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Elasticity

THE EFFECT OF WAVE INPUTS IN AN INHOMOGENEOUS ELASTIC ROD WITH VARIABLE DENSITY

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The effect of wave inputs in an inhomogeneous elastic solid whose elastic parameters depend on one space coordinate only, is considered. The stress and displacement components are assumed to depend on this same space coordinate and time alone. Taking Young's modulus $E = E_0 \sin \alpha x$ and the density $\rho = \rho_0 \sin \alpha x$, we obtain the general solution for the problem of wave input in an inhomogeneous elastic rod. Particular cases in which a modified saw tooth, square and half sine wave rectifier out put as input waves, fed through one end of the rod, are studied and the displacement $u(x, t)$ is obtained. Graphs are drawn to study the behaviour of the input waves at time $t = 2$ Sec., $\alpha = 0.01$ and $\frac{\sqrt{E_0}}{\sqrt{\rho_0}} = 3.5$.

Keywords : Inhomogeneous Elastic Solid; Variable Density; Young's Modulus

INTRODUCTION

THE problem of wave propagation in an inhomogeneous rod has been studied by various authors by taking the elastic parameters and density as function of coordinate x, measured along the rod. Datta (1956) has studied the problem when the rod is of constant density and the elastic coefficients are linear functions of the coordinate measured along the rod. Sur (1961) has taken the elastic coefficient as functions of e^{ax}. Lindholm and Doshi (1965) have discussed the case of a rod when the elastic coefficients are proportional to x^n. All these authors have taken the density to be constant.

In this paper we study the effect of wave inputs in an inhomogeneous elastic rod, whose density $\rho = \rho_0 \sin \alpha x$ and the Young's Modulus $E = E_0$ Sin αx, where x is the space coordinate measured along the length of the rod and obtain the general expression for displacement $u(x, t)$. The stress and displacement components are assumed to depend on the space coordinate x and time t. The general solution is obtained in terms of legendre's Polynomials.

We further discuss the particular cases of the general solution when the wave inputs are modified saw tooth, square and half sine wave rectifier output. The displacement pattern in the above three cases is studied by showing graphs for specific values of the material constants at a fixed time. In all these cases the effect of inhomogenity is noticed in the form of distorsion from the original wave form.

Wave Inputs in a Rod

The stress-strain relation for an elastic rod is

$$\sigma_x = E\frac{\partial u}{\partial x} \qquad ...(1)$$

where x is the space coordinate measured along the rod, σ_x the stress, u the displacement and E is Young's Modulus. In the absence of body forces the equation of motion has the form

$$\frac{\partial \sigma_x}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \qquad ...(2)$$

where t is the time and ρ_0 is the density of rod. Since the rod is of inhomogeneous material depending on x, we assume

$$E = E_0 \sin \alpha x, \ \rho = \rho_0 \sin \alpha x \qquad ...(3)$$

where E_0, ρ_0 and α are real constants.

Simplifying eqn. (2) with the help of eqns. (1) and (3), we get

$$E_0 \sin \alpha x\frac{\partial^2 u}{\partial x^2} + E_0\alpha\,(\cos \alpha x)\frac{\partial u}{\partial x} = \rho_0 \sin \alpha x\ \frac{\partial^2 u}{\partial t^2} \qquad ...(4)$$

Solving this equation, we get

$$u(x, t) = [C_1 \cos kt + C_2 \sin kt]\,[C_3P_n\,(\text{Cos}\ \alpha x) + C_4Q_n(\cos\alpha x)] \quad ...(5)$$

where P_n $(\cos \alpha x)$ and Q_n $(\cos \alpha x)$ are Legender's functions of first and second kind, C_1, C_2, C_3 and C_4 are arbitrary constants. k and n are related by

$$\frac{k^2\rho_0}{\alpha^2 E_0} = n(n+1) \qquad ...(6)$$

n being real number.

Boundary Conditions

Let the rod be of finite length $2c$. We can assume that the elements of the rod are excited by various wave inputs in such a manner that the displacement is always bounded and

$$u_t(x, 0) = 0 \qquad ...(7)$$

This implies that $C_2 = C_4 = 0$

The eqn. (5) gives

$$u(x, t) = \sum_{n=0}^{\infty} B_nP_n\,(\cos \alpha x)\cos kt \qquad ...(8)$$

$$-1 \leqslant \cos \alpha x \leqslant 1.$$

where k is given by eqn. (6)

Making use of the transformation

$$x = \frac{3\pi}{2\alpha} + \frac{\pi}{2\alpha c} z \qquad \ldots(9)$$

eqn. (8) reduced to

$$u(z, t) = \sum_{n=0}^{\infty} B_n P_n \left(\sin \frac{\pi}{2c} z\right) \cos \alpha t \frac{\sqrt{n(n+1)}\, E_0}{\sqrt{\rho_0}} \qquad \ldots(10)$$

where $$B_n = \frac{2n+1}{2} \int_{-1}^{1} u(z, 0)\, P_n\left(\sin \frac{\pi}{2c} z\right) d\left(\sin \frac{\pi}{2c} z\right) \qquad \ldots(11)$$

Particular Cases

Case : I Modified saw tooth wave input

We prescribe modified saw tooth wave input described as follow

$$u(z, 0) = \begin{cases} 0 \text{ for } -C < z < 0 \\ \dfrac{L}{c} z \text{ for } 0 < z < C \end{cases} \qquad \ldots(12)$$

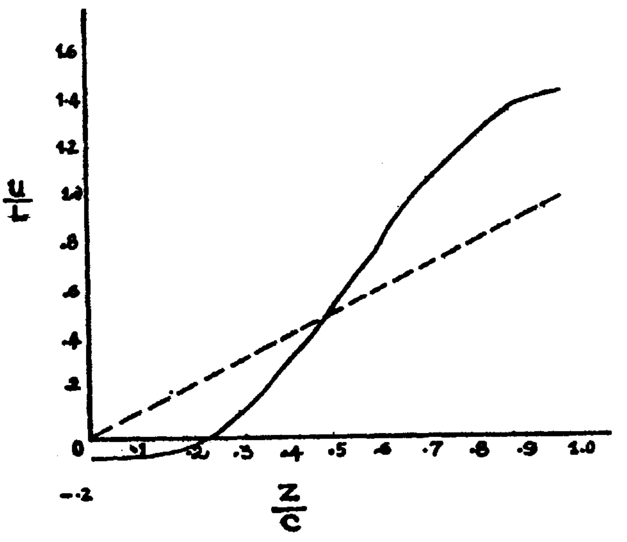

FIG. 1. Displacement patern for saw tooth wave input

With the help of eqns. (9), (10), (11) and (12) the displacement $u(x, t)$ reduces to

$$U(x, t) = \frac{L}{\pi}\left(\frac{\pi}{2} - 1\right) P_0(\cos \alpha x)$$

$$+\frac{9}{8\pi} LP_1(\cos \alpha x) \cos \alpha \frac{\sqrt{2E_0}}{\sqrt{\rho_0}} t$$

$$+\frac{5}{3\pi} LP_2(\cos \alpha x) \cos \alpha \frac{\sqrt{6E_0}}{\sqrt{\rho_0}} t$$

$$+\frac{49}{128} LP_3(\cos \alpha x) \cos \alpha \frac{\sqrt{12E_0}}{\sqrt{\rho_0}} t$$

$$+\frac{9}{120} LP_4(\cos \alpha x) \cos \alpha \frac{\sqrt{20E_0}}{\sqrt{\rho_0}} t$$

$$+ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad \ldots(13)$$

Case : *II* Square Wave input

In this case

$$u(z, o) = \begin{cases} O & \text{for} \quad -C < z < O \\ L & \text{for} \quad O < z < C \end{cases} \qquad \ldots(14)$$

With the help of equs. (9), (10), (11) and (14) the displacement $u(x, t)$ becomes

$$u(x, t) = \tfrac{1}{2} LP_0(\cos \alpha x)$$

$$+ \tfrac{3}{4} LP_1(\cos \alpha x) \cos \alpha \frac{\sqrt{2E_0}}{\sqrt{\rho_0}} t$$

$$-\frac{7}{16} LP_3(\cos \alpha x) \cos \alpha \frac{\sqrt{12E_0}}{\sqrt{\rho_0}} t$$

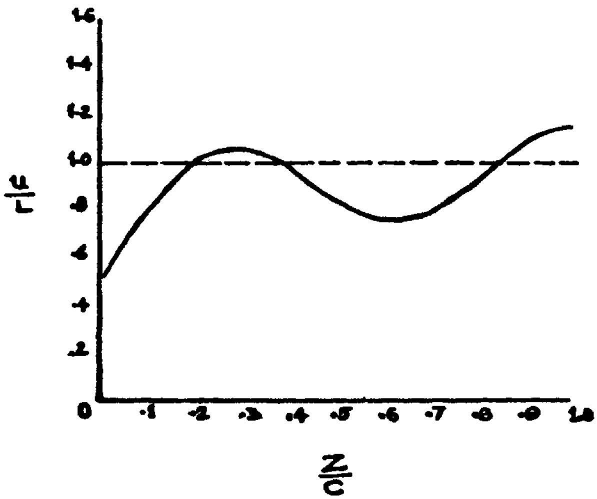

FIG. 2. Displacement patern for square wave input.

$$+\frac{11}{32} LP_5(\cos \alpha x) \cos \alpha \frac{\sqrt{30E_0}}{\sqrt{\rho_0}} t$$

$$+ \quad \cdots \quad \cdots \quad \cdots \qquad \ldots(15)$$

Case : *III* Half sine wave rectifier out put as input

Here

$$u(z, o) = \begin{cases} O \text{ for } -C < z < O \\ L \sin \frac{\pi}{c} z \text{ for } O < z < C. \end{cases} \qquad \ldots(16)$$

With the help of eqns. (9), (10), (11) and (16) the displacement $u(x, t)$ works out to be

$$u(x, t) = \frac{L}{3} P_0(\cos \alpha x)$$

$$+\frac{3\pi}{16} LP_1(\cos \alpha x) \cos \alpha \frac{\sqrt{2E_0}}{\sqrt{\rho_0}} t$$

$$+\frac{L}{6} P_2(\cos \alpha x) \cos \alpha \frac{\sqrt{6E_0}}{\sqrt{\rho_0}} t$$

$$-\frac{7\pi}{64} LP_3(\cos \alpha x) \cos \alpha \frac{\sqrt{12E_0}}{\sqrt{\rho_0}} t$$

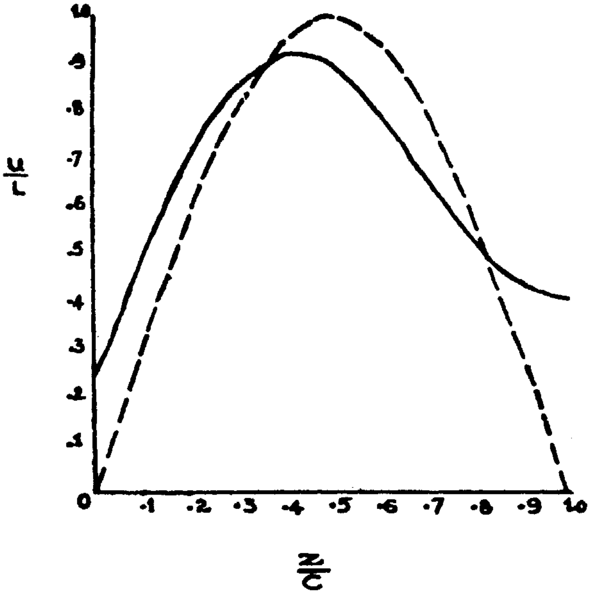

FIG. 3. Displacement patern for half sine wave rectifier output.

$$+ \tfrac{3}{8} LP_4(\cos \alpha x) \cos \alpha \frac{\sqrt{20E_0}}{\sqrt{\rho_0}} t$$
$$+ \quad \ldots \quad \ldots \quad \ldots \qquad \ldots(17)$$

To illustrate the displacement patterns in the above three cases, the graphs for the displacements at a fixed time $t = 2$ Sec. and by taking $\alpha = .01$ and $\frac{\sqrt{E_0}}{\sqrt{\rho_0}} = 3.5$ are shown (graphs 1, 2 and 3). These graphs show the input and output wave. We observe that due to inhomogenity of the material there is a distorsion in the input waves in all the three cases.

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