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ON THE STRUCTURE OF SHOCK WAVES

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The structure of one dimensional shock wave is investigated using the Navier-Stokes equations for the gas phase. The resulting system of three ordinary nonlinear differential equations is reduced to a system of two autonomous nonlinear differential equations which are solved exactly. This solution is obtained formally by neglecting the product term of heat conductivity and viscosity ($\lambda\ \eta = 0$).

Key Words : Shock Waves; Naviear-Stokes Equation; Gas Phase; Heat Conductivity and Viscosity

INTRODUCTION

THE structure of shock waves in a simple gas has been investigated by many authors.[1–13] Early investigations were restricted to the perfect gas equation of state and the Navier-Stokes relations with constant coefficients of heat conduction and viscosity.

Rankine[1] gave an explicit solution for the case of heat conduction only. Later, an explicit solution for viscosity alone was given by G I Taylor,[2] who also gave an explicit formula valid for weak shocks with both viscosity and heat conduction present A specific conclusion was that, for weak shock, the thickness becomes very large, varying as the reciprocal of the strength. Almost simultaneously Rayleigh[3] presented a complicated qualitative discussion of the general case.

The significance of the direction field approach was noted by Becker,[4] who discovered an explicit solution valid for arbitrary shock strength for the special value 3/4 of the Prandtl number.[14–16] He noted that this led to a vanishingly small shock thickness for strong shocks. Later, it was remarked by Thomas[5] that a finite limiting thickness results from the same explicit solution if the viscosity coefficient is taken to vary as the square root of the temperature, and by Morduchow and Libby[6] that a variation of the form T^s results in a zero limiting thickness for strong shocks if $s < \frac{1}{2}$ and a thickness increasing without bound if $s > \frac{1}{2}$. This difference in behaviour is essentially due to the fact that for $s = \frac{1}{2}$ (elastic sphere molecules), the mean free path depends on density alone, and the density ratio before and after a shock remains bounded as the strength increases. On the other hand, for $s \neq \frac{1}{2}$, the ratio of mean free paths before and after the shock approaches zero or infinity as the shock strength increases. It can be remarked that, while such investigations of infinitely strong shocks are certainly interesting, they are almost certainly of little relevance to the physical problem because of the break-down of the Navier-Stokes equations.

The problem of an arbitrary Navier-Stokes fluid (subject to general thermodynamic limitations) has been studied by Weyl,[7] and a rigorous proof of the existence and uniqueness of the solution in this case was given by Gilbarg.[8] Bounds on the thickness of a shock in a perfect gas have been given by von Mises.[9] The Navier-Stokes approximation is, of course, identical to the continuum Navier-Stokes treatment for a monoatomic perfect gas except that the viscosity coefficient is known explicity if a specific molecular model is chosen. An attempt has been made by Wang Chang[10] to solve the problem for the sequence of Chapman-Enskog approximations, but the series obtained is so slowly convergent to be impractical. An approximation has been by Mott-Smith[11] which is possibly significant for strong shocks.

A development of an analytical procedure for approximating the solution of the differential equation has been made by H Grad.[12]

The main purpose of this paper is to present an exact solution for a shock wave, in simple gas with both viscosity and heat conduction.

Basic Equations

The three conservation equations for the one-dimensional steady flow of an arbitrary fluid can be written as

$$\frac{d}{dx}(\rho u = 0) \qquad \ldots(1)$$

$$\frac{d}{dx}(\rho u^2 + p + p_{xx}) = 0 \qquad \ldots(2)$$

and $$\frac{d}{dx}[\rho u(e + u^2/2) + u(p + p_{xx}) + q_x] = 0 \qquad \ldots(3)$$

Here ρ is the mass density, u the flow velocity, p the pressure, e the internal energy, p the x-component of stress, and q_x the x-component of heat flow.

The equations (1)–(3) have to be solved for the boundary conditions:

$$u = u_0,\ T = T_0 \text{ for } x \to -\infty$$

$$u = u_1,\ T = T_1 \text{ for } x \to +\infty$$

Integrating (1)–(3), one obtains :

$$\rho u = m; \qquad \ldots(4)$$

$$\rho u^2 + p + p_{xx} = P; \qquad \ldots(5)$$

and $$\rho u(e + u^2/2) + u(p + p_{xx}) + q_x = E. \qquad \ldots(6)$$

For further treatment explicit expressions for the equations of state, the stress and the heat flux have to be introduced :

$$e = c_v T, \quad p = \rho R T \qquad \dots(7)$$

$$p_{xx} = -\frac{4}{3}\eta\frac{du}{dx}, \; q_x = -\lambda\frac{dT}{dx}, \qquad \dots(8)$$

using these expressions equations (5) and (6) could be put in the following forms :—

$$\frac{4\eta}{3m}\frac{du}{dx} = u + \frac{RT}{u} - \frac{P}{m} \qquad \dots(9)$$

$$\frac{\lambda}{m}\frac{dT}{dx} = c_v T + \frac{u^2}{2} + u(\frac{P}{m} - u) - E. \qquad \dots(10)$$

The integration constants P and E can be expressed in terms of the variables of state a head of the shock wave :

$$\frac{P}{m} = u_0 + \frac{RT_0}{u_0} = \left(1 + \frac{1}{kM_0^2}\right)u_0 \qquad \dots(11)$$

and

$$E = \left\{\frac{1}{(k-1)M_0^2} + \frac{1}{2}\right\} u_0^2, \qquad \dots(12)$$

where M_0 is the Mach number.

The two basic equations (9), (10) are put into the following dimensionless form :

$$\bar{\eta}\frac{dw}{dx} = w + \frac{\theta}{w} - 1, \qquad \dots(13)$$

and

$$\bar{\lambda}\frac{d\theta}{dx} = \theta - \delta\,[(1-w)^2 + \alpha], \qquad \dots(14)$$

where

$$w = \frac{mu}{P}, \; \theta = \frac{m^2RT}{P^2}, \; \phi = \frac{p}{P}, \qquad \dots(15)$$

$$\alpha + 1 = \frac{2\,Em^2}{P^2}, \; \delta = R/(2c_v) = (k-1)/2, \qquad \dots(16)$$

$$k = c_p/c_v, \; R = c_p - c_v, \qquad \dots(17)$$

$$\bar{\eta} = 4\,\eta/(3m), \; \bar{\lambda} = \lambda/(c_v\, m). \qquad \dots(18)$$

These equations are, of course, identical with Gilbarg and Paolucci's equations.[13]

The system of two differential equations (13) and (14) is difficult to solve because of the nonlinearities and the two singularities of the direction field in phase plane. These singularities correspond to the equilbrium conditions a head of and behind the shock wave.

EQUILIBRIUM

Far in front of the wave and far behind the wave all gradients of the variables of state become zero. Under this condition, the equilibrium state can be calculated from equations (13) and (14).

$$\omega_{0,1} = \frac{(1 + 2\delta) \pm \epsilon}{2(1 + \delta}, \qquad ...(19)$$

$$\theta_{0,1} = \frac{1}{4(1 + \delta)^2} [(1 + 2\delta) - \epsilon^2 \mp 2\delta\epsilon] \qquad ...(20)$$

and
$$\phi_{0,1} = \frac{1 \mp \epsilon}{2(1 + \delta)}, \qquad ...(21)$$

where

$$\epsilon^2 = 1 - 4\delta\alpha\ (1 + \delta). \qquad ...(22)$$

The parameter ϵ is a measure for the strength of the change in the variables of state. One has :

$$M_0^2 = \frac{1 + \epsilon/k}{1 - \epsilon}. \qquad ...(23)$$

For very strong shock waves with $M_0 \rightarrow \infty$ one has $\epsilon \rightarrow 1$.

A shock wave cannot occur for $\epsilon = 0$. Equations (19) and (20) are conditions for the two singularities P_0 and P_1 of the direction field.

THE SHOCK PROFILE

Eliminating the temperature θ from differential equations (13) and (14), one obtains the single differential equation :

$$\bar{\lambda}\ \bar{\eta} \frac{d}{dx} \left(w \frac{dw}{dx}\right) - [(\bar{\eta} + 2\bar{\lambda})\ w - \bar{\lambda}] \frac{dw}{dx}$$
$$= -\ (1 + \delta)\ w^2 + (2\delta + 1)\ w - \delta\ (1 + \alpha). \qquad ...(24)$$

By setting $\bar{\lambda}\ \bar{\eta} \rightarrow 0$, where $\bar{\lambda} \neq 0$ and $\bar{\eta} \neq 0$, it follows that :

$$[(\bar{\eta} + 2\bar{\lambda})\ w - \bar{\lambda}] \frac{dw}{dx} = (1 + \delta)w^2 - (2\delta + 1)\ w + \delta\ (1 + \alpha) \qquad ...(25)$$

This equation can be integrated and the solution $w(x)$ is given by :

$$(w_0 - w)^{\alpha}/(w - w_1)^{\beta} = Ae^{\gamma x}, \qquad ...(26)$$

where

$$\left.\begin{aligned} \alpha &= w_0\ (\bar{\eta} + 2\bar{\lambda}) - \bar{\lambda}, \\ \beta &= w_1\ (\bar{\eta} + 2\bar{\lambda}) - \bar{\lambda}, \\ \gamma &= (1 + \delta)\ (w_0 - w_1), \\ A &= w_0^{\alpha}/w_1^{\beta} \end{aligned}\right\} \qquad ...(27)$$

for all values of the heat conductivity $\bar{\lambda}$, viscosity $\bar{\eta}$ and shock strength.

The solution (26) which represents a shock profile of the gas velocity must join the two singularities P_0 and P_1.

Setting $\bar{\lambda} = 0$ in equations (27), one obtains, from (26) Taylor's solution, but for $\bar{\eta} = 0$, one obtains Rankine's solution.

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