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Hydroelasticity

PULSATILE FLOW OF A VISCOUS LIQUID LOADED WITH PARTICLES IN A THIN-WALLED ELASTIC TUBE

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The pulsatile flow of a viscous liquid loaded with particles in a thin-walled elastic tube is studied. The interaction of the solid and liquid is considered by taking into account the inertial forces and the drag forces produced by the liquid on the solid particles and solid-solid interactions are neglected. The phase velocity ratio and reduction of amplitude are determined as a function of the frequency parameter for various specified values of fluid and tube parameters.

Keywords : Elastic Tube; Pulsatile Flow; Viscous Liquid

INTRODUCTION

THE studies in mechanics of blood flows in arteries and veins fall into the domain of Hydroelasticity which is concerned with phenomena involving interactions between hydrodynamic and elastic forces. In order to give a satisfactory mathematical model for blood flows, one has to include the solid-solid interaction in addition to the interaction of the solid and liquid if the particle number density becomes large, and this may involve the technique of statistical mechanics. Attempts have been made to simplify the mathematical model under various assumptions. In a paper considered the oscillatory motion of a viscous liquid in a thin-walled elastic tube (Womersley, 1955). The driving force for the motion of liquid and tube was a pressure gradient periodic in time. Details of the motion and the pulse velocity were calculated for the case of a freely moving tube under the long wavelength approximation. Nayfeh (1966) considered the oscillating two-phase flow through a rigid pipe. Dynamics of the fluid are assumed to be governed by the Navier-Stockes' equations. The forces considered to be acting on the solid particles are the internal forces and the drag forces produced by the liquid, thus the solid-solid interactions are neglected. In the present paper, the authors have studied the pulsatile two-phase flow through a thin-walled elastic tube. The solid particles are assumed spherical and represented by large a number density N, volume concentration ϕ with appreciable mass concentration. It is assumed that the individual particles are so small that the stockes flow approximation to their motion relative to the fiuid is appropriate.

EQUATIONS OF MOTION

Consider the pulsatile two-phase flow of an incompressible viscous fluid through a thin-walled elastic tube. Let us assume that the solid particles are spherical having all the same diameter and distributed uniformly, with a number density N large enough to use average properties of the particles at a point (while neglecting solid-solid interactions). Taking z-axis along the axis of the tube, the equations of motion (Nayfeh, 1966) in cylindrical polar coordinates are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(w) = 0 \quad \ldots(1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_s) + \frac{\partial}{\partial z}(w_s) = 0 \quad \ldots(2)$$

$$\rho_0(1-\phi)\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right] = (1-\phi)\left[-\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right)\right] + Nm\frac{u_s - u}{\tau} \quad \ldots(3)$$

$$Nm\frac{\partial u_s}{\partial t} = -Nm\frac{u_s - u}{\tau} \quad \ldots(4)$$

$$\rho_0(1-\phi)\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right] = (1-\phi)\left[-\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right)\right] + Nm\frac{w_s - w}{\tau} \quad \ldots(5)$$

$$Nm\frac{\partial w_s}{\partial t} = -Nm\frac{w_s - w}{\tau} \quad \ldots(6)$$

where p is the pressure, μ the fluid viscosity, ϕ the volume occupied by the solid per unit volume of the mixture, ρ_0 the fluid density, τ relaxation time of the dust particles ($\tau = \frac{m}{3\pi\mu d}$, where m and d are the mass and diameter of a particle respectively); τ is a measure of the time for the solid particles to adjust to changes in the fluid velocity. For spherical particles of radius a and density ρ_1, we have

$$\tau = \frac{\frac{4}{3}\pi a^3\rho_1}{6\pi a\mu} = \frac{2}{9}\frac{a^2\rho_1}{\nu\rho_0}$$

where ν is the kinematic viscosity of the fluid. The velocity components of the fluid and the equivalent continuum of suspended solid particles in the direction of r, θ and z are designated by (u, o, w) and (u_s, o, w_s), finally we assume independence with respect to θ.

We consider a thin walled elastic tube of radius R, thickness h, which is described by the material properties : ρ = tube density, E = Young's modulus, σ = Poisson's ratio. If ξ and ζ denote the radial and axial displacement of an element of the tube which are functions of z and time t, the equations of motion (Womersely, 1955) in cylindrical polar coordinates for the thin-walled elastic tube are

$$\rho h\frac{\partial^2\zeta}{\partial t^2} = \frac{Eh}{(1-\sigma^2)}\frac{\partial}{\partial z}\left(\frac{\partial\zeta}{\partial z} + \sigma\frac{\xi}{R}\right) - \frac{\rho_0\nu}{R}\left(\frac{\partial w}{\partial y} + R\frac{\partial u}{\partial z}\right)_{y=1}, \quad \ldots(7)$$

$$\rho h \frac{\partial^2 \xi}{\partial t^2} = (p)_{y=1} - \frac{Eh}{R(1-\sigma^2)} \left(\frac{\xi}{R} + \sigma \frac{\partial \zeta}{\partial z} \right) \qquad \ldots(8)$$

where p is the excess fluid pressure on the inner surface of the tube and $y = \frac{r}{R}$.

The boundary conditions at the inner surface of the tube are

$$\left. \begin{aligned} u(y, z, t) &= \frac{\partial \xi}{\partial t} \\ w(y, z, t) &= \frac{\partial \zeta}{\partial t} \end{aligned} \right\} \quad \text{at } y = 1. \qquad \ldots(9)$$

Solution of the Problem

We assume that $\frac{u}{c}, \frac{w}{c}, \frac{nR}{c}$ are all small, then from mass conservation of fluid and solid particles we see that $\frac{u}{w}$ and $\frac{u_s}{w_s}$ are both of $O\left(\frac{nR}{c}\right)$. This allow us to linearize the equations and to look for plane work solutions

$$f(r, z, t) = f_1(r) \exp \left[in \left(t - \frac{z}{c} \right) \right] \qquad \ldots(10)$$

Where f stands for any of $u, w, p, u_s, w_s, \xi, \zeta$ and in place of ξ_1, ζ_1 we get B_1 and B_2 which are constants according to the fact that ξ and ζ do not depend on r. Through straight forward algebra we find

$$\frac{1}{y} \frac{d}{dy} (u_1 y) = \frac{inR}{c} w_1 \qquad \ldots(11)$$

$$\frac{1}{y} \frac{d}{dy} (u_{s1} y) = \frac{inR}{c} w_{s1} \qquad \ldots(12)$$

$$\frac{d^2 u}{dy^2} + \frac{1}{y} \frac{du}{dy} + \left(i^3 \alpha_1^2 - \frac{1}{y^2} \right) u_1 = \frac{R \partial p_1}{\mu \partial y} \qquad \ldots(13)$$

$$\frac{d^2 w_1}{dy^2} + \frac{1}{y} \frac{dw_1}{dy} + i^3 \alpha_1^2 w_1 = - \frac{in}{c} \frac{R^2}{\mu} p_1 \qquad \ldots(14)$$

where $\alpha_1^2 = \frac{nR^2}{\nu} \left(1 + \frac{Nm}{\rho_0 (1 - \phi)(in\tau + 1)} \right) = \frac{nR^2}{\nu} \beta,$

$$\left(\beta = 1 + \frac{Nm}{\rho_0 (1 - \phi)(in\tau + 1)} \right)$$

If we assume $p_1 = A_1 J_0(ky)$, where k is to be determined, the eqns. (13) and (14) can be integrated to give

$$w_1 = C_1 \frac{J_0(\alpha_1 i^{3/2} y)}{J_0(\alpha_1 i^{3/2})} - \frac{inR^2}{c\mu} \frac{A_1}{i^3 \alpha_1^2 - k^2} J_0(ky) \qquad \ldots(15)$$

$$u_1 = C_2 \frac{J_1(\alpha_1 i^{3/2} y)}{J_0(\alpha_1 i^{3/2})} - \frac{Rk}{\mu} \frac{A_1}{i^3 \alpha_1^2 - k^2} J_1(ky) \qquad \ldots(16)$$

where C_1 and C_2 are constants and the constant $J_0(\alpha_1 i^{3/2})$ has been introduced for convenience as Nayfeh (1966). Substitution of (13) and (16) in (11) should reduce to an identity and this gives

$$k = \frac{in\,R}{c}, \frac{C_2}{C_1} = \frac{in\,R}{c\alpha_1 i^{3/2}}. \qquad \ldots(17)$$

Eqn. (16) can be written as

$$u_1 = \frac{in\,R}{c\alpha_1 i^{3/2}} C_1 \frac{J_1(\alpha_1 i^{3/2} y)}{J_0(\alpha_1 i^{3/2} y)} - \frac{Rk}{\mu} \frac{A_1}{i^3\alpha_1^2 - k^2} J_1(ky) \qquad \ldots(18)$$

From (4) and (6) we have

$$w_{s1} = \frac{C_1}{(in\,\tau + 1)} \frac{J_0(\alpha_1 i^{3/2} y)}{J_0(\alpha_1 i^{3/2})} = \frac{in\,R^2}{c\mu(in\,\tau + 1)} \frac{A_1}{(i^3\alpha_1^2 - k^2)} \times J_0(ky) \qquad \ldots(19)$$

$$u_{s1} = \frac{in\,RC_1}{c\alpha_1 i^{3/2}(in\,\tau + 1)} \frac{J_1(\alpha_1 i^{3/2} y)}{J_0(\alpha_1 i^{3/2})} - \frac{Rk}{\mu(in\,\tau + 1)} \frac{A_1}{(i^3\alpha_1^2 - k^2)} J_1(ky). \qquad \ldots(20)$$

Womersely (1955) noted that at the highest frequencies of interest in pulse-wave $(n^2R^2/c^2) \ll \alpha_1^2$ i.e. $k^2 \ll \alpha_1^2$. Therefore $i^3\alpha_1^2 - k^2 \simeq i^3\alpha_1^2$ and due to the smallness of nR/c, $I_0(nR/c) \simeq 1$ and $I_1(nRy/c) \simeq nRy/2c$. Using these approximations eqns. (15), (18), (19) and (20) are reduced to

$$w_1 = C_1 \frac{J_0(\alpha_1 i^3 y)}{J_0(\alpha_1 i^{3/2})} + \frac{A_1}{\beta\rho_0 c} \qquad \ldots(21)$$

$$u_1 = \frac{in\,R}{2c} C_1 \frac{2J_1(\alpha_1 i^{3/2} y)}{\alpha_1 i^{3/2} J_0(\alpha_1 i^{3/2})} + Y \frac{A_1}{\beta\rho_0 c} \qquad \ldots(22)$$

$$w_{s1} = \frac{1}{in\,\tau + 1}\left(\frac{C_1 J_0(i^{3/2} y)}{J_0(\alpha_1 i^{3/2})} + \frac{A_1}{\beta\rho_0 C}\right) \qquad \ldots(23)$$

$$u_{s1} = \frac{in\,R}{2c(in\tau + 1)}\left(C_1 \frac{2J_1(i^{3/2} y)}{\alpha_1 i^{3/2} J_0(\alpha_1 i^{3/2})} + Y \frac{A_1}{\beta\rho_0 c}\right) \qquad \ldots(24)$$

Using (10), (21) and (22) in (7), (8) and (9) after making the some approximations we have

$$\frac{\mu}{2\rho Rh} \alpha_1^2 i^3 F_{10}(\alpha_1) C_1 - \frac{inB\sigma}{\rho Rc} B_1 + n^2\left(1 - \frac{B}{\rho c^2}\right) B_2 = 0 \qquad \ldots(25)$$

$$\frac{A_1}{\rho h} - \frac{c^2}{R^2}\left(\frac{B}{\rho c^2}\right) B_1 + \frac{inB\sigma}{\rho Rc} B_2 = 0 \qquad \ldots(26)$$

$$\frac{inR}{2\beta\alpha_0 c^2} A_1 + \frac{inR}{2c} F_{10}(\alpha_1)\, c_1 - inB_1 = 0 \qquad ...(27)$$

and

$$\frac{A_1}{\beta\rho_0 c} + C_1 - inB_2 = 0 \qquad ...(28)$$

where $B = \dfrac{E}{1-\sigma^2}$ and

$$F_{10}(\alpha_1) = 2J_1(\alpha_1 i^{3/2})/\alpha_1 i^{3/2}\, J_0(\alpha_1 i^{3/2}).$$

Eqns. (25) to (28) are four homogeneous equations in the arbitrary constants A_1, C_1, B_1 and B_2. Eliminating them will give a dispersion relation, which will determine the wave velocity c in terms of the elastic properties of the tube and the frequency parameter. After elimination of A_1, C_1, B_1, and B_2 from (25) to (28) we have

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} F_{10}(\alpha_1) & -1 & 0 \\ \beta\dfrac{\rho_0 R}{\rho h} & 0 & -\dfrac{B}{\rho c^2} & -\dfrac{B\sigma}{\rho c^2} \\ 0 & -\dfrac{\beta R\rho_0}{2\rho h} F_{10}(\alpha_1) & -\dfrac{B\sigma}{\rho c^2} & 1-\dfrac{B}{\rho c^2} \end{vmatrix} = 0 \qquad ...(29)$$

where we have neglected n^2R^2/c^2 in comparison with $B/\rho c^2$. The velocity $(B/\rho)^{1/2}$ of shear waves being greater than the wave speed c and reminding us of Womersely's remark noticed after (20) we may neglect n^2R^2/c^2 in comparison with $B/\rho c^2$. Following Womersley (1955) we define

$$x = \frac{k'B}{\rho c^2},\; k' = \frac{h\rho}{\rho_0 R},\; c_0^2 = \frac{hE}{2\rho_0 R}$$

and write $$\left[\frac{(1-\sigma^2)\,x}{2}\right]^{1/2} = X - iY \qquad ...(30)$$

Where c_0 is the velocity for the non-viscous fluid.
Using (30) in (29) we have

$$(1-\sigma^2)(1-F_{10})\,x^2 - \{(2\beta\sigma + k' - \beta/2)(1-F_{10}) + 5\beta/2 - 2\beta\sigma\}\,x + \beta^2 + 2\beta k' - \beta^2(1-F_{10}) = 0. \qquad ...(31)$$

Here $k' \ll 1$ as we have assumed a thin-walled elastic tube.

The roots of (31) are given by

$$(1-\sigma^2)\,x = G \pm [G^2 - (1-\sigma^2)\,H]^{1/2} \qquad ...(32)$$

Where

$$G = \frac{\beta(5/4-\sigma)}{1-F_{10}(\alpha_1)} + \frac{k'}{2} - \beta(1/4-\sigma) \qquad ...(33)$$

and

$$H = \frac{\beta(\beta + 2k')}{1 - F_{10}(\alpha_1)} - \beta^2, \quad \ldots(34)$$

where following Mclachalan (1955),

$$F_{10}(\alpha_1) = 2J_1(\alpha_1 i^{3/2})/\alpha_1 i^{3/2} J_0(\alpha_1 i^{3/2})$$

$$= \frac{\sqrt{2}}{\alpha_1}\left[\frac{\mathrm{bei}_1\,\alpha_1\,\mathrm{ber}\,\alpha_1 - \mathrm{ber}_1\,\alpha_1\,\mathrm{ber}\,\alpha_1 - \mathrm{ber}_1\,\alpha_1\,\mathrm{bei}\,\alpha_1 - \mathrm{bei}_1\,\alpha_1\,\mathrm{bei}\,\alpha_1}{(\mathrm{ber}\,\alpha_1)^2 + (\mathrm{bei}\,\alpha_1)^2}\right.$$

$$\left. + i\,\frac{\mathrm{ber}_1\,\alpha_1\,\mathrm{bei}\,\alpha_1 - \mathrm{ber}_1\,\alpha_1\,\mathrm{ber}\,\alpha_1 - \mathrm{bei}_1\,\alpha_1\,\mathrm{bei}\,\alpha_1 - \mathrm{bei}_1\,\alpha_1\,\mathrm{ber}\,\alpha_1}{(\mathrm{ber}\,\alpha_1)^2 + (\mathrm{bei}\,\alpha_1)^2}\right].$$

Here $F_{10}(\alpha_1)$ is complex, so x is always complex and the motion is either damped or unstable. This depends on the sign of argument of x, which is determined by that of argument of G. If the sign of argument of x is negative, then the motion is damped otherwise it is unstable.

Particular Cases

Case I. $\beta = 1$, or $Nm \to 0$ i.e. absence of suspended particles.
It reduces to Womersley (1955). The motion is damped.

Case II. When $n\tau \ll 1$, $\beta \simeq 1 + \dfrac{Nm}{\rho_0(1-\phi)}$.

In the case of blood, we have

	Number	Diameter
R.B.C.	5–5.5 million/mm³	5.5–8.8×10^{-6} cm
W.B.C.	6000 to 8000/mm³	8–15×10^{-6} cm
Platelets	250,000 to 500,000/mm³	2.5×10^{-4} cm. approx.

Viscosity of the plasma $\mu = 1.2$ CP
Density of the plasma $\rho_0 = 1.1$ gm/cm³
Mass of the packed cells $= 5.0212 \times 10^{-4}$ gm/cc.
Taking the volume concentration for a normal blood of the order of 45 per cent

$$\frac{Nm}{\rho_0(1-\phi)} = 8.2995 \times 10^{-4}$$

and $\tau = 1.057 \times 10^{-7}$ sec.

Thus for a normal blood, $n\tau \ll 1$.
In this case argument of G is always negative. Hence the motion is damped.

We infer from (30) that the ratio of the velocity for the non-viscous fluid to that for the wave-velocity $c_0/c = X$ and over a distance of one wave-length the amplitude will be reduced in the ratio exp. $(-2\pi Y/X)$. The following table gives the values of $1/X$ and $\exp(-2\pi Y/X)$ for variable frequency parameter α_1 which correspond to the first four harmonics with $R^2/\nu = 1/9 \times 10^2$ as used by Womersley (1955) and for $\beta = 1$ and 1.5; calculated for $\sigma = 0.5$ and $k' = 0.1$. The effects of an added mass and an elastic constraint can be controlled simply by varying k'.

TABLE I

$\sigma = 0.5, k' = 0.1$

α_1		$1/X$		$\exp(-2\ Y/X)$	
$\beta = 1$	$\beta = 1.5$	$\beta = 1$	$\beta = 1.5$	$\beta = 1$	$\beta = 1.5$
3.34	4.08	0.913	0.753	0.274	0.385
4.72	5.78	0.924	0.765	0.472	0.554
5.78	7.07	0.936	0.773	0.565	0.640
6.67	8.17	0.942	0.777	0.636	0.688

In Fig. 1, 2 and 3 for small values of α_1 (i.e. upto 3) there is a rapid change in $1/X = c/c_0$; the change in phase velocity ratio with α_1 is quite small for $\sigma = 0,0.25$ and 0.5 for $\alpha_1 > 3$ which are of practical interest. Fig. 4 shows the relationship between α_1 and the damping factor $\exp(-2\pi Y/X)$, we find that the damping factor increases with the increase of α_1 and σ for fixed β.

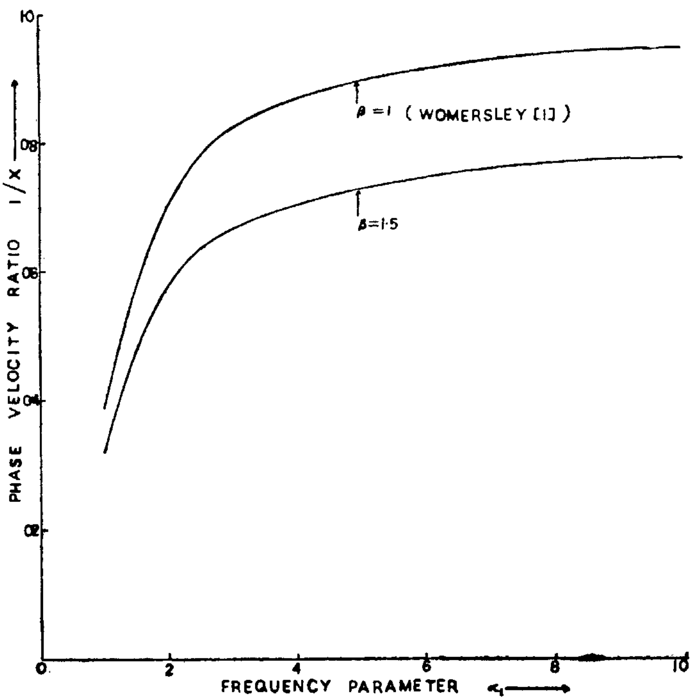

FIG. 1. Variation of Phase Velocity Ratio with the Frequency Parameter $\alpha_1 = R\sqrt{\frac{n\beta}{\nu}}$, for $\sigma = 0$, $k' = 0.1$.

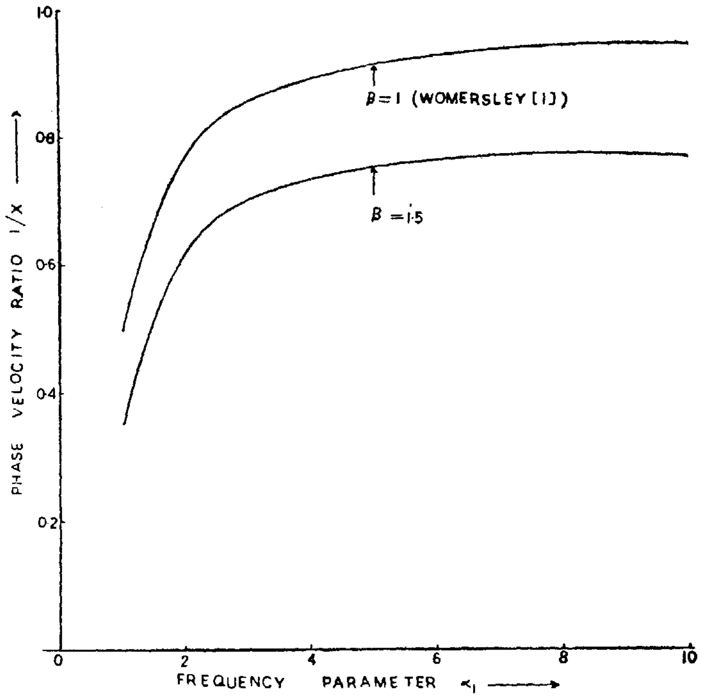

FIG. 2. Variation of Phase Velocity Ratio with the Frequency Parameter $\alpha_1 = R\sqrt{\frac{n\beta}{\nu}}$, for $\sigma = 0.25$, $k' = 0.1$.

Case III. $n\tau > 1$, $\beta \simeq 1$ reduces to case *I* (Womersley, 1955).

Case IV. $n\tau = 1$, $\beta = \left(1 + \frac{Nm}{2(1-\phi)\rho_0}\right) - i\left(\frac{Nm}{2(1-\phi)\rho_0}\right)$.

Therefore, $\alpha_1 i^{3/2} = r^{1/2} e^{i\theta/2}$,

where $$r^2 = n^2R^4 \left\{\frac{N^2m^2}{4(1-\phi)^2\rho_0^2} + \left(1 + \frac{Nm}{2(1-\phi)\rho_0}\right)^2\right\}\nu^2.$$

(i) If $$e^{i\theta/2} = i^{3/2} \Rightarrow \theta = 3\pi/2 = \tan^{-1}\left(1 + \frac{2(1-\phi)\rho_0}{Nm}\right) \Rightarrow Nm \to 0,$$

this case also reduces to case I(Womersley, 1955).

(ii) When $e^{i\theta/2} \neq i^{3/2}$, the particles lag behind the fluid, and the amplitude and the phase of the velocities are influenced significantly by the presence of suspended solid particles.

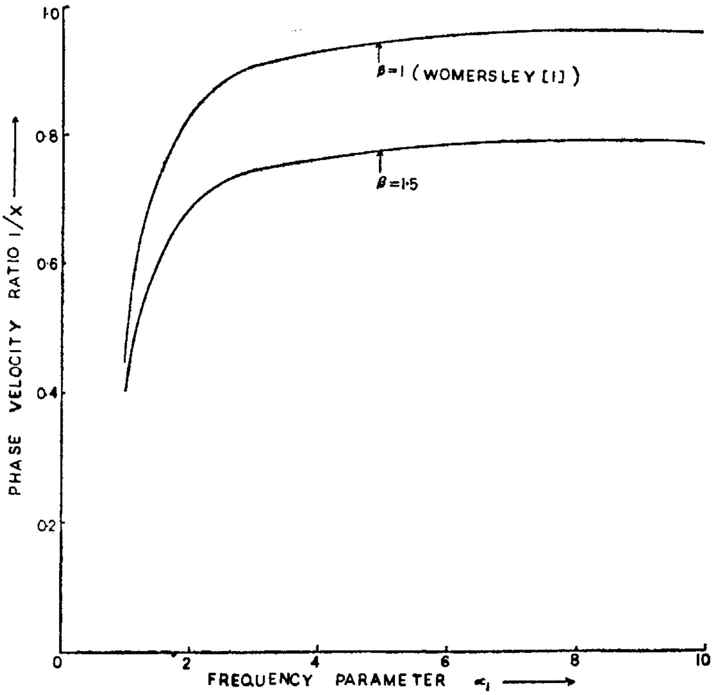

FIG. 3. Variation of Phase Velocity Ratio with the Frequency Parameter $\alpha_1=\sqrt{\frac{n\beta}{\nu}}$, for σ=0.5, k'= 0.1.

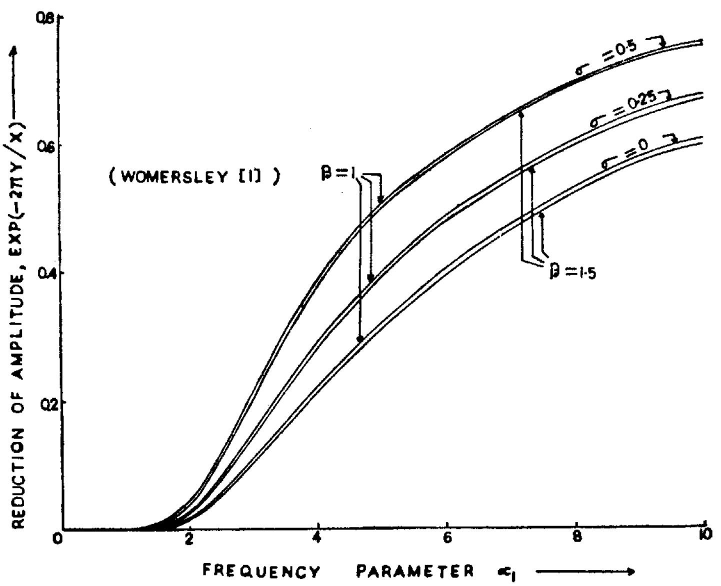

FIG. 4. Variation of Reduction of Amplitude with the Frequency Parameter $\alpha_1=R\sqrt{\frac{n\beta}{\nu}}$, for k'=0.1.

Conclusion

The particle material density is large compared with that of the fluid. Also, we have assumed that nR/c is a small quantity. Therefore, the terms of the order nR/c and n^2R^2/c^2 are neglected from the equation of motion. The equation for the wave valocity will then involve two non-dimensional parameters, α_1 and nR/c. The fluid is free of particles, for $\beta = 1$ and the motion is damped. There are changes in the amplitude and the phase velocity for $\beta \neq 1$. When the relaxation time is smaller than the period of pulse wave, the motion is damped, otherwise the flow is unaffected by the particles. Finally when the relaxation time is of order of the period of pulse wave, the amplitudes and the phase velocities are affected and the particles lag behind the fluid.

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