, a Sommerfeld-Watson transform for $\bar{A}(k, z)$ is still feasible. Mukherjee and Shastry (1967) and Klarsfeld (1967) studied the analytic continuation of $S(\lambda, k)$ in the complex angular momentum using a procedure similiar to that adopted by Bottino *et al.* (1962) and showed that an analytic continuation of $\bar{S}(\lambda, k)$ exists such that

$$| \bar{S}(\lambda, k) - 1 |_{|\lambda| \to \infty} = O(\exp(-\operatorname{Re} \lambda \xi) \mid \lambda \mid^{-1/2})_{\operatorname{Re} \lambda \geqslant 0} \qquad \ldots(12)$$

provided the nuclear potential is of Yukawa type. This shows that one can carry out the Sommerfeld-Watson transform on $\bar{A}(k, z)$ in the same way as for $A(k, z)$. It should be noted here that $| Sc(\lambda, k) |_{|\lambda| \to \infty} = O(1)$.

Thus we can write the Regge representation for the 'Nuclear amplitude' as

$$\bar{A}(k, z) = \frac{1}{2k} \int_{-\infty}^{\infty} d\lambda' \lambda' \, Sc(i\lambda', k) \frac{[\bar{S}(i\lambda', k) - 1]}{\cosh(\pi\lambda')} P_{i\lambda'-(1/2)}(-z)$$

$$- i\pi \sum_n \lambda_n \beta_n Sc(\lambda_n, k)\, P_{\lambda_n-(1/2)}(-z)/\cos(\pi\lambda_n) \qquad \ldots(13)$$

where β_n is the residue of $\bar{S}(\lambda, k)$ at $\lambda = \lambda_n$. Eqns. (3) and (13) provide a basis for the application of Regge pole theory to nuclear scattering problems.

It is well known (Newton, 1960) that the poles in the momentum plane, when they are in the 3rd and 4th quadrants just below the real axis, with small imaginary parts, can be associated with resonances. Similarly, Regge poles in the λ-plane

which have real parts equal to positive half integers, *viz* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc, and small positive imaginary parts can describe resonances. This we see as follows: In eqn. (3) if we neglect the background integral and make a partial wave projection using (Erdelye, 1953)

$$\int_{-1}^{1} dz P_l(z) P_{\lambda_n-(1/2)}(-z) = -\frac{2}{\pi} \frac{\cos \pi\lambda_n}{\lambda^2 - \lambda_n^2} \qquad \ldots(14)$$

we obtain $a_l(k)$, the partial wave amplitude in the Regge approximation

$$a_l(k) = a(\lambda, k) = \frac{S(\lambda, k) - 1}{2ik} = \sum_n \frac{\beta_n}{2ik} \frac{2\lambda_n}{\lambda^2 - \lambda_n^2}. \qquad \ldots(15)$$

Expanding $\lambda_n(E)$ as a function of energy around E_0 for which Re $\lambda_n(E_0) = l + \frac{1}{2}$, we get (cf. Newton, 1964)

$$(\lambda - \lambda_n)^{-1} = \left[\frac{d}{dE} \lambda_n(E) \mid_{E=E_0}\right]^{-1} \frac{1}{E - E_0 + \Delta E + i\frac{\Gamma}{2}} \qquad \ldots(16)$$

where

$$\Delta E = I_m\lambda_n I_m\lambda_n' / [(\text{Re } \lambda_n')^2 + (I_m\lambda_n')^2] \qquad \ldots(17)$$

$$\Gamma = 2I_m\lambda_n \ \text{Re } \lambda_n' / [(\text{Re } \lambda_n')^2 + (I_m\lambda_n')^2] \qquad \ldots(18)$$

and primes indicate derivatives with respect to E evaluated at $E = E_0$. Insertion of (16) in (15) leads to a standard Breit-Wigner formula for the dominant energy dependence of the partial wave amplitude near E_0. ΔE is the shift energy and Γ is the resonance width. $I_m\lambda_n$ is closely related to Γ and ΔE. Thus, when ΔE is larger, and $I_m\lambda_n$ large, a sharp resonance cannot occur. It may be noted that in Regge representation for the partial wave amplitude the energy dependence of the latter comes from the energy dependence of λ_n and $\beta_n \cdot \lambda_n$ as a function of energy describes a Regge trajectory. If one knows the Regge trajectories, corresponding scattering amplitude can be evaluated at any given energy. We will briefly describe the Regge trajectories in the next section.

Regge Type Representations

Let us consider the Regge representation (R) for the scattering amplitude in the pole approximation

$$A(k, z) = -\frac{i\pi}{k} \sum_n \beta_n\lambda_n P_{\lambda_n-(1/2)}(-z)/\cos(\pi\lambda_n). \qquad \ldots(19)$$

The corresponding expression for the partial wave amplitude is given by (15). This representation is useful only when R is a good approximation to $A(k, z)$. Equivalently, one may ask whether the partial wave amplitude obtained from R is a good

approximation to $a_l(k)$. From eqn. (19), it is clear that in Regge representation, $A(k, z)$, will diverge in the forward angle $\theta = 0$ because $P_\alpha(-z)$ for non-integer α is not a polynomial and has a singularity at $\theta = 0$. This divergence is not present in the $A(k, z)$ for Yukawa type potentials. Alternatively speaking, this divergence occurs because the partial wave series

$$\sum_{l=0}^{\infty} (2l + 1)\, a_l(k) P_l(z)$$

obtained from the Regge representation in R, is not absolutely convergent. Furthermore, corresponding $a_l(k)$ does not have the necessary physical properties. $S_l(k)$, the S-matrix should obey a set of properties like (*i*) threshold behaviour in energy (*ii*) asymptotic behaviour in angular momentum (*iii*) unitarity; and (*iv*) correct positions of poles and zeros. For short range potentials, the threshold behaviour of $S_l(k)$ can be obtained by noting the threshold behaviour of phase shift $\delta_l(k)$.

$$\delta_l(k)_{k\to 0} = O(k^{2l+1}) \qquad \ldots(20)$$

which means

$$\text{Re } a_l(k)_{k\to 0} = O(k^{2l}),$$

$$Im a_l(k)_{k\to 0} = O(k^{4l+1}). \qquad \ldots(21)$$

The unitarity condition for $S(\lambda, k)$ is

$$S(\lambda, k)\, S^*(\lambda^*, k^*) = 1$$

for real potential. This means if the pole λ_n is the pole of $S(\lambda, k)$, then λ_n^* will be the zero of $S(\lambda, k)$. It is desirable that the representation of $a_l(k)$ takes into account the unitarity together with the pole and zero structure of the S-matrix. The asymptotic behaviour that the partial wave amplitudes should obey as a function of λ is evident from eqn. (5).

In order to construct suitable Regge type representations for the partial wave amplitude, it is desirable to incorporate as many physical features of the S-matrix as possible. In spite of this, it is possible that the representations so constructed do not reproduce the amplitude to a reasonable accuracy. Let us examine this situation in more detail: if $a(\lambda, k)$, the analytically continued $a_l(k)$ ($a_l(k) = a(l+\frac{1}{2}, k)$ for $l = 0, 1, 2, \ldots$ is a meromorphic function of λ for positive energy, it can be expressed in the form (cf. Mukherjee, 1967)

$$a(\lambda, k) = \frac{1}{2ik} \sum_n \frac{\beta_n}{2ik} \frac{1}{\lambda - \lambda_n} + f_1(\lambda, k) \qquad \ldots(22)$$

For Yukawa type potential for which $| a(\lambda, k) | = O[(| \lambda | ^{-1/2} \exp(-\text{Re } \lambda\xi)]$ we have

$$f_1(\lambda, k) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda' a(\lambda', k)/(\lambda - \lambda'). \qquad \ldots(23)$$

Similarly, we may write

$$a(\lambda, k) = \frac{1}{2ik} \sum_n \beta_n \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} + f_2(\lambda, k)$$

$$= \frac{1}{2ik} \sum_n \beta_n \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda' \frac{F(\lambda', \lambda)\, a(\lambda', k)}{\lambda - \lambda'} \quad \ldots(24)$$

where $F(\lambda', \lambda)$ is a function such that $F(\lambda, \lambda) = 1$ and $F(\lambda', \lambda)$ is analytic for $\mathrm{Re}\,\lambda \geqslant 0$, $\mathrm{Re}\,\lambda' \geqslant 0$ and $F(\lambda', k)\, a(\lambda', k)$ is bounded on the right half plane so that the integral in (24) can be carried out by closing the contour in the right half plane. It is easily seen that

$$\sum_n \frac{\beta_n}{2ik} \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} = \sum_n \frac{\beta_n}{2ik} \frac{1}{\lambda - \lambda_n} + [f_1(\lambda_1\, k) - f_2(\lambda, k)]. \quad \ldots(25)$$

This means the Regge pole representation $\sum_n \frac{\beta_n}{2ik} \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n}$ can be obtained from $\sum_n \frac{\beta_n}{2ik(\lambda - \lambda_n)}$ by adding a term $[f_1(\lambda, k) - f_2(\lambda, k)]$, which is analytic for $\mathrm{Re}\,\lambda \geqslant 0$. But this is obviously part of the background integral in (23). This gives a general procedure for the construction of different types of representation of $a(\lambda, k)$ in terms of Regge poles. Thus we have for example, the usual Regge representation (R) if we set

$$F(\lambda', \lambda) = 2\lambda'/(\lambda + \lambda') \quad \ldots(26)$$

and the Khuri representation (Khuri, 1963) if we use

$$F(\lambda', \lambda) = e^{(\lambda' - \lambda)\xi}. \quad \ldots(27)$$

It should be noted that in the original derivation of the Khuri representation, part of the background integral is added to the Regge pole terms. The method of Mukherjee gives a general approach to extract contributions from the background integral (Mukherjee, 1967).

From the above discussion, it is clear that one can make good use of $F(\lambda', \lambda)$ to ensure the desired physical properties of $a_l(k)$ and perhaps also the smallness of background integral. For this purpose one can use $F(\lambda', \lambda)$ that damps out the integrand in the λ' variable and that at the same time ensures the correct asymptotic behaviour in the λ variable. This can be done in many ways. For example, the factor in $F(\lambda', \lambda)$, which guarantees the correct threshold behaviour in both the real and the imaginary parts of $a_l(k)$ and the correct asymptotic behaviour in the λ variable may be taken as $F_1(\lambda', \lambda)$ and the damping factor as $F_2(\lambda'\, \lambda)$ where

$$F_1(\lambda', \lambda) = g(\lambda)/g(\lambda') \quad \ldots(28)$$

$$g(\lambda) = \exp\left[-\lambda\xi_1 + i\exp\left(-\lambda\xi_2\right)\right] \qquad \ldots(29)$$

$$\cosh \xi_i = 1 + \mu_i/(2k^2). \qquad \ldots(30)$$

μ_1, μ_2 are suitable parameters and

$$F_2(\lambda', \lambda) = [2\lambda/(\lambda + \lambda')]^m \qquad \ldots(31)$$

which for large λ' can be expected to damp out the background term contribution.

The scheme described above, synthesises a large class of Regge type representations. One can obtain, for example, modified Regge representations which take into account the Born terms (Abbe *et al.*, 1965; Abbe & Garg, 1967; Ahmadzadeh, 1964; and Suprokash Mukherjee, 1967) separately. Thus one can define,

$$b(\lambda, k) = a(\lambda, k) - B(\lambda, k) \qquad \ldots(32)$$

where $B(\lambda, k)$ is the Born term. As usual, we may write

$$b(\lambda, k) = \sum_n \frac{\beta_n}{2ik} \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(\lambda', \lambda)\, B(\lambda', k)\, d\lambda'/(\lambda - \lambda')$$

$$+ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(\lambda', \lambda)\, a(\lambda', k)\, d\lambda'/(\lambda - \lambda'). \qquad \ldots(33)$$

$F(\lambda', \lambda)$ should have the asymptotic behaviour of $b(\lambda', k)$. If $b(\lambda, k)$ is known explicitly as in the case of Yukawa potential the second term in eqn. (33) can be calculated and one obtains the modified Regge representation. When

$$F(\lambda', \lambda) = \exp\left[(\lambda' - \lambda)\bar{\xi}\right]$$

with $\cosh \bar{\xi} = 1 + 2\mu^2/k^2$ in the case of Yukawa potential one gets the following 'Modified Khuri' representation (MK) for the partial wave amplitude (*vide loc. cit*):

$$a(\lambda, k) = \sum_n \frac{\beta_n}{2ik} \frac{\exp\left[(\lambda_n - \lambda)\bar{\xi}\right]}{\lambda - \lambda_n} + \frac{G}{2k^2} Q_l(\cosh \xi)$$

$$- \frac{G}{2k^2} \sum_{n=1}^{\infty} P_{n-1}(\cosh \xi) \frac{\exp\left[-(l+n)\bar{\xi}\right]}{(l+n)}. \qquad \ldots(34)$$

In the representation that one can write using the above procedure, it is possible to incorporate the pole structure in the right hand plane, threshold and asymptotic behaviour, but they do not incorporate unitarity or the zeros of the S-matrix. These are well taken into account in the product representation of $S(\lambda, k)$ constructed by Cheng. These representations also can be incorporated in the present framework. Thus, the Cheng and modified Cheng representation (Hung Cheng, 1966) can be obtained from the analysis of the following contour integral:

$$I = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda' \frac{F(\lambda', \lambda)}{\lambda - \lambda'} [\ln \{S(\lambda', k)\} - \bar{B}(\lambda, k)]. \quad \ldots(35)$$

In the modified Cheng representation (denoted by MC), $\bar{B}$ is the Born amplitude for the partial wave ($S_l - 1$). We may close the contour in the right half λ' plane Re $\lambda' \geqslant 0$ if the integrand is bounded well enough for suitable functions $F(\lambda', \lambda)$ and $\bar{B}(\lambda, k)$. When closed in the right half plane we have for the integral I_1,

$$I = [\ln S(\lambda, k)] - \bar{B}(\lambda, k) + \sum_n \int_{\lambda_n}^{\lambda_n^*} d\lambda' F(\lambda', \lambda)/(\lambda - \lambda') \quad \ldots(36)$$

or

$$S(\lambda, k) = \exp [\bar{B}(\lambda, k)] \prod_n \exp \left[\int_{\lambda_n}^{\lambda_n^*} d\lambda' F(\lambda', \lambda)/(\lambda - \lambda') \right]$$

$$\times \exp \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda' \frac{F(\lambda', \lambda)}{\lambda - \lambda'} \{\ln [S(\lambda', k)] - \bar{B}(\lambda', k)\} \right] \quad \ldots(37)$$

where the number of terms in the summation in (36) or product in (37) extends over the number of Regge poles in the right half plane. We have assumed $F(\lambda, \lambda) = 1$ and that $F(\lambda', \lambda)$ and $\bar{B}(\lambda, k)$ are analytic functions in λ in the right half-plane. We get the Cheng representation (C) by putting $\bar{B}(\lambda, k) = 0$ and using $F(\lambda', \lambda) = \exp [(\lambda' - \lambda)\xi]$ and the modified Cheng representation by using

$$\bar{B}(\lambda, k) = \frac{i}{k} G \, . \, Q_l(\cosh \xi)$$

and $F(\lambda', \lambda) = \exp [(\lambda' - \lambda)\xi]$. In fact $F(\lambda', \lambda)$ defines a general class of products representations. It is clear that the unitarity of the S-matrix elements, their threshold properties, and the asymptotic properties in λ-plane can be guaranteed in many ways by proper choice of $\bar{B}(\lambda, k)$ and $F(\lambda', \lambda)$. In other words, the conditions imposed by unitarity, threshold properties for small k and asymptotic behaviour for large λ are not enough to fix the representation for $S(\lambda, k)$ or $a(\lambda, k)$ uniquely. Therefore, the only criteria for establishing the suitability of one representation over another is the relative contribution of the background integral. Since $F(\lambda', \lambda)$ incorporates the effects of background integral as explained earlier, one has to construct suitable $F(\lambda', \lambda)$. It should however be noted that unlike in the sum type representation, in the Cheng type product representation, the product of pole terms may very well generate $S(\lambda, k)$ without the background integral being small. Because this background can be integral multiple of $2\pi i$ instead of zero to make the last factor in (37) unity. But in the 'sum type' representation the smallness of the

background integral implies the smallness of the contribution of the left half-plane singularities which it signifies.

After an examination of the analytic properties of the S-matrix for a class of potentials like the superposition of Yukawa and square well, Desai and Newton have obtained a product representation using Weierstrass factorization procedure (Desai & Newton, 1963)

$$S(\lambda, k) = S(0, k) \exp(-2i\lambda I_m \bar{a}(k) \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^*} \frac{\lambda - \lambda_n^*}{\lambda - \lambda_n} \times \exp(2i\lambda I_m(\lambda_n^{-1}), \; E > 0 \quad \ldots(38)$$

where $\bar{a}$ is a function of k and obeys $\bar{a}(-k) = \bar{a}^*(k)$. Note that in this representation there is no background term and all the poles of $S(\lambda, k)$ are incorporated.

It is possible to incorporate the zero structure of $S(\lambda, k)$ in the sum type representation that we have discussed (Mukherjee & Shastry, 1967). Choose

$$F(\lambda', \lambda) = F_0(\lambda', \lambda) \prod_m (\lambda - \lambda_m^*) \Big/ \prod_m (\lambda' - \lambda_m^*) \quad \ldots(39)$$

such that F_0 does not have poles in the right half plane. Then we can get

$$a(\lambda, k) = \sum_n \frac{\beta_n}{2ik} \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} - \frac{1}{2ik} \sum_n F_0(\lambda_n^*, \lambda) \times \prod_{m \neq n} (\lambda - \lambda_m^*) \Big/ \prod_{m \neq n} (\lambda_n^* - \lambda_m^*) + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda' \frac{a(\lambda', k)}{\lambda - \lambda'} F(\lambda', \lambda). \quad \ldots(40)$$

It is easy to verify that

$$a(\lambda_m^*, k) = -\frac{1}{2ik}, \; S(\lambda_m^*, k) = 0.$$

It may be noted that the second term in (40) is an analytic function in the right half λ-plane. Thus here if one neglects the background integral, the representation that results has an analytic term in addition to pole terms.

From now on we shall denote the Regge type representations in which the pole terms are in the sum form as 'sum type' representation and the representations in which they are expressed as product as the 'product type' representation. Both the sum type and product type representations can be utilised to calculate the residues of $S(\lambda, k)$ at the Regge poles in approximate way. For example, if we introduce unitarity at $\lambda = \lambda_m^*$ for a sum type representation like (24) we get the following set of linear equations for the β_n :

$$\sum_{n=1}^{N} \beta_n \frac{F(\lambda_n, \lambda_m^*)}{\lambda_m^* - \lambda_n} = -1, \; m = 1, 2, \ldots, N. \qquad \ldots(41)$$

Solving this set of equations we can get β_n in term of pole positions. In a good Regge type representation when the background integral is small, one may expect to generate the residues satisfactorily. In the Cheng type product representation, one can get expressions for residues by taking the limit

$$\operatorname*{Lt}_{\lambda \to \lambda_n} (\lambda - \lambda_n) \, S(\lambda, k) = \beta_n.$$

The construction of Regge type representation needs further modifications for $S(\lambda, k)$ in the Coulomb-nuclear problem. In particular, it is desirable that a Coulomb threshold factor is incorporated. This is given by (cf. Treacy, 1967):

$$\xi^2_{\lambda-(1/2)} = e^{-\pi\eta} \frac{\Gamma(\lambda + \frac{1}{2} + i\eta) \, \Gamma(\lambda + \frac{1}{2} - i\eta)}{(\Gamma(\lambda + \frac{1}{2}))^2} \qquad \ldots(42)$$

Thus in the pole approximation one can construct the two following classes of representations (Shastry & Satpathy, 1968):

$$\bar{S}(\lambda, k) - 1 = \sum_n \frac{\beta_n}{\lambda - \lambda_n} F(\lambda_n, \lambda) \, \xi^2_{\lambda-(1/2)} / \xi^2_{\lambda_n-(1/2)} \qquad \ldots(43)$$

It should be noted that when $\eta = 0$, $S(\lambda, k)$ is the S-matrix for the nuclear potential only. A 'sum type' representation which takes into account unitarity at $\lambda = \lambda_n$ is

$$\begin{aligned}
\bar{S}(\lambda, k) - 1 = {} & \sum_{n=1}^{N} \beta_n \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} \left(\frac{2\lambda_n}{\lambda + \lambda_n} \right)^{m_0} \frac{\xi^2_{\lambda-(1/2)} \prod_m (\lambda - \lambda_m^*)}{\xi^2_{\lambda_n-(1/2)} \prod_m (\lambda_n - \lambda_m^*)} \\
& - \sum_{n=1}^{N} F(\lambda_n, \lambda) \left(\frac{2\lambda_n^*}{\lambda = \lambda_n^*} \right)^N \left(\frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda_n^*-(1/2)}} \right) \frac{\prod_{m \neq n} (\lambda - \lambda_m^*)}{\prod_{m \neq n} (\lambda_n^* - \lambda_m^*)} \\
& + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\lambda' \, \frac{[S(\lambda', k) - 1]}{\lambda - \lambda'} F(\lambda', \lambda) \left(\frac{2\lambda'}{\lambda + \lambda'} \right)^{m_0} \frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda'-(1/2)}} \\
& \times \frac{\prod_m (\lambda - \lambda_m^*)}{\prod_m (\lambda' - \lambda_m^*)}.
\end{aligned} \qquad \ldots(44)$$

This representation is denoted by M and modified Khuri representation (Khuri, 1963 for $S(\lambda, k)$ is

$$\bar{S}(\lambda, k) - 1 = \frac{iG}{k} Q_{i\eta, \lambda-(1/2)}(\cosh \xi)/\xi^2_{\lambda-(1/2)}$$

$$+ \sum_n \beta_n \frac{\exp[(\lambda_n - \lambda)\bar{\xi}]}{\lambda - \lambda_n} \frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda_n-(1/2)}}$$

$$- \frac{iG}{k} \sum_n \frac{\exp[-(\lambda - \frac{1}{2} + n)\bar{\xi}]}{(\lambda - \frac{1}{2} + n)} P_{i\eta, n-1}(\cosh \xi) \quad ...(45)$$

where $P_{i\eta,\, n-1}$ are the residues of $Q_{i\eta,\, n-1}(\cosh \xi)$.

Cheng or modified Cheng representation for $\bar{S}(\lambda, k)$ can be constructed using

$$\frac{\ln \bar{S}(\lambda, k)}{\xi^2_{\lambda-(1/2)}} = \sum_{n=1}^{N} \int_{\lambda_n}^{\lambda_n^*} d\lambda' \exp[(\lambda' - \lambda)\xi]/[\lambda' - \lambda)\xi^2_{\lambda'-(1/2)}] \quad ...(46)$$

$$\left[\ln \bar{S}(\lambda, k) - \frac{iG}{k} Q_{i\eta, \lambda-1/2}(\cosh \xi\right] \Big/ \xi^2_{\lambda-(1/2)}$$

$$= \sum_{n=1}^{N} \int_{\lambda_n}^{\lambda_n^*} d\lambda' \frac{\exp[(\lambda' - \lambda)\bar{\xi}]}{(\lambda' - \lambda)\, \xi^2_{\lambda'-(1/2)}}$$

$$- \frac{iG}{k} \frac{\exp[-(\lambda - \frac{1}{2} + n)\bar{\xi}]}{(\lambda - \frac{1}{2} + n)} P_{i\eta, n-1}(\cosh \xi). \quad ...(47)$$

All the representations mentioned above reduce to the representation of $S(\lambda, k)$ when $\eta = 0$.

A comparative study of various representations has been done by several authors for a range of energy in the case of Yukawa potential. Except at low energy none of the representations is good for quantitative analysis. However, in general, the representations which isolate the Born term and take into consideration unitarity at least at $\lambda = \lambda_n^*$, or explicitly like in the Cheng representations, are found to be superior to other representations. Figs. 2, 3, 4 and 5 exhibit the S-matrix calculation from various representations together with the exact results. In Figs. 2 and 3 the representations designated by $N_m (m = 1, 2)$ have the corresponding $F(\lambda', \lambda)$ given by

$$F(\lambda', \lambda) = F_1(\lambda', \lambda)\, F_2(\lambda', \lambda)$$

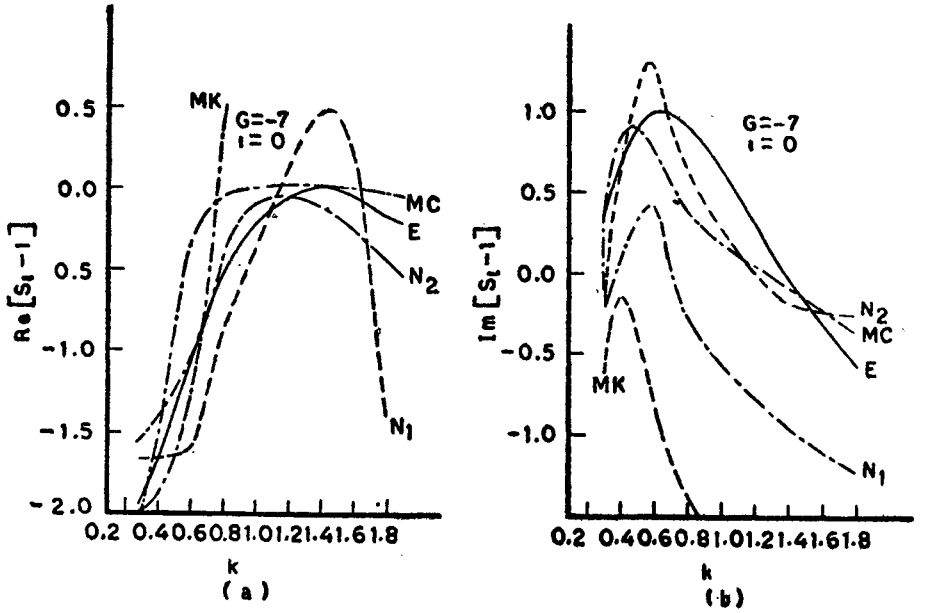


FIG. 2(*a-b*). Plot of $[S_l(k) - 1]$ *vs.* k for $l = 0$, from various Regge pole approximations. The potential used is $V(r) = -7 \exp(-r)/r$. E denotes exact results.

where

$$F_1(\lambda', \lambda) = g(\lambda)/g(\lambda')$$

$$g(\lambda) = \exp[-\lambda\xi_1 + i \exp(-\lambda\xi_2)]$$

$$F_2(\lambda', \lambda) = [2\lambda/(\lambda + \lambda')]^m.$$

In Figs. 4 and 5, the representation N_m has form factor $F(\lambda', \lambda)$ given by

$$F(\lambda', \lambda) = \left(\frac{2\lambda}{\lambda + \lambda'}\right)^m [\exp(ie^{-\lambda\xi}) - 1]/[\exp(ie^{-\lambda'\xi}) - 1]$$

and symbols R, C, MC, K and MK denote the Regge, Cheng, modified Cheng, Khuri and modified Khuri representations corresponding to Coulomb-nuclear problem as discussed earlier.

The results also indicate that, from the practical point of view, the number of trajectories needed in a representation seems to depend very much on the representation itself and on the strength of interaction.

REGGE TYPE REPRESENTATION FOR SCATTERING OF SPIN ZERO PARTICLE BY SPIN $\frac{1}{2}$ PARTICLES

In the nucleon-nucleus optical model type calculations, the nuclear spin is generally ignored, but the nucleon spin and the corresponding spin-orbit interaction is incorporated. This procedure also helps to calculate the polarization of the scattered

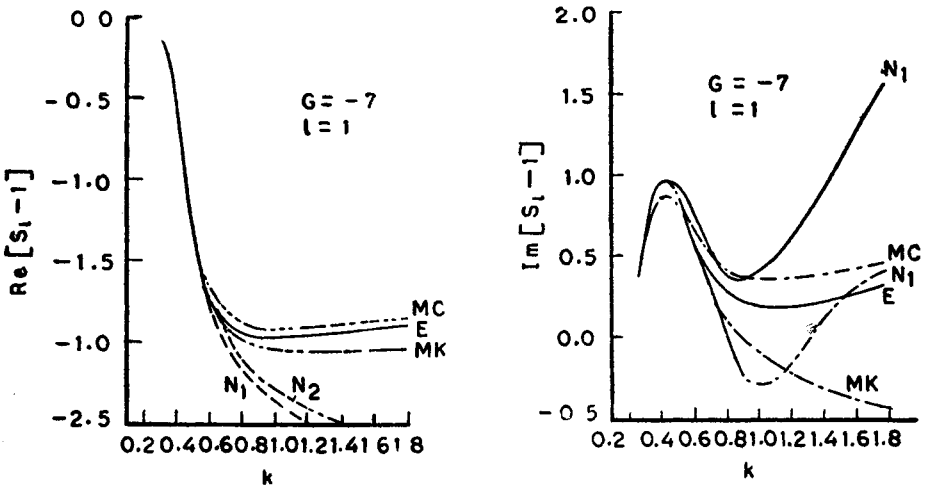


FIG. 3(*a-b*). Plot of $[S_l(k) - 1]$ *vs.* k for $l = 1$ from various Regge pole approximations. The potential used is the same $V(r) = -7 \exp(-r)/r$. E denotes exact results.

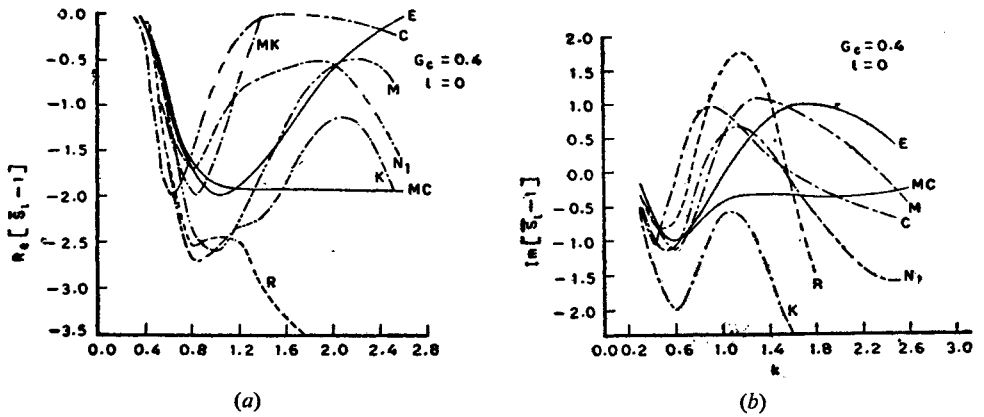
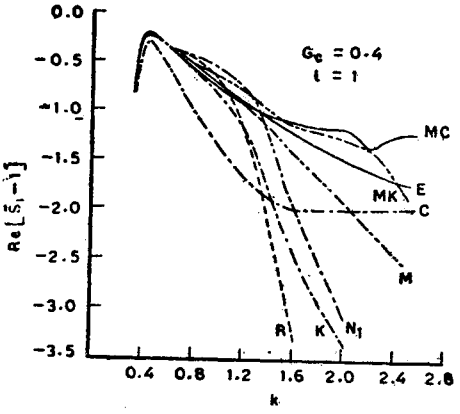


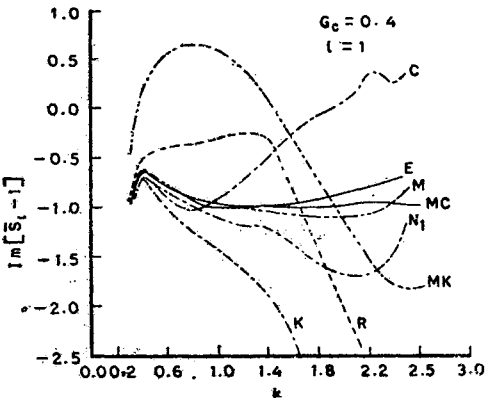
FIG. 4(*a*). Plot of Re $[S_l(k) - 1]$ generated by various Regge type representations for $l = 0$ as a function of k along with exact results. Nuclear potential used is $V(r) = -10 \exp(-r)/r$. G_c denotes the strength of repulsive Coulomb potential.

FIG. 4(*b*). Plot of I_m $[S_l(k) - 1]$ generated by various Regge type representations as a function of k along with exact results for $l = 0$. Nuclear potential is same as in Fig. 4(*a*).

nucleon. In these cases, when spin-orbit interaction is present, one has decoupled radial equations corresponding to total angular momentum $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$. The scattering cross-section and polarization can be obtained from the following two amplitudes:



(a)



(b)

FIG. 5(a). Same as 4(a) for $l = 1$.
FIG. 5(b). Same as 4(b) for $l = 1$.

$$A_1 = A_c(k, z) + \sum_{l=0}^{\infty} \exp(2i\sigma_l)\,[(l+1)\,a_l^+(k) + l a_l^-(k)]\,P_l(z)$$

$$A_2 = \sum_{l} \exp(2i\sigma_l)\,[a_l^+(k) - a_l^-(k)]\,P_l^1(z) \qquad \ldots(48)$$

where

$$a_l^{\pm}(k) = [S_l^{\pm}(k) - 1]/(2ik).$$

The Regge representation, neglecting the background (cf. Shanta & Satpathy, 1970) for $A_1(k, z)$ is

$$A_1(k, z) = \frac{\pi}{2ik}\left[\sum_n \beta_n^+ \frac{(\lambda_n^+ + \tfrac{1}{2})}{\cos \pi\lambda_n^+} P_{\lambda_n^+ - (1/2)}(-z) + \sum_n \beta_n^- \frac{\lambda_n^- - \tfrac{1}{2}}{\cos \pi\lambda_n^-} P_{\lambda_n^- - (1/2)}(-z)\right] \qquad \ldots(49)$$

and the Regge representation for $A_2(k, z)$ is

$$A_2(k, z) = \frac{\pi}{2ik}\left[\sum_n \beta_n^+ \frac{P'_{\lambda_n^+ - (1/2)}(-z)}{\cos \pi\lambda_n^+} +\right.$$

(equation continued on p. 390)

$$+\sum_n \frac{\beta_n^-}{\cos \pi\lambda_n^-} P'_{\lambda_n^+ - (1/2)}(-z)\Bigg]. \qquad \ldots(50)$$

One can obtain the partial wave projection of $A_1(k, z)$ by using (14) and get

$$\frac{\lambda \pm \frac{1}{2}}{2\lambda} a^{\pm}(\lambda, k) = \sum_n \beta_n^{\pm} \frac{\lambda_n^{\pm} \pm \frac{1}{2}}{[\lambda^2 - (\lambda_n^{\pm})^2]} \qquad \ldots(51)$$

and partial wave projection of $A_2(k, z)$ can be carried out by using

$$\tfrac{1}{2} \int_{-1}^{1} dz\, P_l^m(z)\, P_\alpha^m(-z) = \frac{\sin \pi\alpha \Gamma(\alpha + m + 1)}{\pi \Gamma(\alpha - m + 1)(\alpha - l)(\alpha + l + 1)}$$

and obtain

$$\frac{\lambda^2 - 1/4}{2\lambda} a^{\pm}(\lambda, k) = \sum_n \beta_n^{\pm} \frac{[(\lambda_n^{\pm})^2 - 1/4]}{[\lambda^2 - (\lambda_n^{\pm})^2]}. \qquad \ldots(52)$$

It is clear that the "Regge representation" obtained from A_1 and A_2 are different. Using the procedure of Khuri one can show in the pole approximation that

$$\frac{\lambda \pm \frac{1}{2}}{\lambda^2} a^{\pm}(\lambda, k) = \sum_n \frac{\lambda_n^{\pm} \pm \frac{1}{2}}{(\lambda_n^{\pm})^2} \beta_n \frac{\exp(\lambda_n^{\pm} - \lambda)\xi}{\lambda - \lambda_n^{\pm}} \qquad \ldots(53)$$

$$a^{\pm}(\lambda, k) = \sum_n \beta_n^{\pm} \frac{\lambda}{\lambda_n^{\pm}} \frac{\exp(\lambda_n^{\pm} - \lambda)\,\xi}{\lambda - \lambda_n^{\pm}}. \qquad \ldots(54)$$

It should however be noted that one can construct a host of representations for $a^{\pm}(\lambda, k)$ using the general procedure formulated by Mukherjee:

$$a^{\pm}(\lambda, k) = \sum_n \beta_n^{\pm} \frac{F(\lambda_n^{\pm}, \lambda)}{\lambda - \lambda_n^{\pm}} \qquad \ldots(55)$$

with $F(\lambda, \lambda) = 1$ and having the right analytic and asymptotic behaviour. For scattering of spin-one particles from spin-zero nuclei, one has to take the five amplitudes A, B, C, D and E related to scattering cross-section.

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{3}[\,|A|^2 + |B|^2 + C|^2 + |D|^2 + |E|^2]. \qquad ..(56)$$

Following the above procedure, Satpathy and Mishra (1975) have given Regge-type representation for $a^{\pm}$ and a^0 obtained from these five amplitudes. The five amplitude terms A, B, C, D and E can be obtained from the partial wave amplitudes $a^+(\lambda, k)$, $a^-(\lambda, k)$ and $a^0(\lambda, k)$ corresponding to $j = l + 1$, $l - 1$ and $j = l$ respectively. For example, Regge representation for $a^{\pm}(\lambda, k)$ and $a^0(\lambda, k)$ are

$$[(\lambda \pm \tfrac{1}{2})^3 + 2]\,\frac{a^{\pm}(\lambda, k)}{2\lambda} = \sum_n \beta_n^{\pm}\,[(\lambda_n^{\pm} \pm \tfrac{1}{2})^3 + 2]/[\lambda^2 - (\lambda_n^{\pm})^2]$$

$$(\lambda^2 - \tfrac{1}{4})\,a^0(\lambda, k) = \sum_n \beta_n^0\,[(\lambda_n^0)^2 - \tfrac{1}{4}]/[(\lambda^2 - \lambda_n^0)^2]$$

Similarly, one can work out the Khuri and other ampitudes. In the case of Coulomb-nuclear amplitude we have to incorporate the Coulomb factor $\xi^2_{\lambda-(1/2)}$.

Regge-Trajectories for Nuclear Systems

The calculations of cross-sections based on Regge poles require the knowledge of the Regge trajectories corresponding to the interacting system. This can be obtained either doing explicit calculations using the interaction potential or it can be generated phenomenologically from the knowledge of the bound states and resonances of the interacting system. Let us briefly summarise the essential features of the Regge pole for potentials of Yukawa type in the half plane Re $\lambda > 0$ (*see* Newton, 1964). If the energy is negative, the Regge poles are confined to the real axis. This is analogous to the corresponding situation in the E-plane when the negative energy poles are confined to the negative real axis. For real potentials of Yukawa type when the energy becomes complex, the poles in right half λ-plane are confined to the first quadrant excluding the real axis. As the energy increases, when the energy is negative, the poles move forward along the real axis and when the energy becomes positive they leave the real λ-axis and migrate to the first quadrant. Regge poles, as a function of energy, generate Regge trajectories. The behaviour of the trajectories as they leave the real axis is interesting. As $E \to 0^+$ for the Regge trajectory

$$\begin{aligned}\frac{d}{dE}\,\mathrm{Re}\,\lambda_0(E) &= \text{const, Re } \lambda_0 > 1 \\ &= -a \ln E, \text{ Re } \lambda_0 = 1 \\ &= \text{const } E^{\lambda_0 - 1} \cos \pi(\lambda_0 - 1),\ 0 < \text{Re } \lambda_0 < 1 \qquad ...(57)\end{aligned}$$

with $E \to 0$. The angle γ with which the trajectory leaves the real axis is,

$$\begin{aligned}\cot \gamma &= O(E^{1-\lambda_0}\ ,\ \lambda_0 > 1 \\ &= O(1) \quad\ ,\ 0 < \lambda_0 < 1\end{aligned}$$

(equation continued on p. 392)

$$= O(\ln E), \quad \lambda_0 = 1$$

as $E \to 0$.

Thus when $0 < \lambda_0 < \frac{1}{2}$, the trajectory leaves the real axis at a finite angle towards the left; when $1 > \lambda_0 > \frac{1}{2}$, towards the right, and when $\lambda_0 > 1$, is osculates the real axis and more closely for larger λ_0. This explains in the present context why an almost bound s-state causes no low-energy resonance but an almost bound state of $l > 0$ does, and the higher its l value the sharper the resonance. As the energy increases further, for Yukawa type potentials the trajectory turns back towards the left. In the case of square-well or other potentials of finite range they usually do not. For Yukawa type potentials as the energy increases, the trajectories turn towards the imaginary axis and eventually cross there, thereby entering the left half plane. In general, the stronger the potentials (attractive), the more is the number of trajectories in the right half plane.

If one is interested in calculating Regge trajectories for nuclear two body systems having Coulomb potential in addition to nuclear potential it is necessary to calculate $\bar{S}(\lambda, k)$, the 'nuclear S-matrix' and determine its poles in the λ-plane for various energies. It is clear that the role of repulsive Coulomb potential is to pull the trajectories towards the left half λ-plane. A study of the variation of Regge

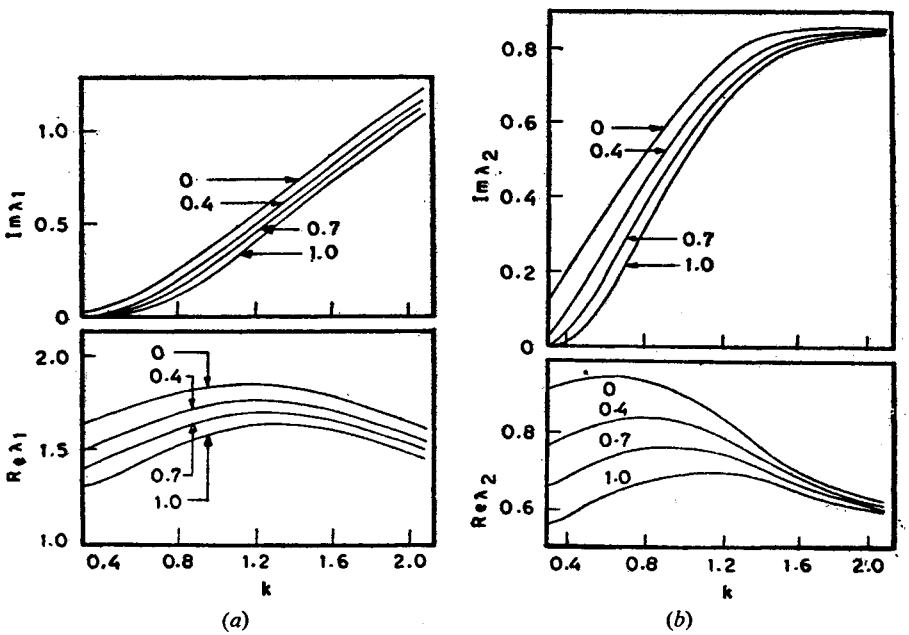


FIG. 6(*a*). Variation of the real and imaginary parts of the leading pole $\bar{S}(\lambda, k)$ corresponding to nuclear potential $V(r) = -10 \exp(-r)/r$. The numbers in the figure indicate G_c, the strength of the Coulomb potential.

FIG. 6(*b*). Variation of the real and imaginary parts of the second pole of $\bar{S}(\lambda, k)$ corresponding to nuclear potential $V(r) = -10 \exp(-r)/r$. The numbers in the figure indicate the values of G_c.

trajectory with the strength of Coulomb potential is carried out by Shastry and Satpathy (1969) for typical Coulomb nuclear system. It is found that the Regge trajectories for Coulomb nuclear system form a smooth extension of the Regge trajectories for pure short range potential (Fig. 6).

We have seen so far that it is possible to represent two body bound states and resonances as points in the Regge trajectory. Following the spirit of Chew-Frautschi plots (cf. Frautschi, 1964) one can classify the resonance of nuclear scattering in terms of Regge Trajectory. This procedure is very reasonable, when the nuclear scattering problem, which actually is a many body problem, can be appoximated by a two body problem, particularly so below the inelastic threshold. We illustrate this procedure with several examples in this section. For practical reasons, the classification of trajectory from the compound nuclear levels is to be done from a phenomenological point of view. We shall illustrate this with several examples.

Let us consider the scattering between two spin zero particles. Some well-known examples are $\alpha - \alpha$, $\alpha - C^{12}$ and $\alpha - O^{16}$. One can construct empirical Regge trajectories for the system from the knowledge of the corresponding compound nuclear levels namely those of Be^8, O^{16} and Ne^{20} respectively. Treacy was the first to study the resonance of $\alpha = \alpha$ system in terms of Regge trajectories (Treacy, 1965, 1967). In general the following procedure can be adopted to construct a trajectory after incorporating the essential features of the Regge trajectory (Shanta & Shastry, 1968). The energy levels forming a trajectory are grouped according to increasing energy and relative orbital angular momentum. Knowing the resonance energies and the corresponding angular momentum values, the real part of the trajectory is drawn using a graphical method and assuming a polynomial expansion for the real part of the trajectory:

$$\text{Re } \lambda_n(E) = \text{Re } \lambda_n(E_0) + a_1^{(n)} (E - E_0) + \dots . \qquad \dots(58)$$

Using the known widths of the resonances and the slope of the trajectory, i.e., $\frac{d}{dE}$ Re λ_n, $I_m\lambda_n$ is calculated corresponding to each resonance. Then this set of $I_m\lambda_n$, corresponding to various resonances is connected in a smooth way using a graphical method and assuming a smooth polynomial dependence with respect to energy.

$$I_m\lambda_n(E) = I_m\lambda_n(E_0) + b_1^{(n)} (E - E_0) + \dots . \qquad \dots(59)$$

This procedure is similar to the one used by Treacy and gives an empirical set of Regge trajectories. From the knowledge of these trajectories and unitarity and residues for each pole corresponding to various representations can be evaluated, as discussed earlier.

Table I shows the grouping of the three levels of Be^8 in order to obtain the Regge trajectory for the $\alpha - \alpha$ system. This is a system of two identical spinless particles obeying Bose-Einstein statistics and therefore, the resonance of elastic scattering occur only at even values of the orbital angular momentum. Fig. (7) shows the Regge trajectory of this system calculated following the method of Treacy described earlier.

TABLE I

Level parameters of Be^8 *for the Regge trajectory of the* α-α *system*

E_α (lab) (MeV)	E_x (MeV)	E_α (c.m.) (MeV)	J^π	Γ (c.m.)
0.190	0	0.095	0^+	6.8 ± 1.7 eV
6.55	3.18	3.275	2^+	1.3 MeV
22.99	11.4	11.495	4^+	7.0 MeV

Table II shows the grouping of energy levels of O^{16} which mainly decay by α-emission from which the Regge trajectories of α-C^{12} system are determined. Figs. 8(*a*) and 8(*b*) show the trajectories. The full circles correspond to actually observed levels and the open circles refer to tentatively assigned levels. Extrapolation of some of these trajectories in the region where experimental findings of different

TABLE II

Level parameters of O^{16} *for the Regge trajectories of the* α-C^{12} *system*

Trajectory	E_α (lab) (MeV)	E_x (MeV)	E_α (c.m.) (MeV)	J^π	Γ (keV)
I		5.66[a]	—1.49[a]	0^{+a}	
		7.1	—0.05	1^-	
	3.582	9.84	2.69	2^+	1
	11.08	15.45	8.3	3^-	380
	13.9	17.55	10.4	4^+	225
II		6.13	—1.02	3^-	
	4.28	10.36	3.21	4^+	36
	8.15[a]	13.27[a]	6.12[a]	5^{-a}	189[a]
	12.1	16.2	9.05	6^+	380
	18.4	20.9	13.75	7^-	750
III	3.24	9.58	2.43	1^-	860
	5.82	11.51	4.365	2^+	106
	11.4	15.1	8.53	3^-	525
	16.13[a]	19.25[a]	12.1[a]	4^{+a}	1074[a]
	18.5	21.0	13.85	5^-	1200
IV		6.036	—1.114	0^+	
	7.04	12.43	5.28	1^-	230
	12.35	16.41	9.26	2^+	60
	15.46[a]	18.75[a]	11.6[a]	3^{-a}	120[a]
	17.75	20.44	13.29	4^+	150
V		6.92	—0.23	2^+	
	3.066[a]	9.45[a]	2.3[a]	3^{-a}	137[a]
	8.97	13.88	6.73	4^+	90
	15.2	18.55	11.4	5^-	190
	18.7	21.2	14.05	6^+	450

[a]Tentative assignments.

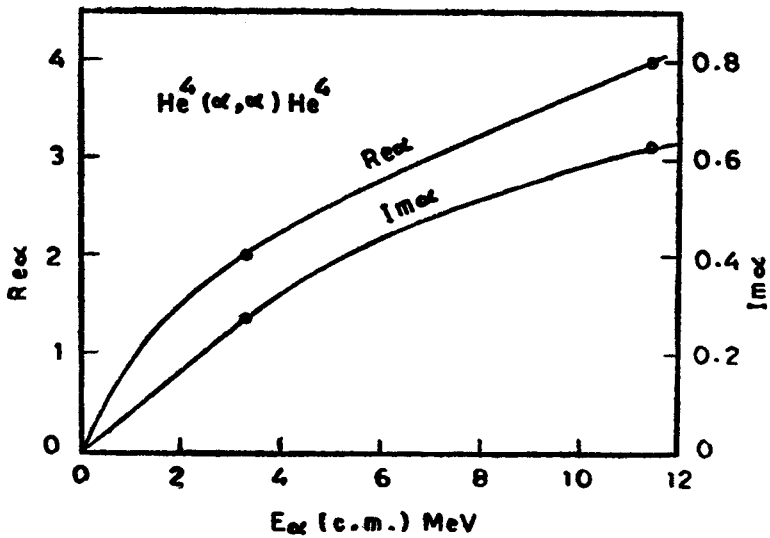
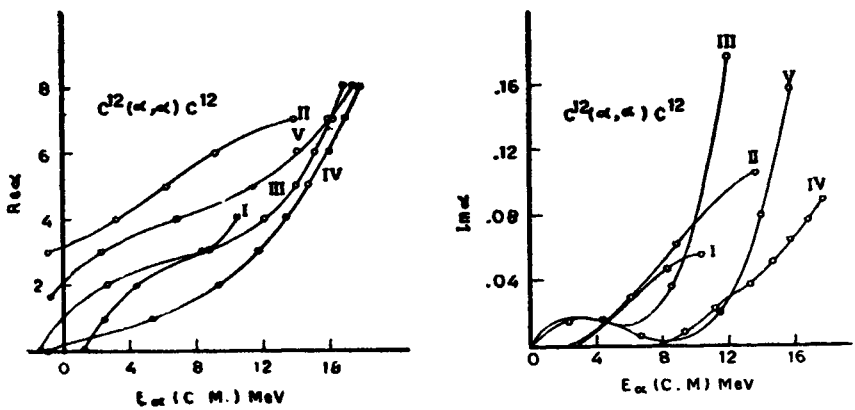


FIG. 7. Regge trajectory in l plane for the low lying levels of Be^8, considered as the trajectory for the α-α system.



(*a*) (*b*)

FIG. 8(*a*). The real part of the Regge trajectories in l plane as a function of incident energy in the c.m. system for the $C^{12}(\alpha, \alpha)C^{12}$ system obtained from the levels of O^{16}. (x axis-0 to 16)

FIG. 8(*b*). The imaginary part of the Regge trajectories in l plane corresponding to Fig. 8(*a*).

resonances and their quantum numbers being inconclusive are interesting. For example, extrapolation of trajectory IV in Fig. 8(*a*) indicates a series of narrow resonances at excitation energy $E_x = 21.8, 22.95, 23.05$ MeV etc. Rasmussen *et al.* (see Shanta & Shastry, 1968) observed a resonance at $E(\text{lab}) = 21.85$ MeV, which indicates the existence of an intermediate state of $C^{12} + \alpha$ at 23.5 MeV. Extrapolation of trajectory V also indicates the presence of an excited level at energies 23 and 24.3 MeV

of spin and parity 7^-, 8^+ and approximate widths 6.35 and 770 KeV, respectively. From rotational band systematics, Carter *et al.* also expected an 8^+ level between 24 and 30 MeV of width equal to 400 KeV (*see* Shanta & Shastry, 1968).

Table III contains the energy density of Ne^{20} grouped in order to obtain the trajectories of $\alpha - O^{16}$ system. Figs. 9(a) and 9(b) show these trajectories. When extrapolating trajectory I, it is seen to pass through the levels at excitation energies 14.24, 14.43 and 15.52 MeV of spin and parity 3^-, 6^+ and 7^- and approximate widths 86, 193 and 212 KeV, respectively. The later two may correspond to the levels at 15.72 and 16.483 obtained by Mehta *et al.* (1967) whose spin and parities were not determined. Trajectory IV passes through a tentative level 3^- at $E_x = 5.653$ MeV of width 269 KeV. This may well correspond to the 3^- level observed by Pearson *et al.* (1964) at 5.62 MeV excitation energy. On extrapolation, trajectory IV passes through an 8^+ level at $E_x = 14.463$ MeV which may correspond to level observed by Mehta *et al.* (1967) at 14.38 MeV, its parity and spin were not assigned.

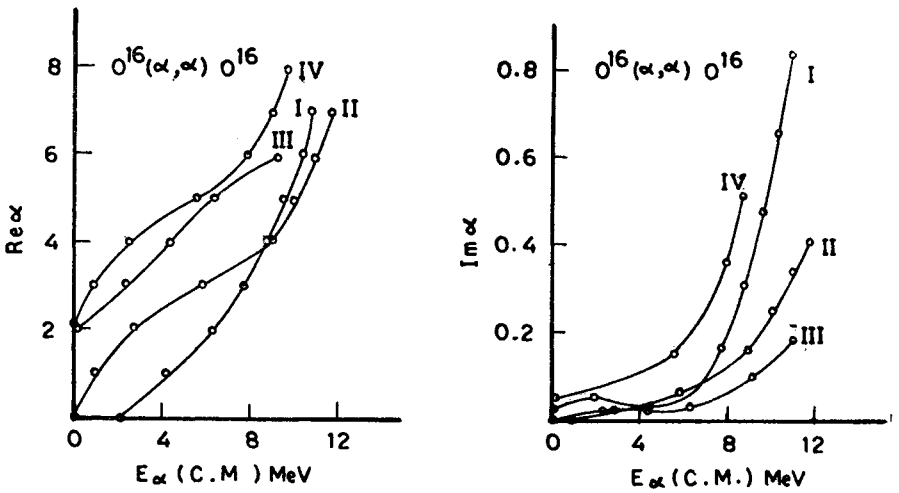
Sen and Shastry have reconsidered α-C^{12} and α-O^{16} systems and have drawn Regge trajectories as in Figs. 10 and 11 respectively (Dilip Sen & Shastry, 1971). In case of (C^{12}-α) system they obtain three trajectories

TABLE III

Level parameters of Ne^{20} *for the first set of Regge trajectories of the* α-O^{16} *system*

Trajectory	E_x (lab) (MeV)	E (MeV)	E_α(c.m.) (MeV)	J^π	Γ KeV
I	2.46	6.722	1.969	0^+	19
	5.19	8.905	4.152	1^-	23
	7.76[a]	10.963[a]	6.21[a]	2^{+a}	49[a]
	9.58	12.39	7.637	3^-	100
	10.87	13.43	8.677	4^+	140
II	1.3	5.79	1.037	1^-	2.5
	3.45	7.434	2.681	2^+	10
	7.2	10.49	5.737	3^-	55[a]
	10.546	13.19	8.437	4^+	75
III	0.271	4.97	0.217	2^+	2.3
	3.016	7.166	2.413	3^-	10
	5.403	9.076	4.323	4^+	10
	7.88[a]	11.063[a]	6.31[a]	5^{-a}	16.5[a]
	11.483	13.94	9.187	6^+	125
IV	0.037[a]	4.783[a]	0.03[a]	2^{+a}	9[a]
	1.125[a]	5.653[a]	0.09[a]	3^{-a}	269[a]
	2.834	7.02	2.267	4^+	740[a]
	6.934	10.3	5.547	5^-	150
	9.82	12.58	7.827	6^+	130
	11.2	13.69	8.937	7^-	400
	12.137[a]	14.463[a]	9.71[a]	8^{+a}	595.5[a]

[a]Tentative assignments.



(*a*) (*b*)

FIG. 9(*a*). The real part of the Regge trajectories in *l* plane versus the incident energy in the centre of mass system for the $O^{16}(\alpha, \alpha)\ O^{16}$ system obtained from the levels of Ne^{20}.

FIG. 9(*b*). The imaginary part of the Regge trajectories corresponding to Fig. 9(*a*).

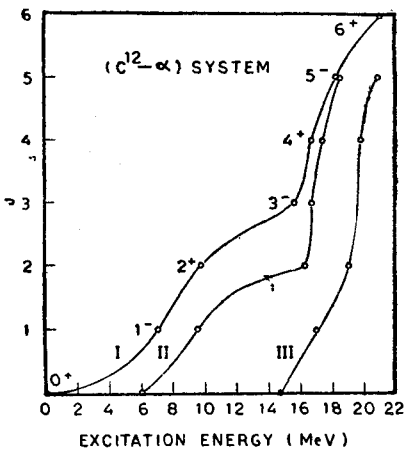


FIG. 10. Real part of the Regge trajectories constructed by Sen and Shastry from the levels of O^{16} as a function of excitation energy.

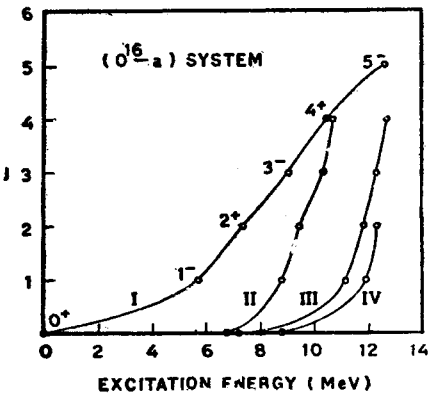


FIG. 11. Real part of the Regge trajectories constructed by Sen and Shastry from the levels of Ne^{20}, as a function of excitation energy.

in the right half plane ($\mathrm{Re}\ \lambda \geqslant 0$) where as in the case of ($O^{16} - \alpha$) system they obtain four trajectories in the right half-λ-plane. This indicates that the potential between O^{16} and α is more attractive than that between C^{12} and α inspite of more repulsive Coulomb potential in the former case. However, it should be

mentioned that the trajectories shown are not necessarily unique for a given system, particularly when the compound nucleus has many closely packed energy levels. Because, even after restricting to the general prescription discussed earlier, one may still be able to obtain more than one set of trajectories. This is indeed a limitation of the present procedure. However, in the absence of knowledge of the realistic target nucleus potential which can generate the actual trajectories, the choice of a particular set of trajectories should be based on the fits it gives to the experimental data.

Following the above procedure Sen and Shastry (1971) have also analysed ($O^{16} - n$) ($O^{16} - p$), ($C^{12} - n$) and ($C^{12} - p$) systems which are in spin-zero spin $\frac{1}{2}$ channels (Dilip Sen & Shastry, 1971). As a result of spin-orbit coupling one expects splitting of Regge trajectories. Grouping the energy levels of O^{17} and F^{17} Regge trajectories are drawn as shown in Figs. 12(a) and 12(b). It is clearly seen

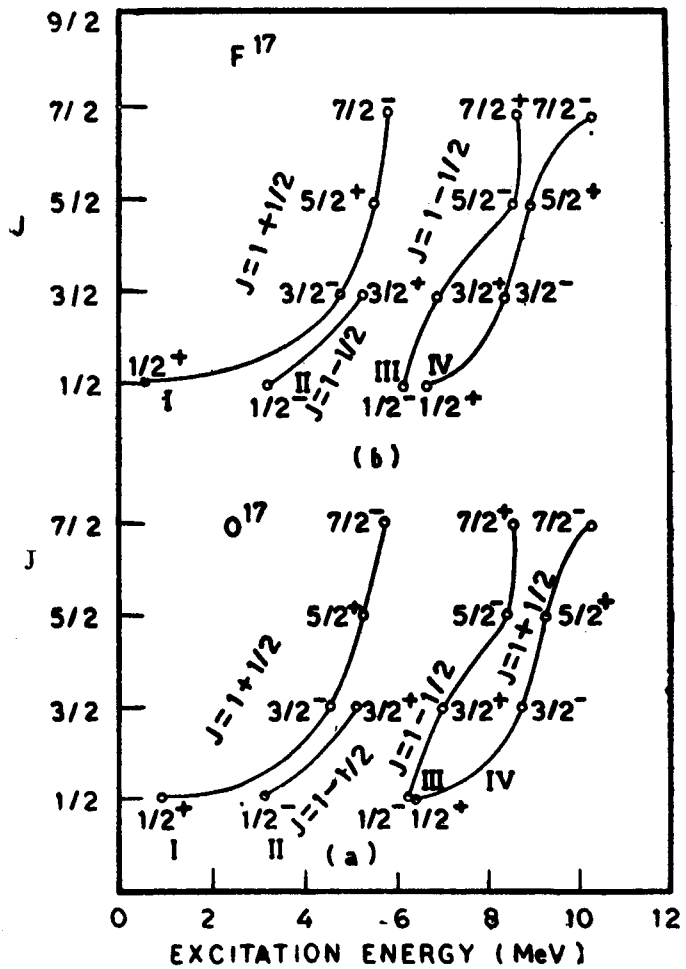


FIG. 12(*a*), Regge trajectories of (O^{16}-*n*) system which are obtained from the levels of O^{17}.
FIG. 12(*b*). Regge trajectories of (O^{16}-*p*) system obtained from the levels of F^{17}.

that in both the cases there are two distinct trajectories for $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$. The second set of trajectories III and IV for $O^{16} + n$ channel starting at $E = 6.24$ MeV for $j = l \pm \frac{1}{2}$ can be interpreted as the trajectories abtained by coupling the neutron with the first excited 0^+ state of 0^{16}. The corresponding iso-baric analogue trajectories for F^{17} can be interpreted correspondingly as the coupling of a proton to the same excited state (0^+) of 0^{16}. An interesting consequence of these studies was that spin and parity of levels at $E = 5.22$ and $E = 8.70$ MeVs and O^{17} have been assigned by comparing the isobaric analogue trajectories for $0^{16} - n$ and O^{16} — p systems.

In the same way ($C^{12} - n$) and ($C^{12} - p$) are also spin-zero-spin-$\frac{1}{2}$ systems. These systems have trajectories corresponding to the $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ cases as shown in Fig. 13. The spin-parity assignment of $\frac{3}{2}^+$ to the C^{13} level at $E = 7.49$ MeV could be determined by comparing the isobaric analogue trajectories.

From the analysis made in Refs. Shanta and Shastry (1968) and Dilip Sen & Shastry (1971) one can summarise the results as follows:—

Considering a nucleus as a two-body system in its predominant channel, a large number of levels of such nuclei are found to fall into well behaved Regge trajectories. The trajectories are approximately linear in energy unlike the trajectories in elementary-particle physics. In the case spin-zero-spin-$\frac{1}{2}$ systems, the spin-orbit coupling gives rise to splitting of trajectories. For mirror nuclei, it is shown that one can construct isobaric analogue trajectories. However, for spin-zero-spin-$\frac{1}{2}$ systems, the ground state could not be incorporated into the Regge trajectory. The reason may be that ground state spin is obtained by treating a nucleus as a many-body system (not a two-body system) with a shell-model scheme for filling the single particle states with nucleons and incorporating the Pauli principle explicitly.

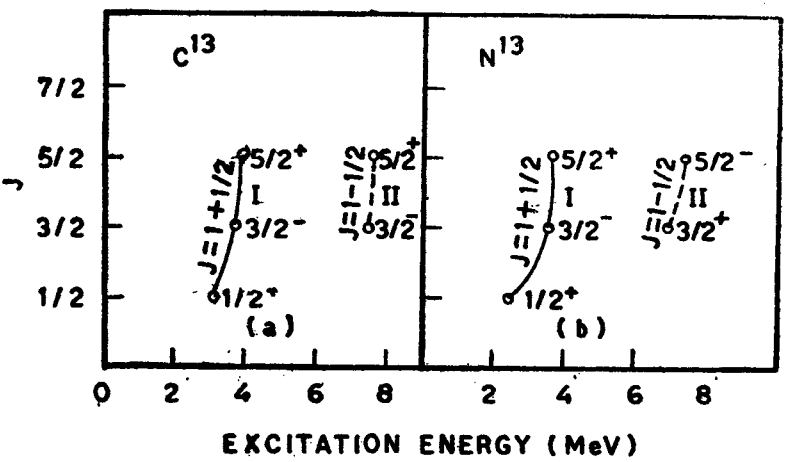


FIG. 13. Regge trajectories of ($C^{13} - n$) system and ($C^{13} - p$) system obtained from the levels of C^{13} and N^{13} respectively.

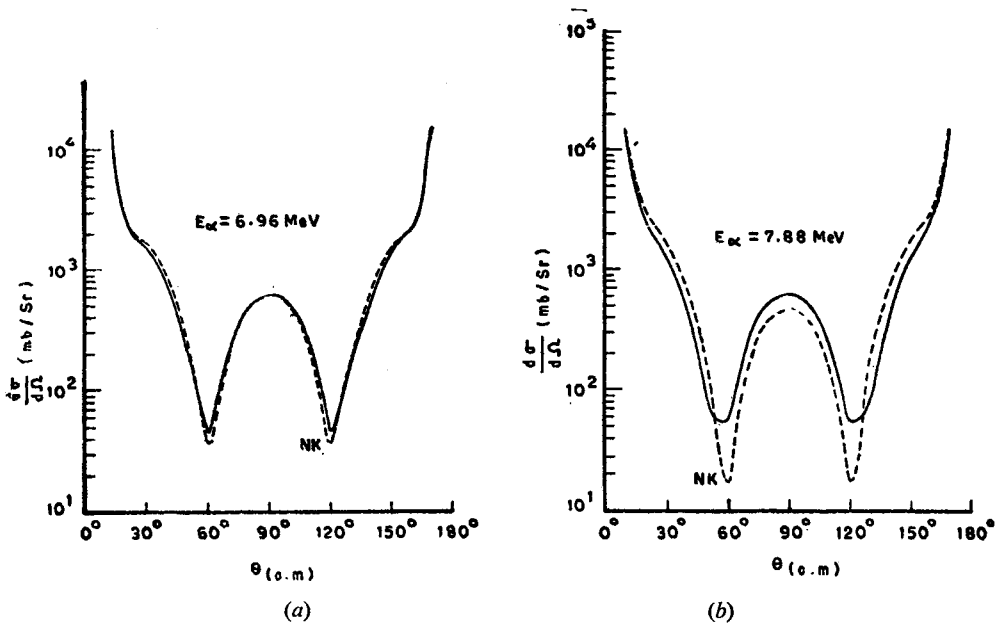


FIG. 14(*a* & *b*). $He^4(\alpha, \alpha)He^4$ angular distributions at incident energies 6.96 MeV and 7.88 MeV respectively. Solid lines correspond to angular distributions from experimental phase shifts and the dashed line indicates the same from the Regge pole approximation. The symbols designating the various representations are described in the text.

In connection with the phenomenological Regge trajectories, obtained from various nuclear scattering systems, one important observation needs to be made (Viano, 1973; and McVoy, 1967). This arises due to the fact that the trajectories we obtained continue to show positive slope as a function of energy whereas for well-behaved potentials, we expect the trajectories to turn back towards the left half plane (Newton, 1964). Corresponding to rising trajectories, when the phase shift crosses $\pi/2$ we get the resonances whereas phase shifts crossing the value $\pi/2$ corresponding the Regge trajectory which is moving backwards towards the left half plane, correspond to echos. Although an echo occurs when phase shift crosses $\pi/2$, it is not a resonance. Whereas the width of the resonance measures (inversely) the time delay of a scattered wave packet, there is no trapping of a wave at the echo energy and its width instead measures the time advance of the wave packet. If one tries to fit the scattering data using only rising trajectories, the echo contribution to the scattering is not properly taken into account. However, in the absence of information regarding the backward moving portion of the trajectory, it is still worthwhile to analyse cross-sections using parameters of rising trajectories.

REGGE POLES IN THE ANALYSIS OF NUCLEAR SCATTERING

As compared to high energy physics, up to now the use of complex angular momentum methods in nuclear and atomic scattering has been rather limited. Viano (1968) anb Abhe (1972) contain references for these applications till 1972. Rebolia

and Viano (1962), Shastry (1969), Grushin and Nikitin (1972), Ceuleneer *et al.* (1966), Bryant and Jarmie (1968), Cowley and Heymann (1970), McVoy (1971), Frahn (1967), Longergan and McVoy (1973), Fuller and Dragun (1974), Fuller and Avishai (1974), Fuller (1973), Anni and Taffara (1974), Abel-Magd (1974), Tamura and Wolter (1972), Rowley and Marty (1976), Nussenzweig (1975), Fuller and Moffa (1977), Fuller and McVoy (1974), Knoll and Schaeffer (1976), Takemasa and Tamura (1978) and Lee and Takigawa (1978) list a typical set of papers on the application of Regge poles in nuclear scattering in addition to the references cited earlier (e.g., Treacy, 1967; Shanta & Satpathy, 1970; Shanta & Shastry 1968; and Dilip Sen & Shastry, 1971). In this Section, we describe few typical set of calculations in nuclear scattering including heavy ion scattering.

Analysis of Nuclear Scattering using Regge Poles

An exhaustive set of calculations have been carried out by Shanta and Shastry (1968). The trajectories deduced from the levels of Be^8, O^{16} and Ne^{20} to study the alpha-alpha, alpha-C^{12} and alpha-O^{16} scattering are described in the earlier section. The main idea behind this work is to use those pole parameters in the Regge type representations of the form

$$S(\lambda, k) - 1 = \sum_n \beta_n \frac{F(\lambda_n, \lambda)}{\lambda - \lambda_n} \qquad \ldots(60)$$

in order to calculate the 'nuclear' partial wave amplitudes and hence cross-sections. The functional form of $F(\lambda', \lambda)$ will be so chosen that the salient physical properties of $a(\lambda, k)$ are taken into account, and background effects are reduced. The following functional forms for $F(\lambda', \lambda)$ are chosen:

$$(i) \qquad F(\lambda', \lambda) = e^{(\lambda'-\lambda)\xi} \frac{\lambda}{\lambda'} \left[\frac{e^{-\lambda'/ka_0} + e^{-R/a_0}}{e^{-\lambda/ka_0} + e^{-R/a_0}}\right] \frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda'-(1/2)}}. \qquad \ldots(61)$$

The representation with the above form for $F(\lambda', \lambda)$ is denoted by NK. The term in the square bracket in eqn. (61) originates from the fact that the effective nuclear potential in nuclear scattering has Woods-Saxon shape. The two parameters R and a are held fixed. R is of the order of $(1.45A^{1/3} + 1.3)$ fm and a_0 is of the order of 0.55 fm. ξ is calculated using $\cosh \xi = 1 + 1/(2a_0^2 k^2)$

$$(ii) \qquad F(\lambda', \lambda) = \left(\frac{2\lambda}{\lambda+\lambda'}\right)^m \frac{\exp(ie^{-\lambda\xi}) - 1}{\exp(ie^{-\lambda'\xi}) - 1} \cdot \frac{\lambda}{\lambda'}$$

$$\times \frac{e^{-\lambda'/ka_0} + e^{-R/a_0}}{e^{-\lambda/ka_0} + e^{-R/a_0}} \cdot \frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda'-(1/2)}}. \qquad \ldots(62)$$

The representation corresponding to this form of $F(\lambda', \lambda)$ is denoted by N_m. The residues of the Regge poles are obtained from the pole parameters by solving the following equations which are derived using unitarity:

$$\sum_{n=1}^{N} \beta_n F(\lambda_n, \lambda_m^*)/(\lambda_m^* - \lambda_n) = -1, \; m = 1, 2, \ldots, N \qquad \ldots(63)$$

Figs. 14, 15, 16 and 17 show the differential cross-section for $\alpha - \alpha$, $\alpha - C^{12}$ and $\alpha - O^{16}$ scattering. The cross-sections obtained from Regge type representations are compared with those of cut-off model or Ackhizer-Pomeranchuk-Blair-McIntyre (APBM) model (Mehta *et al.*, 1967; and Carter *et al.*, 1964) in the case of $\alpha - C^{12}$ and $\alpha - O^{16}$ scattering. The results show fairly good qualitative agreement between the cross-sections obained from Regge pole approach and the experimental observation. However, it is clear that, in general there is no quantitative agreement. It could be remembered in this connection that the Regge pole parameters, R and a are not used as variable parameters in the calculation of cross-section. Secondly,

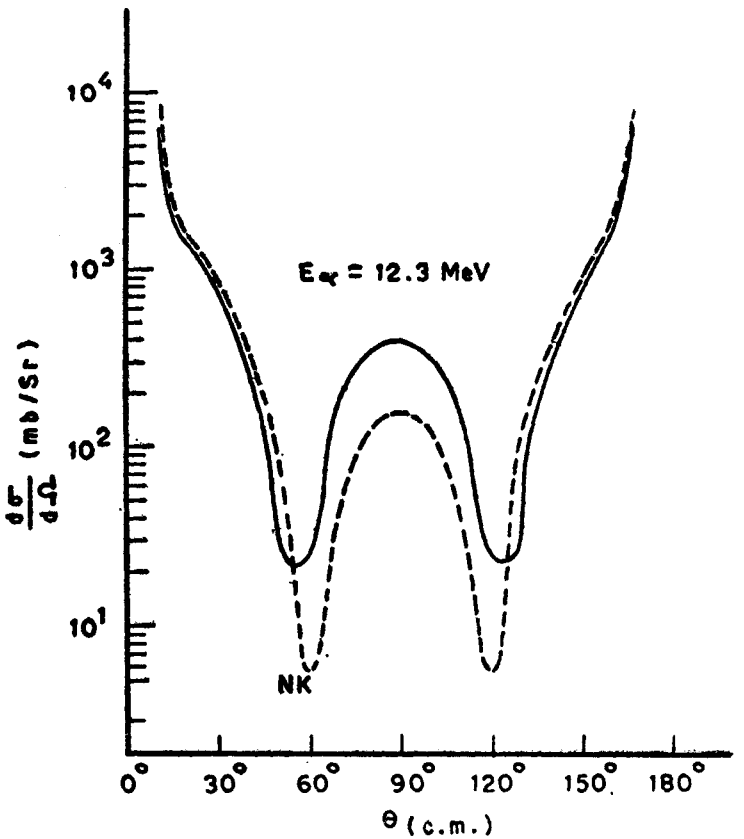


FIG. 15. Same as Fig. 14 at a higher energy, $E_\alpha = 12.3$ MeV.

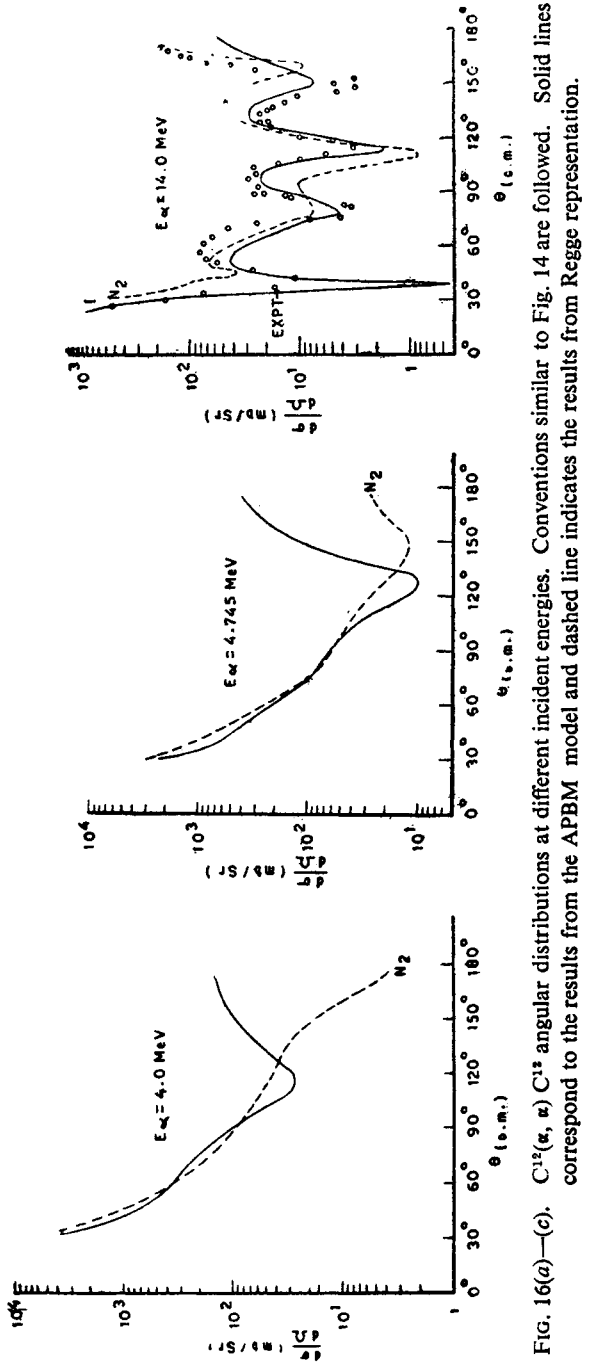


FIG. 16(*a*)—(*c*). $C^{12}(\alpha, \alpha)C^{12}$ angular distributions at different incident energies. Conventions similar to Fig. 14 are followed. Solid lines correspond to the results from the APBM model and dashed line indicates the results from Regge representation.

it should be mentioned that in this scheme of calculations particularly in the calculation of residues, one has tacitly assumed that only the elastic channel is open. When one makes calculations at energies where the inelastic channels are also open, this assumption can lead to errors. For example in the $O^{16} - \alpha$ scattering inelastic channels become significant for $E_\alpha > 10$ MeV. The comparatively worse results for $\alpha - O^{16}$ scattering, particularly at higher energies may be attributed to this source of error. Calculations of the above type also indicate that, even modified Regge type representations, without the inclusion of background effects, may not be able to give quantitative agreement to the differential cross-sections at all angles.

Another type of calculation is where one obtains the Regge pole parameters from experimentally known phase shifts or from the solution of the Schrödinger equation with the appropriate optical potential (Shanta & Satpathy, 1970). It is well known that phenomenological optical potentials used for the analysis of nuclear scattering are not unique. However, they may essentially belong to the same set of physical Regge trajectories. Using the trajectories obtained by phase shifts (Reich *et al.*, 1956) or optical potentials (Shanta & Satpathy, 1970), Satpathy and Shanta have studied differential cross-sections for the elastic scattering of nucleons from C^{12}, O^{16} and Ca^{40}. One interesting aspect of these calculations is the comparison of the energy levels deduced from the calculated Regge trajectories with those

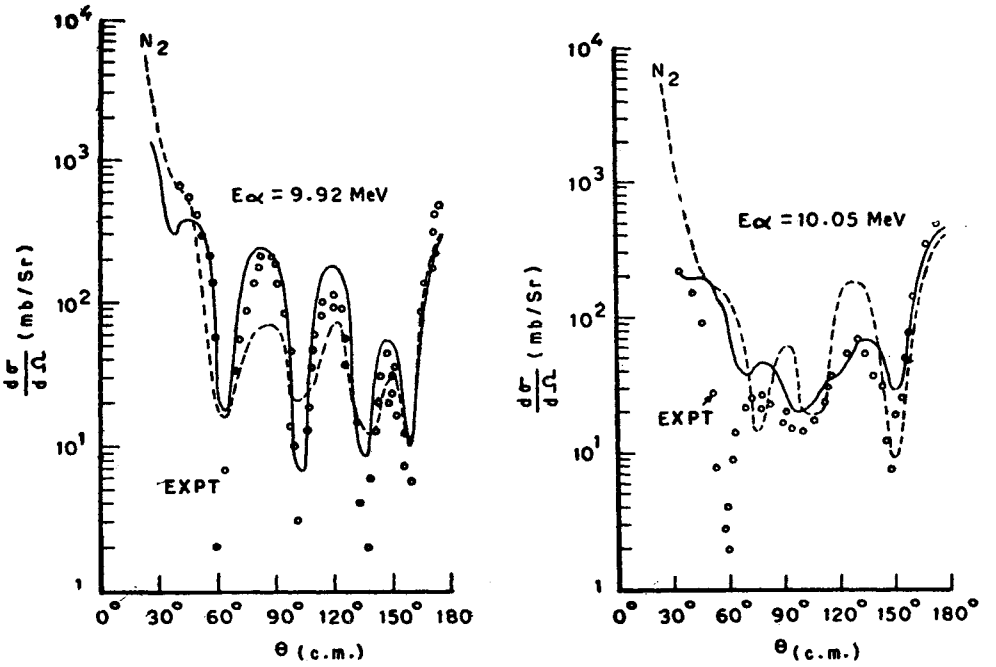


FIG. 17(*a-b*). $O^{16}(\alpha, \alpha)\, O^{16}$ angular distributions. The conventions are similar to those in Figs. 14 to 16. Solid lines indicate the results from a phase shift analysis.

obtained experimentally. In Tables IV and V, we compare the levels predicted by Regge trajectories and those observed experimentally for the nuclei N^{13} and F^{17}. We find that several of the levels agree well with those deduced from the trajectories. Other levels may correspond to those which are to be experimentally confirmed, or it may also be that the levels suggested from phenomenological phase shifts or potentials may not actually exist. However, the fact that extrapolated Regge trajectories can indicate the presence of unknown levels is also shown by Shastry (1969).

In the calculation of cross-sections they (Shanta & Satpathy, 1970) use Regge type representations designated by R_1, R_2, R_3 and R_4. R_1 denotes the representation

$$\frac{(\lambda \pm \tfrac{1}{2})^2 a^{\pm}(\lambda, k)}{2\lambda} = \frac{1}{2ik} \sum_n \beta_n^{\pm} \frac{(\lambda_n^{\pm} \pm \tfrac{1}{2})^2}{[\lambda^2 - (\lambda_n^{\pm})^2]} \frac{\xi^2_{\lambda-1/2}}{\xi^2_{\lambda_n^{\pm}-1/2}} \qquad \ldots(64)$$

TABLE IV

Levels in N^{13} *obtained from the Regge trajectory for* C^{12} $(p, p)^{12}$ *compared with experimental data*

Trajectory	E(lab) (MeV)	l_p	J^{π}	E_x (Calc) (MeV)	E_x (Expt) (MeV)
Re λ_1^+ *vs.* E	5.0	0	$1/2^+$	6.556	6.60
	12.37	1	$3/2^-$	13.36	13.5[a]
Re λ_1^- *vs.* E	12.05	1	$1/2^-$	13.06	...
Re λ_2^+ *vs.* E	9.5	3	$7/2^-$	10.71	10.36
Re λ_2^- *vs.* E	5.7	2	$3/2^+$	7.2	6.908
	11.75	3	$5/2^-$	12.78	12.08[a]

[a]Spin and parity not assigned.

TABLE V

Levels in F^{17} *obtained from the Regge trajectory for* O^{16} (p, p) O^{16} *compared with experimental data*

Trajectory	E (lab) (MeV)	lp	j^{π}	E_x (Calc) (MeV)	E_x (Expt) (MeV)
Re λ_1^+ *vs.* E	6.35	0	$1/2^+$	6.564	6.56
Re λ_2^+ *vs.* E	13.3	3	$7/2^-$	13.11	13.03[a]
Re λ_2^- *vs.* E	7.75	2	$3/2^+$	7.29	7.44
	14.6	3	$5/2^-$	14.337	14.3[a]

[a]Spin and parity not assigned.

R_2 is given by

$$\frac{a^{\pm}(\lambda, k)}{\lambda^2} = \frac{1}{2ik} \sum_n \beta_n \frac{1}{(\lambda_n^{\pm})^2} \frac{e^{(\lambda_n^{\pm}-\lambda)\xi}}{\lambda - \lambda_n^{\pm}} \frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda_n^{\pm}-(1/2)}} \frac{e^{-\lambda_n^{\pm}\xi} + e^{-R/a_0}}{e^{-\lambda\xi} + e^{-R/a_0}} \frac{\lambda}{\lambda_n^{\pm}} \quad \ldots(65)$$

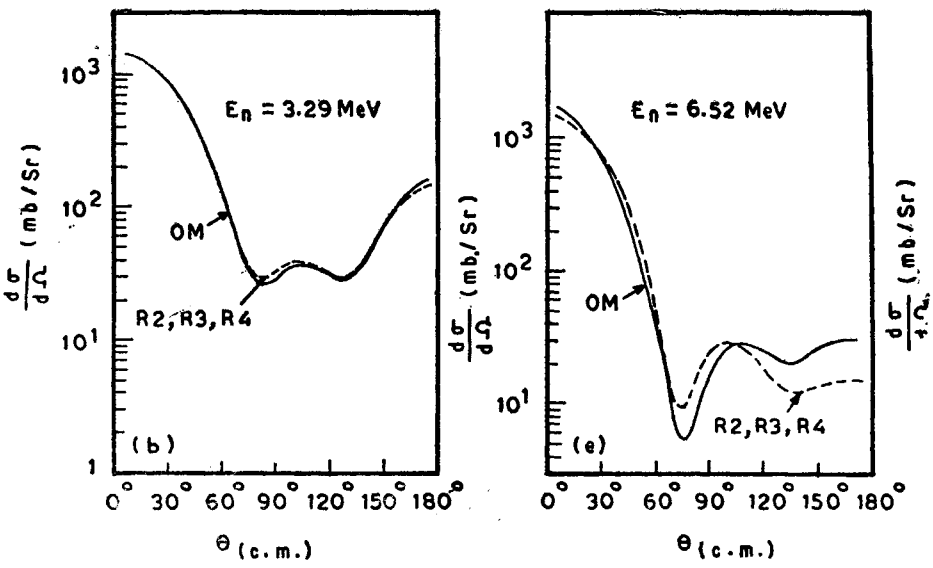
The representation

$$\frac{a^{\pm}(\lambda, k)}{\lambda^2} = \sum_n \xi_n^{\pm} \frac{1}{\lambda - \lambda_n^{\pm}} \frac{1}{(\lambda_n^{\pm})^2} \left(\frac{2\lambda}{\lambda + \lambda_n^{\pm}}\right)^m \frac{\exp(ie^{-\lambda\xi}) - 1}{\exp(ie^{-\lambda n\xi}) - 1}$$

$$\times \frac{\xi^2_{\lambda-(1/2)}}{\xi^2_{\lambda_n^{\pm}-(1/2)}} \frac{e^{-\lambda_n^{\pm}\xi} + e^{-R/a_0}}{e^{-\lambda\xi} + e^{-R/a_0}} \frac{\lambda}{\lambda_n^{\pm}} \quad \ldots(66)$$

with $m = 0$ and $m = 1$ denotes R_3 and R_4 respectively.

A typical set of results obtained for differential cross-sections from the Regge pole parameters calculated by Shanta and Satpathy (1970) are shown in Fig. 18. The common feature of these results and the results obtained by Shanta and Shastry (1968) is that Regge pole representations, even in several modified forms fail to give quantitative agreement with the cross-section data in general. This perhaps indicates that there may be a limit for the modification of Regge representations beyond which it may not be possible to reduce the background effects further.



(*a*) (*b*)

FIG. 18(*a* & *b*). $Ca^{40}(n, n)Ca^{40}$ angular distribution at incident energies 3.29 MeV and 6.52 MeV respectively. OM is the result from theoptical model potential and R_2, R_3, R_4 indicate the results from Regge representations explained in the text.

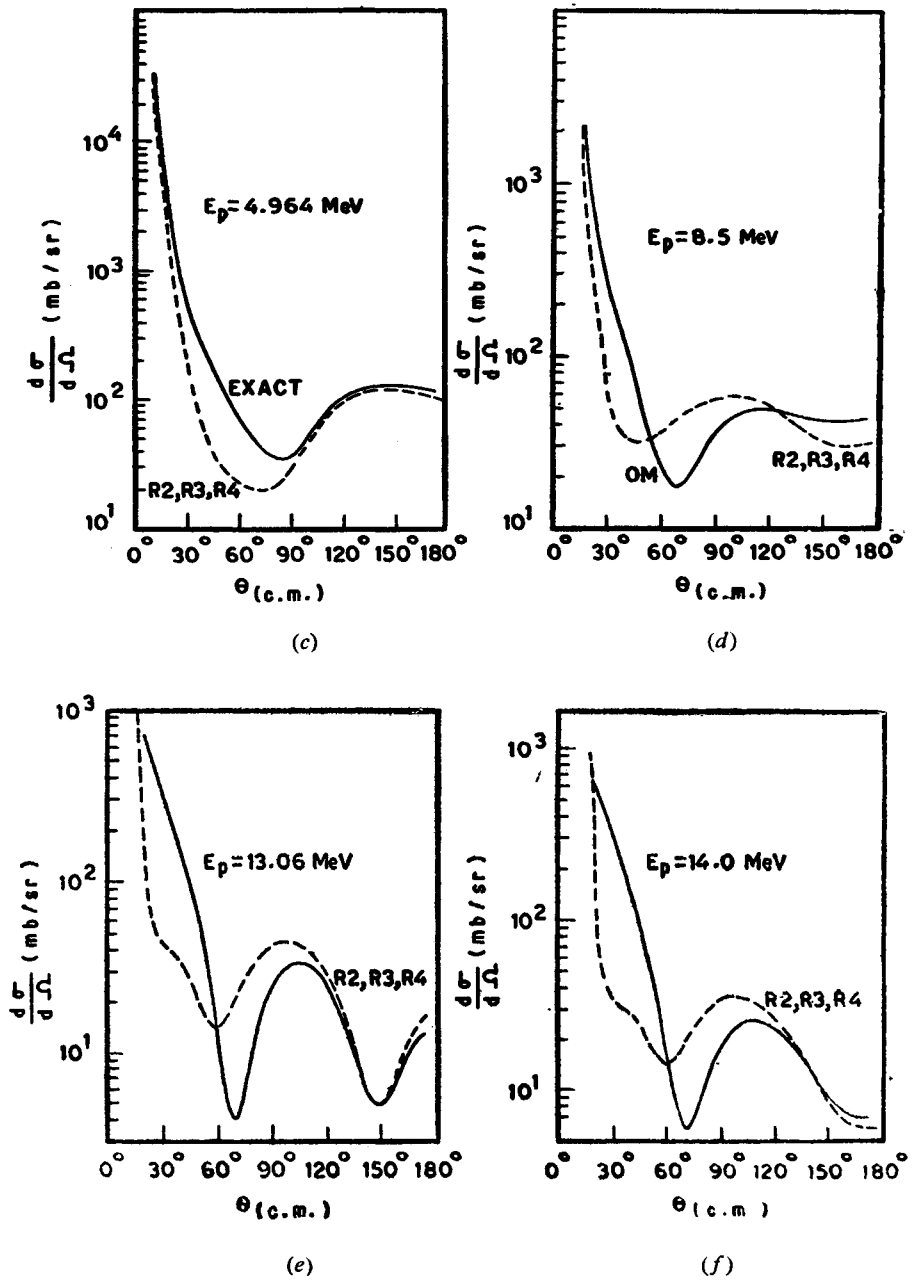


FIG. 18(*c*)–(*f*). $C^{12}(p, p)C^{12}$ angular distributions, from optical model and Regge representations.

In this connection, Regge description of optical model of elastic scattering carried out by Tamura and Wolter (1972) is of interest. They also find that the approximations that retain only pole terms are generally poor and converge slowly with increasing number of pole terms. They showed that by adopting simple analytic

expressions to incorporate background effects in addition to pole terms improve the results significantly, under favourable conditions. The work of Mukherjee and Shastry (1969) also shows that in the case of optical model of elastic scattering, the manipulation of pole terms alone is not sufficient to obtain quantitative fits to the differential cross-sections.

Applications to Heavy Ion Scattering

In heavy ion physics the main emphasis of most work using Regge poles was to see if the salient characteristics of the cross-sections can be reproduced using suitable Regge amplitudes along with background terms. Heavy ion scattering is characterised by short de Broglie wave lengths, quite appreciable absorption of the incident flux inside the strong interaction region, and comparatively strong Coulomb absorption (Frahn & Rehm, 1978). The ratio $\sigma(\theta)/\sigma_{Rutn}(\theta)$, the ratio of differential cross-section to the Rutherford cross-section oscillates around unity in the forward angles and then it rapidly falls as θ increases. If Coulomb potential is not large, as θ increases the average differential cross-section switches over from one slope to another with large oscillations in the region where two slopes meet. Also near $\theta \sim \pi$, $\sigma(\theta)$ has a relative maximum.

In heavy ion scattering also it is useful to separate the Rutherford amplitude $A_c(\cos\theta)$ from the full amplitude $A(\cos\theta)$ thereby isolating the 'nuclear amplitude' $\bar{A}(\cos\theta) \equiv \bar{A}(k, z)$

$$\bar{A}(\cos\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)\, e^{2i\sigma_l} (\bar{S}_l(k) - 1]\, P_l(\cos\theta). \qquad \ldots(67)$$

In order to discuss the role of Regge poles in heavy ion scattering, it is important to note the salient properties of $\bar{S}_l$ as a function of l. Due to strong absorption for low partial waves, we have

$$\bar{S}_l(k) \simeq 0,\ l \text{ small} < l_0$$

$$\bar{S}_l(k) \to 1,\ l \to \infty. \qquad \ldots(68)$$

The shape of $|\bar{S}_l(k)|$ as a function of l has a shape similar to Woods-Saxon potential form (Fig. 19). In order to carry out optical model type calculations one uses different parametric froms of the $\bar{S}_l(k)$ (*See* Rowley & Marty, 1976). One of the familiar parametric representations used is

$$\bar{S}_l(k) = [1 + \exp\{(L - l)/\Delta\}]^{-1},\ L = \Lambda - \tfrac{1}{2}. \qquad \ldots(69)$$

In this representation L is a complex parameter and hence L, Δ correspond to three real parameters. The form of $\bar{S}_l$ given above has poles in the right half plane, however these poles originate from the particular functional form chosen and do not correspond directly to physical resonances. However, it is found to be useful to represent amplitudes in terms of these poles also. Again, in the 'nuclear

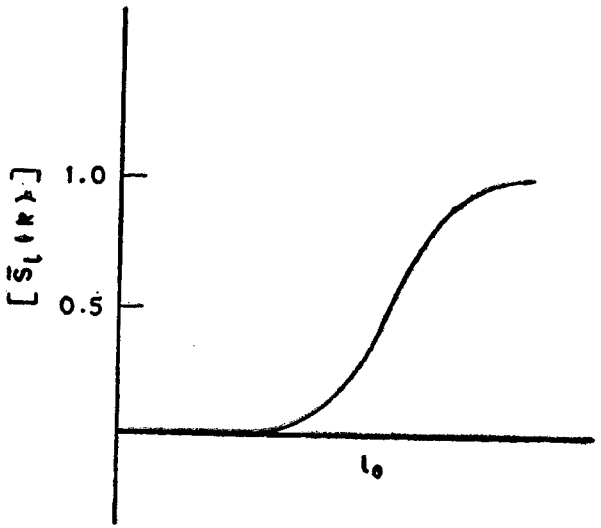


FIG. 19. Qualitative l dependence of $|\bar{S}_l(k)|$ in the heavy ion scattering.

amplitude' $\tilde{A}(\cos\theta)$, the Coulomb S-matrix $e^{2i\sigma l}$ gives rise to poles in the lower half of the complex l-plane at positions l_q such that

$$l_q + 1 + i\eta = -q,\ l_q + \tfrac{1}{2} = \lambda_q$$

and η in the usual Coulomb parameter.

In the analysis of scattering amplitude in heavy ion physics one introduces the concept of running waves or surface waves. This is done by expressing the Legendre polynomial as a sum of functions $Q_l^{\pm}(\cos\theta)$

$$P_l(\cos\theta) = Q_l^{+}(\cos\theta) + Q_l^{-}(\cos\theta). \qquad \ldots(70)$$

The functions $Q_l^{\pm}$ have the following asymptotic behaviour for large l:

$$Q_l^{\pm}(\cos\theta) \to \frac{1}{\sqrt{2\pi\lambda \sin\theta}}\, e^{\pm i[\lambda\theta - \pi/4]}. \qquad \ldots(71)$$

The exact definition of $Q_l^{\pm}$ is:

$$Q_l^{\pm} = \tfrac{1}{2}\left[P_l(\cos\theta) \mp \frac{2i}{\pi} Q_l(\cos\theta) \right] \qquad \ldots(72)$$

where Q_l denotes the Legendre functions of the second type. Using eqn. (70), the amplitude $\tilde{A}(\cos\theta)$ can be expressed as a sum of $A^{+}(\cos\theta)$ and $A^{-}(\cos\theta)$ where

$$A^{\pm}(\cos\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)\, e^{2i\sigma_l} [\bar{S}_l(k) - 1]\, Q_l^{\pm}(\cos\theta)$$

Using the method of Poisson decomposition (Rowley & Marty, 1976), we can write

$$A_m^{\pm}(\theta) = \int_{-a}^{\infty} e^{2i\pi m l \pi} \frac{2l+1}{2ik} [\bar{S}_l(k) - 1] e^{2i\sigma_l} Q_l^{\pm}(\cos\theta)\, dl. \qquad ...(73)$$

Using the asymptotic forms of $Q_l^{\pm}$ (cos θ) it is easy to show that

$$(2\pi \sin\theta)^{1/2} A_m^{\pm}(\theta) = \frac{(-1)^m}{ik} e^{\mp i\pi/4} I_m^{\pm}(\theta)$$

where

$$I_m^{\pm}(\theta) = \int_0^{\infty} \sqrt{\lambda}\, \bar{S}(\lambda, k)\, e^{2i\sigma_l}\, e^{i\lambda(\pm\theta + 2m\pi)} d\lambda. \qquad ...(74)$$

Corresponding to the parametric representation given by (69), we have the poles of $\bar{S}_l$ at λ_p:

$$\lambda_p = l_p + \tfrac{1}{2} = L + \tfrac{1}{2} + i\pi(2p+1)\Delta = \Lambda + i\pi(2p+1)\Delta. \qquad ...(75)$$

Using the method of contour integration, it is easy to show (Marty, 1978) that

$$I_0^{+} = 2\pi i \,\Sigma \text{ residues at } \lambda_p \text{ above the real axis} \qquad ...(76)$$

$$I_m^{\pm} = 2\pi i \,\Sigma \text{ residues at } \lambda_p \text{ above the real axis, } m > 0 \qquad ...(77)$$

$$I_0^{-} = -2\pi i \{\Sigma_n \text{ residues at } \lambda_p \text{ below the real axis} + \Sigma_n \text{ residues at } \lambda_q\}. \qquad ...(78)$$

Similar method can be used for $m < 0$.

This procedure gives a method for expressing the heavy ion scattering amplitude in terms of pole parameters of parametric $\bar{S}_l(k)$. It should be stressed again that λ_p's are not genuine Regge poles: they do not correspond directly to resonances or bound states. In spite of this, the above scheme gives a powerful method for the analysis of heavy ion scattering. The amplitude $A_m^{\pm}$ for $m \geqslant 0$ are dominated by pole at $\Lambda + i\pi\Delta$ whereas the amplitude A_0^{-} has main contribution from the pole at $\Lambda + i\pi\Delta$ and also the poles of Coulomb S-matrix. In the backward angles, the amplitude A_0^{-} is dominated by the pole at $\Lambda - i\pi\Delta$. Similarly, the amplitudes $A_m^{\pm}$ for $m < 0$ are dominated by pole at $\Lambda - i\pi\Delta$.

The decomposition of nuclear amplitude into A^{+} and A^{-} and their representation in terms of poles and residues of parametric S-matrix can be further improved if one considers the S-matrix more realistically. In order to do this, the S-matrix should be derived from the corresponding optical potential. In this case S-matrix will have genuine Regge poles (Shanta & Satpathy, 1970; and Tamura & Wolter,

1972) which generate meaningful Regge trajectories in the usual way. We will briefly outline this procedure here.

Based on earlier discussions, we write, when scattering angle θ is not too large,

$$A^{\pm}(z) \simeq A_0^{+}(\theta),\ z = \cos\theta \qquad \ldots(78a)$$

and in backward angles

$$A^{+}(\cos\theta) \simeq A_0^{+}(\theta) + A_{-1}^{+}(\theta) \qquad \ldots(79)$$

$$A^{-}(\cos\theta) \simeq A_0^{-}(\theta) + A_1^{-}(\theta). \qquad \ldots(80)$$

Evaluating the integrals for $A_m^{\pm}$ using contour integration, one can write them in terms of Regge pole contributions and background contribution. Noting that even for a complex potential Regge poles occur only in the upper half plane (Mukherjee & Shastry, 1969), one obtains

$$A^{+} = (A_0^{+})_{\text{pole}} + (A_0^{+})_{\text{back}} \qquad \ldots(81)$$

and

$$A^{-} = A_0^{-} = (A_0^{-})_{\text{back}}. \qquad \ldots(82)$$

Note that $(A_0^{-})_{\text{back}}$ does not have contributions from Regge poles of $\bar{S}_l(k)$, but may incorporate the pole contributions from the Coulomb S-matrix. The analysis in terms of poles is useful only when the background contribution is small. When pole terms dominate we have

$$(A_0^{+})_{\text{pole}} = \frac{2\pi}{k} \sum_n \lambda_n \beta_n e^{2i\sigma(\lambda_n)} Q_{\lambda_n - (1/2)}(\cos\theta).$$

If there is asingle pole (λ_1) dominance, it can be shown that Longergan & McVoy, 1973; and Fuller & Avishai, 1974)

$$(2\pi \sin\theta)^{1/2} A^{+} = \text{const} \exp(i\lambda_1\theta).$$

Defining local angular momentum $l^{\pm}$ as

$$l^{\pm} = \frac{d}{d\theta}(\arg A^{\pm}) \qquad \ldots(83)$$

we get $l^{\pm} = \text{Re}\,\lambda_1$

when we have a single Regge pole dominance.

A good example of single Regge pole dominance is provided in thc $^{16}O - {}^{16}O$ scattering analysis using the optical potential adopted by (Gobbi *et al.*, 1973). It is of Woods-Saxon form and has the following parameters:

$$V = 17\ \text{MeV},\ r_R = 1.35\ \text{fm},\ a_R = 0.49\ \text{fm},\ \text{W} = (0.8 + 0.2\ \text{E})\text{MeV}$$

$$r_I = 1.27\ \text{fm},\ a_I = 0.15\ \text{fm},\ r_c = 1.35\ \text{fm}.$$

In Fig. (20) the plot of $l^{\pm}$ as a function of θ illustrates clearly the region of single pole dominance. The pole dominance is clearly energy dependent and at higher energies the pole dominance is quite impressive. The trajectory corresponding to this dominating Regge pole is shown in Fig. (21).

Similar studies have been carried out for $O^{16} + {}^{28}Si$ system (Marty, 1978). At energy $E_L = 55$ MeV, the potential which fits the data has only one Regge pole in the right half plane. Plot of $\ln[\sqrt{\sin\theta}\, A^{\pm}]$ *vs.* θ provides a good test for pole dominance (Fig. 22). One can see that in the forward direction the plot is practically a straight line for $\ln[\sqrt{\sin\theta}\, A^{\pm}]$ and the pole parameters expected from this are $\mathrm{Re}\,\lambda_1 \simeq 24$–$25$, $I_m\lambda_1 \simeq 4 - 5$ which compares very well with exact values $\lambda_1 = 24.29 + i\,4.2$.

It should be however stated that the ion-ion scattering cross-sections are, in general dominated by more than one Regge poles. For example different potentials used by Takemasa and Tamura (1978) show the importance of first three poles and the background term. For $\theta > 90°$ there is strong cancellation between the two oscillating components. From these studies one finds that in general heavy ion scattering cross-section may not be dominated by a single pole or even a few poles, and background effect should be considered for quantitative analysis. It is found that A^{+} is pole dominated and A_0^{-} is predominantly background dependent. In a

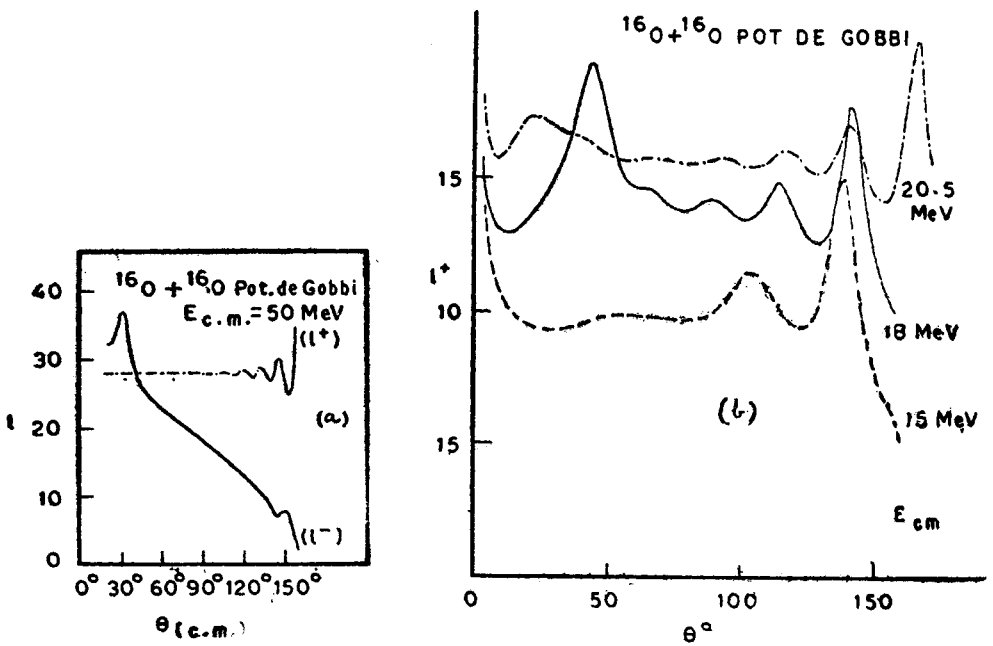


FIG. 20(*a* & *b*). Dependence of l^{+} and l^{-} on scattering angle θ, in the $O^{16} - O^{16}$ system using the potential due to Gobbi *et al.* (1973).

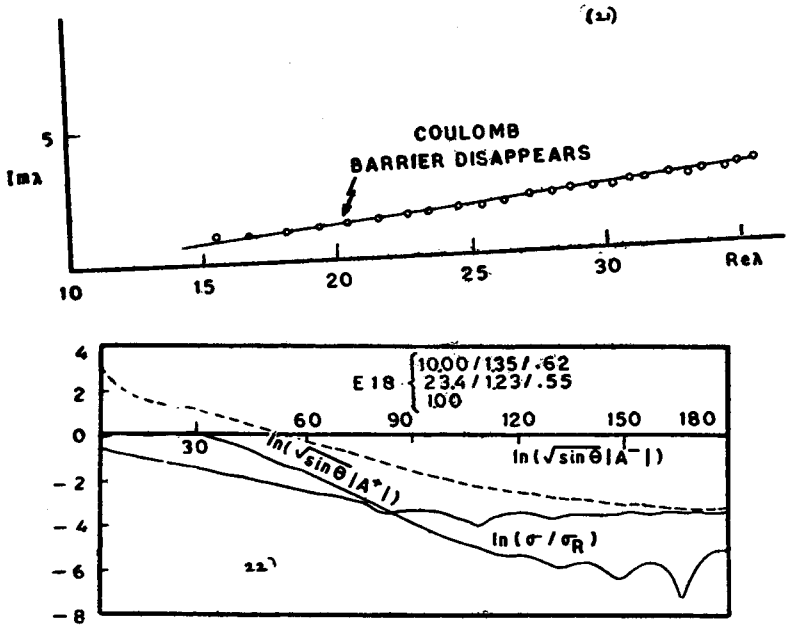


FIG. 21. Regge trajectory corresponding to the dominant pole, for $O^{16} - O^{16}$ system using Gobbi potential.

FIG. 22. Behaviour of ln $[\sqrt{\sin\theta}\, A\pm]$ as a function of θ. The potential used in this is designated by E 18 and the parameters of the potential are $V = 10.00$ MeV, $r_R = 1.35$ fm, $a_R = 0.62$ fm, $w = 23.4$ MeV, $r_e = 1.23$ fm, $a_I = 0.55$ fm, $r_c = 1.00$ fm.

Regge pole analysis only the poles near the grazing values need be considered explicitly and rest of the poles can be lumped into the background term which acts mainly at background angles via terms like A_0^-.

The role of Regge pole in heavy ion scattering can be appreciated in another manner. (Knoll & Schaffer, 1976; Koeling & Malfliet, 1975, and Landowne *et al.*, 1976). The strong absorption nature of heavy ion processes introduces effects such as diffraction due to which the wave can propagate in classically forbidden regions. A semiclassical analysis of this problem will lead to the notion of complex trajectories with complex angular momentum and complex turning points. The use of complex angular momentum and Regge pole is a way of replacing single given real angular momentum l by a packet of angular momenta centred around l, with an imaginary part singnifying the width of the packet.

Consider Regge amplitude in the single pole (α) approximation in the l plane.

$$A(\cos\theta) \sim \beta P_\alpha(-\cos\theta)/\sin\pi\alpha \qquad \ldots(84)$$

For large α

$$A(\cos\theta) \sim \beta \sqrt{\frac{2i}{\pi\alpha \sin\theta}}\; e^{i[\alpha+(1/2)]\theta} \qquad \ldots(85)$$

The factor $(\sin\theta)^{1/2}$ indicates that the amplitude decreases as angles deviate from $\pi/2$. The representation is expected to be more meaningful at backward angles. If such a representation can fit cross-sections in the backward angles, it can be understood as due to the trapping of the incident particle which rotates for some time along the surface of the target nucleus with a damping factor exp $(-I_m\alpha\theta)$. In this way, one understands the role of Regge pole in the analysis of backward angles (Viano, 1968). In optics, we have an analogous phenomenon. In glory scattering of light, there is a strong enhancement in near backward scattering of light by small water droplets. This is due to surface waves which travel around the droplet, sending tangential rays which are in phase in backward and forward directions (Van de Hulst, 1957). The semiclassical analysis of heavy ion collision (Koeling & Malfliet, 1975) also can be carried out using the phenomenon of glory scattering.

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