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Classical Physics

ASYMPTOTICALLY FLAT STATIONARY SOLUTIONS FROM LAPLACE'S SOLUTIONS

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Asymptotically flat stationary solutions of Einstein and Einstein-Maxwell equations including the Kerr solution are generated from the solutions of Laplace's equation.

Keywords : Einstein-Maxwell equation; Laplace equations; Kerr Solution

INTRODUCTION

THE axially symmetric exact solutions with physical relevance are the powerful tools for better understanding the implications of general relativity besides its growing necessity in tackling astrophysical problems. A lot of work has therefore, been done in this field which falls into the category of stationary axially symmetric solutions. Nevertheless, only a few of them possesses required relevance. On the other hand the situation for eletrovac fields is not also much better.

Several transformation techniques have been applied by Eduard Herlt (1978) to Van Stockum solution and lastly as a consequence the Kerr solution is obtained. Even then the central problem remains unsolved and no further progress therefore has yet been done perhaps due to the non-availability of preassigned solution of a linear eqn. (2.2) which actually plays an important role of generating function. However Herlt's technique beautifully applies to electrovac cases where the problems are some what easy. As regards the generation technique of stationary Einstein-Maxwell solutions given by Israel-Wilson (1972) and Perjes (1971) the situation is not also much better for the reason that no suitable solution of Laplace's equation is yet known except a few dealt with earlier which ultimately entails a physically realistic axially symmetric stationary solution of Einstein-Maxwell equations.

In this paper we show that physically realistic axially symmetric vacuum stationary solutions are obtainable from the solutions of Laplace's equation by the same combinations of transformations given by Eduard Herlt. A new stationary axially symmetric vacuum solution of Einstein equations is presented here starting from a solution of Laplace's equation and as a consequence it is shown that the Kerr solution can be obtained similarly from the Laplace's equation. The same solution of Laplace's equation is used as an input to the Israel-Wilson and Perjes technique and another new axially symmetric stationary solution of Einstein-Maxwell equations is derived. Thirdly a new set of static axially symmetric electrovac solution is also presented.

In section (2), Herlt's procedure is briefly reviewed and it is shown that how Laplace's equation plays the vital role instead of eqn. (2.2). Section (3) gives the new stationary solution of Einstein equation and also the electrovac solution derived from the Laplace's solution used in the former case. We, in section (4), present the new stationary solution of Einstein-Maxwell equations with a brief review of the technique of Israel-Wilson and Perjes. Conclusion follows.

HERLT'S PROCEDURE

Van Stockum metric can be written as

$$ds^2 = \rho^{-1/2}(d\rho^2 + dz^2) + 2\rho\, d\varphi\, dt - \chi_\rho dt^2 \qquad ...(2.1)$$

and all the metric coefficients are derivable from the solution of a single equation (Herlt, 1978).

$$\chi_{\rho\rho} + \chi_{zz} - \frac{1}{\rho}\chi_\rho = 0 \qquad ...(2.2)$$

Ernst's (1968) axially symmetric stationary metric on the other hand is given by

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt + \omega d\varphi)^2 \qquad ...(2.3)$$

Ernst (1968) simplified the Einstein field equations and showed that the metric (2.3) can be obtained from the solution of a single equation.

$$(\xi\xi^* - 1)\,\Delta^2\xi = 2\xi^*\Delta\xi.\Delta\xi \qquad ...(2.4)$$

where $\epsilon = (\xi - 1)/(\xi + 1) = f + i\phi$, ϕ is known as twist potential. Herlt has shown that if one obtains the solution of eqn. (2.2) he can get the solution of eqn. (2.4) and consequently f and ϕ can be determined accordingly. After performing a series of transformations for obtaining realistic solutions Herlt finally has given f and ϕ in terms of the solution of eqn. (2.2), which read as

$$f = \beta_1\beta_2\left\{\frac{1}{\chi} + \frac{1}{\rho(\chi_\rho)^{-1}[\chi_\rho^2 + \chi_z^2] - \chi}\right\} \qquad ...(2.5)$$

$$\phi = i\beta_1\beta_2\left\{\frac{1}{\chi} - \frac{1}{\rho(\chi_\rho)^{-1}[\chi_\rho^2 + \chi_z^2] - \chi}\right\} \qquad ...(2.6)$$

Suffix after χ indicates partial differentiation. χ is a complex solution of eqn. (2.2). Unfortunately, all the complex solutions of eqn. (2.2), it is seen, do not give f and ϕ real simultaneously which is essential for obtaining realistic axially symmetric stationary vacuum solution of Einstein equation. Herlt was able to obtain only one set of solution which ultimately leaves f and ϕ real. This solution is as

$$\chi = A\{2 + e^{i\lambda}r_1 + e^{-i\lambda}r_2\} \qquad ...(2.7)$$

where $r_1 + r_2 = 2x,\ z = xy$ ...(2.8)

x and y are the prolate spheroidal coordinates and λ and A are two arbitrary constants. Herlt actually obtained two real solutions of eqn. (2.2) and made complex with the use of $e^{\pm i\lambda}$.

Equation (2.2) can be transformed to Laplace's equation and therefore solution to Laplace's epuation can be used as generating function to obtain axially symmetric stationary vacuum fields instead of eqn. (2.2).

(i) If we put

$$\chi_\rho = \rho\psi \qquad \ldots(2.9)$$

Eqn. (2.2) reduces to

$$\psi_{\rho\rho} + \psi_{zz} + \frac{\psi_\rho}{\rho} = 0 \qquad \ldots(2.10)$$

This is Laplace's equation in two dimension.

(ii) If we consider $\chi_\rho = \rho\zeta_z$ and $\chi_z = -\rho\zeta_\rho$ then also eqn. (2.2) is satisfied and we get back the Laplace's equation as integrability condition for χ.

(iii) Again, if ψ is Laplace's solution, ψ_z is also another solution and therefore ψ_z also constructs χ of eqn. (2.2).

Here we only explore the possibilities of constructing realistic solution of axially symmetric vacuum stationary Einstein fields with the use of eqns. (2.9) and (2.10) only. In prolate spheroidal coordinates (x, y) the solutions eqn. (2.10) come out as

$$\psi = \frac{1}{x + y} \text{ and } \psi = \frac{1}{x - y} \qquad \ldots(2.11)$$

where $$\rho = (x^2 - 1)^{1/2}(1 - y^2)^{1/2}, z = xy \qquad \ldots(2.12)$$

From eqns. (2.9) and (2.11) χ can be evaluated as

$$\chi = A[2 + e^{i\lambda}(x + y) + e^{-i\lambda}(x - y)] \qquad \ldots(2.13)$$

Here A and $e^{\pm i\lambda}$ are two arbitrary constants, the latter is used only to make χ complex. It is seen that χ obtained from the above procedure actually satisfies eqn. (2.2).

We obtain f and ϕ from eqns. (2.5) and (2.6) by using eqn. (2.13). The solution gives the Kerr solution in Ernst's notation as

$$f = -\frac{i\beta_1\beta_2}{2A} \cdot \frac{x^2\cos^2\lambda + y^2\sin^2\lambda - 1}{(x\cos\lambda + 1)^2 + y^2\sin^2\lambda} \qquad \ldots(2.14)$$

$$\varphi = -\frac{i\beta_1\beta_2}{2A} \cdot \frac{-2iy\,\mathrm{Sin}\,\lambda}{(x\cos\lambda + 1)^2 + y^2\,\mathrm{Sin}^2\lambda} \qquad \ldots(2.15)$$

Thus Laplace's solutions (2.11) have given rise to the Kerr metric. All solutions of Laplace's equation, however, do not give f and ϕ real simultaneously and therefore

the central problem remains where it was. We, however, have found out another set of Laplace's solution which incidentally give f and ϕ real. The new stationary vacuum solution generated from this set is given in sec. (3). It is interesting to note that this set in combination with other known sets may generate Tomimatsu and Sato metric for $\delta = 2$. But this awaits further investigation.

Stationary Gravitational and Static Electrovac Fields

In section (2) it is shown that how Laplace's solution can be used to generate stationary gravitational fields. The central problem lies in finding out suitable solutions of Laplace's equation which ultimately in combination leaves f and ϕ real. All arbitrary real Laplace's solutions will give rise to realistic static electrovac fields, no doubt, which fall into the functionally independent group, an interesting class in exact solutions. Thus any solution of Laplace's equation either generates realistic stationary vacuum fields or static electrovac fields and we therefore, require only a large number of Laplace's solutions at hand.

The second set of Laplace's solution in prolate spheroidal coordinates which we have obtained for generating stationary gravitational solution is

$$\psi = (xy + 1)/(x + y)^3 \qquad \ldots(3.1)$$

This solution in combination with the previous one gives from eqn. (2.9).

$$\chi = A(x + y) + iB(xy + 1)/(x + y) + C \qquad \ldots(3.2)$$

By using (2.5) and (2.6) we obtain f and ϕ in Ernst-Notation.

$$f = \beta_1\beta_2 \frac{B^2(x^2 - 1)(1 - y^2) - A^2(x + y)^4}{A^2C(x + y)^4 + 2AC^2(x + y)^3 + C^3(x + y)^2 + C^3(xy + 1)^2} \qquad \ldots(3.3)$$

$$\phi = 2\beta_1\beta_2 C \frac{(xy + 1)(x + y)}{(x + y)^2(Ax + Ay + C)^2 + C^2(xy + 1)^2} \qquad \ldots(3.4)$$

It is to be noted that in the above neither f nor ϕ is complex although χ in (3.2) is complex. Ernst (1968*a*) has already shown that the above two potential f and ϕ are sufficient to determine the metric (2.3) uniquely. β_1 and β_2 are two necessary arbitrary constants which are connected with A, B and C intimately in making the metric coefficients of (2.3) flat at spatial infinity with proper signature. The constants B and C in eqn. (3.2) and D in eqn. (2.6) are suitably adjusted to make f and ϕ real, otherwise no realistic stationary gravitational field is obtainable

$$B = C = K \text{ and } D = -1/K \qquad \ldots(3.5)$$

The above asymptotically flat stationary gravitational solution is new, f and ϕ when expanded asymptotically take the following forms

$$f = -\beta_1\beta_2\left[1 + \frac{2}{Ax} - \frac{8A^2y(2y + A) + (6y + 1)}{A^2x^2} + \ldots\right] \qquad \ldots(3.6)$$

$$\phi = 2\beta_1\beta_2\left[\frac{y}{x^2} + \frac{1}{A^2x^2}\{(1+y^2)A^2 - y(4y+2A)\} + \ldots\ldots\right] \quad \ldots(3.7)$$

$$K = 1.$$

These expansions show that $\beta_1\beta_2 = -1$ so that f approaches unity at spatial infiity ($x \to \alpha$). ϕ contains dipole as well as other multipole moments which vanish at spatial infinity and f contains mass term as well as other higher mass-moments.

In the above we obtained asymptotically flat stationary gravitational fields with the presence of mass term from just a combination of Laplace's solutions which served the purpose of a generating function. Again we know that Laplace's equation is nothing but the simplified form of the Einstein field equations for static vacuum metric. Therefore, the discussed technique maps static gravitational fields into stationary fields with no functional relationship between f and ϕ. The only limitation of the technique is that it does not always give realistic stationary gravitational fields whereas it always operate successfully for obtaining static electrovac fields with the interesting property of functional independency. In the following we discuss briefly the technique of obtaining electrovac fields with an illustration by using the same solutions of the Laplace's equation (3.1).

Static axially symmetric metric is written as

$$ds^2 = e^{-2u}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - e^{2u}dt^2 \quad \ldots(3.8)$$

Bonnor (1966) and Kramer-Neugebauer (1968) discussed a theorem which maps stationary vacuum fields into static Einstein-Maxwell one. It connects the twist potential ϕ and f eqn. (2.3) by

$$f \to e^u, \ \phi \to i\,(k0/2)^{1/2}\,\psi \quad \ldots(3.9)$$

with electric potential ψ and metric coefficient e^{2u} of the static Einstein-Maxwell fields eqn. (3.8).

Equation (3.9) shows that if ϕ be originally real, then electric potential ψ becomes imaginary. Even then elimination of imaginary term is achieved by changing certain constant parameter parameters/present in f and ϕ (Das & Banerji, 1978).

In the following case we choose χ (eqn. 3.2) as real function and therefore get with the use of eqns. (2.5) and (2.6) f as real but ϕ as imaginary. f and ϕ can be transformed into e^u and ψ for static electrovac metric (3.8). ψ, the electric potential becomes real although ϕ, the twist potential was originally imaginary. Thus any real solution of eqn. (2.2) or alternatively eqn. (2.10) gives rise to electrovac metric uniquely. Here also no functional relationship exists between e^{2u} and ψ. Considering

$$\chi = A(x+y) + B(xy+1)/(x+y) + C \quad \ldots(3.10)$$

as generating function we get e^{2u} and ψ as

$$e^{u} = -\beta_1\beta_2 \frac{A^2(x+y)^4 + B^2(x^2-1)(1-y^2)}{A^2C(x+y)^4 + A(C^2-B^2)(x+y)^3 - B^2C[(x^2-1)\times(1-y^2) + 2(xy+1)^2]} \quad ...(3.11)$$

$$\psi = 2\beta_1\beta_2(2/k_0)^{1/2} \frac{BC(x+y)(xy+1)}{A^2C(x+y)^4 + A(C^2-B^2)(x+y)^3 - B^2C[(x^2-1)\times(1-y^2) + 2(xy+1)^2]} \quad ...(3.12)$$

The above electrovac solution of metric (3.8) is asymptotically flat and mass as well as electric monopole term with higher multipoles are present whereas in the stationary case no monopole term is present in the expression for ϕ. There, ϕ varies as x^{-2} at spatial infinity which is essential for vanishing of ω at infinity (2.3) (Das & Banerji, 1978). Moreover in the solution given by eqns. (3.11) and (3.12) no restriction on constants B and C is necessary, which on the other hand, is necessary for stationary axially symmetric case (3.5) to make f and ϕ real. As it is easy to tackle the problem of static electrovac case, several new electrovac solutions have been given by Herlt. As stationary solution, we get only one such solution from his work which is already known as Kerr-solution. We, in this paper, have presented a new stationary gravitational fields (2.3) and (2.4) and observe that its counterpart can similarly be constructed with the following solution of Laplace's equation instead of (3.1)

$$\psi = A + B/(x-y) + C(xy-1)/(x-y)^3 \quad (3.13)$$

There exists hopeful possibility of getting a series of new stationary or electrovac fields from a combination of the above series of Laplace's solution and other known solutions. A third set of Laplace's solution in oblate spheroidal coordinates exists as

$$\psi = (3x^2y^2 - x^2 - 4ixy + y^2 - 3)/(x-iy)^5 \quad ...(3.14)$$

It is hoped that all these solutions of two dimensional Laplace's equation in suitable combination may lead to very interesting solutions of stationary gravitational fields yet unexplored.

Stationary Einstein-Maxwell Fields

Israel and Wilson (1972), and Perjes (1971) independently published a method of generating solution of stationary Einstein-Maxwell equations from Laplace's solution. They have shown that the generated solution can be interpretated as exterior fields of static or steadily moving distributions of charged dust having numerically equal charge and mass densities.

The stationary axially symmetric line element in Ernst's form

$$ds^2 = f^{-1}[e^{2\gamma}\{(d\rho^2 + dz^2) + \rho^2 d\varphi^2\}] - f(dt - \omega d\varphi)^2 \quad ...(4.1)$$

is known if we know Ernst's potentials $\epsilon = f - \Phi\Phi^* + i\phi$ in detail. f, Φ and ϕ are considered for this investigation as functions of ρ and z only. Φ and ϕ are known as complex electromagnetic scalar potential and twist potential respectively.

Israel-Wilson and Perjes showed that ϵ can be obtained directly from Laplace's solution when mass and charge densities become numerically equal.

$$\Delta^2(1 + \epsilon)^{-1} = 0 \qquad \ldots(4.2)$$

where Δ^2 is the Laplacian operator in ρ, z coordinates. Eqn. (4.2) being linear we can introduce arbitrary constants in suitable positions as

$$\epsilon = (1 - L^2 - M^2 - 2iM)/\{(1 + L)^2 + M^2\} \qquad \ldots(4.3)$$

$$\Phi = e^{i\alpha}(L + iM)/(1 + L + iM) \qquad \ldots(4.4)$$

$$\phi = -2M/\{1 + L)^2 + M^2\} \qquad \ldots(4.5)$$

$$f = \{(1 + L)^2 + M^2\}^{-1} \qquad \ldots(4.6)$$

where α is a constant. L and M are two independent real solutions of Laplace's equation as

$$\psi = 1 + L + iM \qquad \ldots(4.7)$$

ψ is obtained by the following equations in prolate spheroidal coordinates, defined as

$$\rho = (x^2 - 1)^{1/2}(1 - y^2)^{1/2},\ Z = xy \qquad \ldots(4.8)$$

$$\omega_x = -2(1 - y^2)[ML_y - (L + 1)M_y] \qquad \ldots(4.9)$$

$$\omega_y = 2(x^2 - 1)[ML_x - (L + 1)M_x] \qquad \ldots(4.10)$$

where suffices x and y indicate partial differentiations w.r.t x and y only.

The central problem, therefore, reduces to find out two suitable Laplace's solutions only. The different metric coefficients of (4.1) can be obtained, then from eqns. (4.4)–(4.6) and (4.9)–(4.10).

We now take

$$\psi = 1 + A/(x + y) + iB(xy + 1)/(x + y)^3, \qquad \ldots(4.11)$$

where x and y are prolate spheroidal coordinates and A, B are arbitrary constants. Then the metric coefficients become

$$f = (x + y)^6/\{(x + y)^4[x + y + A]^2 + B^2(xy + 1)^2\} \qquad \ldots(4.12)$$

$$\omega = -2(1 - y^2)\left[-\frac{AB(1 - y^2)}{2(x + y)^4} - \frac{3ABy + 3B(1 - y^2)}{3(x + y)^3} - \frac{4By - AB}{2(x + y)^2} + \frac{B}{x + y}\right] + \text{Const.} \qquad \ldots(4.13)$$

Asymptotic expansion of f takes the following form

$$f = 1 - \frac{2A}{x} + \frac{2Ay}{x^2} - \frac{B^2}{(x+y)^2} - \frac{2xy}{(x+y)^2} - \frac{x^2y^2}{(x+y)^2} \quad \ldots(4.14)$$

This shows that $f \to 1$ when $x \to \alpha$ i.e. at spatial infinity f goes to asymptotically flat form. At infinity, ω and Φ also vanish and therefore the derived stationary Einstein-Maxwell field is well behaved at infinity.

Perjes (1971) has pointed out that if one chooses for large values of the radial coordinates r,

$$\psi = 1 + M/r + iJ \text{ Cos } \theta/r^2 + \eta \quad \ldots(4.15)$$

when η stands for real terms of order r^{-2} as for imaginary ones of order r^{-3} and θ is the polar angle, then the derived stationary Einstein-Maxwell field is well behaved in the sense that the derived metric is asymptotically flat at large distance and ω or Φ vanishes accordingly. Eqn. (4.11) uniquely fits the above requirements. The limitation of Israel-Wilson-Perjes method is that we cannot switch off the electromagnetic field and get therefore its gravitational analogue. In our derived solution this limilation exists.

Rotating stars take the form of oblate spheroids and therefore, solutions in oblate spheroidal coordinates are of much interest. In this coordinate system two other Laplace's solutions exist as

$$\psi = A/(x - iy) \quad \ldots(4.16)$$

$$\psi = i(xy - i)/(x - iy)^3 \quad \ldots(4.17)$$

Stationary Einstein-Maxwell solution derived from Laplace's solution (4.16) (with suitable constants added) by Israel-Wilson and Perjes method is also given as

$$f = \frac{x^2 + y^2}{(x+1)^2 + y^2}, \quad \Phi = \frac{e^{i\alpha}}{(x+1) - iy},$$

$$\phi = -\frac{2y}{(x+1)^2 + y^2}, \quad \omega = \frac{(1 - y^2)(2x + 1)}{x^2 + y^2}. \quad \ldots(4.18)$$

This solution do not have its gravitational analogue although it is well behaved in the sense that $f \to 1$, $\Phi \to 0$, $\omega \to 0$ at spatial infinity. Different solutions in oblate spheroidal coordinates may also be derived from the Laplace's solutions given in (3.14), (4.16) and (4.17) sparately or in proper combinations.

Conclusion

In fact no physically realistic axially symmetric and asymptotically flat stationary gravitational solution other than the Kerr and Tomimatsu-Sato (1973) series containing mass term was avilable in the literature till few years back. Recently Kninnersley and Chitre (1978), Hoensaelaers *et al.* (1979) Yamazaki *et al.* (1980) have

added a few to the list but their solutions are more complicated than the former and therefore it is difficult to handle those properly. The stationary gravitational field given by eqns. (3.3) and (3.4) is some what simple and no doubt an addition to the list of stationary gravitational solutions. Moreover, the Laplace's solutions (2.11) and (3.1) applied as input to the transformation eqns. (2.5) and (2.6) is the marked second example in the Herlt's procedure, which leaves f and ϕ as real functions. Primarily it is expected that the suitable combinations of the stated solutions of Laplace's equations may be selected and mingled with $e^{\pm i\lambda}$ to give rise to Tomimatsu and Sato series. Further investigation along these lines is therefore promising. With little effort, solutions like (3.3) and (3.4) may be constructed from the following combination of Laplaces's solutions

$$\psi = A/(x - y) + iB\,(xy - 1)/(x - y)^3. \qquad \ldots(3.13a)$$

This will be the third example of Laplace's solution which gives f and ϕ real. With similar procedure discussed in the previous sections the above solution (3.13*a*) used as input to Israel-Wilson-Perjes method will entail another interesting solution of stationary Einstein-Maxwell equations like (4.12) and (4.13). It is needless to say that each of the Laplace's solutions stated separately or in combination shall definitely give electrovac fields. Thus transformation eqns. (2.5) and (2.6) are very promising and opened up an avenue towards generation of new staitonary gravitational fields. Very little work has yet been done and a lot remains unexplored.

Electrovac solution given by eqns. (3.11) and (3.12) has been derived directly from real Laplace's solution by transformation eqns. (2.5) and (2.6). Another class of electrovac fields may be obtained from solutions (3.3) and (3.4) by changing certain constants (Das & Banerji, 1978). Thus this procedure may lead to new solutions other than eqns. (3.11) and (3.12).

In oblate spheroidal coordinates three sets of complex Laplace's solutions (3.14) (4.16) and (4.17) shall also generate stationary Einstein-Maxwell fields. We have just shown one of them (4.18). In Prolate coordinates we have obtained stationary Einstein-Maxwell fields from two sets of Laplace's solutions. The remaining may be utilized too.

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