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STANDARD MONOMIAL THEORY AND GEOMETRY OF SCHUBERT VARIETIES

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I would like to give a brief survey of the work that I have been doing during the last few years. Most of this work appears in a series of papers entitled 'Geometry of G/P,' written in collaboration with V Lakshmibai and C Musili.[8–11,19]

In a classical work, Hodge gave canonical basis for the homogeneous coordinate rings for Grassmannians and its Schubert varieties (for the canonical Plucker imbedding into projective space) by what he called 'standard monomials'.[4] Our aim has been to generalize this work in the context of flag varieties and its Schubert varieties, associated to any semi-simple, simply-connected, algebaric group G. We have succeeded in getting a good generalization in the case when G is a *classical group*. We have also partial results when G is exceptional. Our results give, in particular, explicit canonical bases for finite dimensional irreducible representations of classical groups (the base field being of characteristic zero).

An important problem would be to generalize our results to the case of any *exceptional group* G. Recently, Lakshmibai has succeeded in extending our results to the case of the group G_2.[6]

There are many applications of standard monomial theory, for example, to the following : (1) Kodaira type of vanishing theorems for flag varieties and their Schubert varieties and their Cohen-Macaulay nature (2) classical invariant theory (3) variety of complex and (4) determination of the singular locus of a Schubert variety. All these applications stem from the fact that standard monomial theory gives explicit generators for the ideal (sheaf) defining a Schubert variety in the flag variety.

Recently, in a series of beautiful papers, A Ramanathan and others has proved the Kodaira type of vanishing theorems and Cohen-Macaulay properties for Schubert varieties associated to *any* semi-simple, simply-connected algebraic group G.[13,16–18] It fact, he has proved many other qualitative properties for general Schubert varieties which had been deduced previously as consequences of standard monomial theory (and hence valid only for Schubert varieties associated to classical groups).

Note : The Lecture was delivered at INSA, New Delhi.

1. Work of Hodge

We shall now recall the classical work of Hodge, as is done in the paper of Musili.[14]

Let G be the special linear group $SL(n)$, B its *Borel subgroup* consisting of the upper tringular matrices and T its *maximal tones* consisting of the diagonal matrices. Let P be a *maximal parabolic subgroup* of G with respect to B (i.e. an algebraic subgroup G containing B, $\neq G$ and maximal for these properties). Then G/P (the space P cosets of the form $\{gP\}$, $g \in G$, is the Grassmannian of r-dimensional linear subspaces of an n-dimensional vector space (for some r, $1 \leqslant r \leqslant n-1$). One knows that Pic G/P (the group of isomorphism classes of line bundles on G/P) $\simeq Z$ and that there is a *very ample* generator L of this group. Its sections give an imbedding of G/P into a projective space (called the Plucker imbedding). How $H^0(G/P, L)$ is a G module and the T stable vectors (to be precise, the one-dimensional space spanned by the vector is T stable) are called the *weight vectors* in this G module. One has the notion of *highest* and *lowest weight vectors* (the highest weight vector being **B** stable and the lowest weight vector being $\mathbf{B}^-$ stable) where $\mathbf{B}^-$ is the Borel subgroup, *opposite* to **B** i.e. consisting of the lower triangular matrices). Let f be the lowest weight vector in $H^0(G/P, L)$ (f is uniquely determined upto a constant). Let $W = W(G)$ denote the *Weyl group* of G. One sees that the subgroup of W fixing the one-dimensional linear space spanned by f, is precisely the Weyl group $W(P)$ of P. The translates of f by W can therefore be indexed by $W/W(P)$ and we set

$$p(\tau) = \tau.f, \quad \tau \in W/W(P).$$

The translates of the highest or lowest weight vector by W are called *extremal weight vectors* so that $\{\mathbf{p}(\tau)\}$ are extremal weight vectors. Now $H^0(G/P, L)$ is a *fundamental representation* of G and it is known that in this case ($G = SL(n)$) its basis is formed of extremal weight vectors i.e. $\{\mathbf{p}(\tau)\}$, $\tau \in \mathbf{W/W(P)}$, form a basis of $H^0(G/P, L)$.

The **B** stable closed subvarieties of G/P are called *Schubert varieties* (in G/P). It is known that the number of Schubert varieties in G/P is equal to $\# W/W(P)$, in fact, the Schubert varieties in G/P can be naturally indexed as $X(\tau)$, $\tau \in W/W(P)$. We have a canonical *partial order* in $W/W(P)$ namely

$$\tau_1 \geqslant \tau_2 \Leftrightarrow X(\tau_1) \supseteq X(\tau_2).$$

This partial order can be defined combinatorially. For example, we have a canonical identification of $W/W(P)$ with the following set of indices:

$$I_n(r) = \{(i_1,\ldots\ldots i_r) | 1 \leqslant i_1 < i_2 < \ldots\ldots < i_r \leqslant n\}$$

and that if $\tau_1 = (i_1,\ldots\ldots,i_r)$ and $\tau_2 = (j_1,\ldots\ldots,j_r)$,

$$\tau_1 \geqslant \tau_2 \Leftrightarrow i_k \geqslant j_k;\ 1 \leqslant k \leqslant r.$$

We have also a *length function* l on $W/W(P)$ and we have

$$\mathrm{l}(\tau) = \dim X(\tau),\ \tau \in W/W(P).$$

It can also be shown that

$$P(\lambda)|X(\tau) \not\equiv 0 \Leftrightarrow \tau \geqslant \lambda.$$

Definition 1 :

We call a *standard tableau* of length m on G/P (resp. $X(\tau)$) a subset of m elements of $W/W(P)$ of the form

$$\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m \text{ (resp. } \tau \geqslant \tau_1 \geqslant \ldots \geqslant \tau_m).$$

We call a *standard monomial of length m on G/P (resp. $X(\tau)$)* on expression of the form :

$$\begin{cases} p(\tau_1)\ldots\ldots p(\tau_m) \text{ such that} \\ \tau_1 \geqslant \ldots\ldots \geqslant \tau_m \text{ (resp. } \tau \geqslant \tau_1 \geqslant \ldots\ldots \geqslant \tau_m) \end{cases}$$

Now a standard monomial of length m on G/P (resp. $X(\tau)$) represents an element of $H^0(G/P, L^m)$ (resp. $H^0(X(\tau), L^m)$. The main result of Hodge is the following.[4]

Theorem 2 : Standard monomials of length on G/P (resp. $X(\tau)$) form a basis of $H^0(G/P, L^m)$ (resp. $H^0(X(\tau), L^m)$).

2. Main Results

Let G be a semi-simple, simply-connected, algebraic group G and P a *parabolic subgroup* i.e. an algebraic subgroup P of G, $P \supseteq B$. Then most of the definitions as in §1 can be made; in particular, we can define Schubert varieties in G/P, which are indexed by $W/W(P)$. If P is a maximal parabolic group, then Pic $G/P \simeq Z$ and has a very ample generator L. In general, it is *not* true that $H^0(G/P, L)$ has a basis formed of extremal weight vectors. However, this holds when P is minuscule i.e associated to a *minuscule* fundamental weight and all the definitions as in §1 can be made and we can ask whether Theorem 2 holds i.e. whether standard monomials on $X(\tau)$ of length m, form a basis of $H^0(X(\tau), L^m)$. This is indeed the case and this was proved in $G/P - I$.[19] This suggested the possibility that Hodge's theorem (Theorem 2) could admit of a vast generalization in the context of flag varieties associated to semi-simple groups. However, having a standard monomial theory just for the minuscule case is not good enough, for apart from the case of $SL(n)$, in general, few fundamental representations are minuscule. In fact, there are exceptional groups for which no fundamental representation is minuscule.

Let then P be a maximal parabolic subgroup in G, which need *not* be minuscule. The first steps in edifying a standard monomial theory is to be able to index a basis of *non-extremal* weight vectors in $H^0(G/P, L)$ in a nice manner, relating it two W and its partial order. Guided by the work of De Concini and Procesi on classical invariant theory,[2] we found that in the case when P is of *classical type* (this concept is explained below), a basis of non-extremal weight vectors can be indexed by certain pairs of elements in $W/W(P)$. Then we can generalize the definition of standard monomials and prove the analogue of Theorem 2. For a classical group,

we see that every maximal parabolic group is of classical type and for an exceptional group, there is at least one maximal parabolic subgroup which is of classical type. We shall now give the precise results.

Definition 3: Let θ be a fundamental weight and P a maximal parabolic subgroup of G associated to θ. We say that θ or P is of *classical type* if for every root α

$$\left|\frac{2(\theta, \alpha)}{(\alpha, \alpha)}\right| \leqslant 2.$$

Let now Ch G/P denote the *Chow ring* of G/P. For a Schubert variety $X(\tau)$ in G/P, we denote by $[X(\tau)]$ the class in G/P determined by $X(\tau)$. We denote by $[H]$ the class in G/P determined by the unique codimension one Schubert variety H in G/P. Then if $\bullet$ denotes multiplication in G/P, we find that

$$[X(\tau)] \bullet [H] = \Sigma\, d_i[X(\varphi_i)],\ d_i > 0,$$

where $X(\varphi_i)$ is a Schubert divisor in $X(\tau)$ i.e., it is contained in $X(\tau)$ and $X(\varphi_i)$ is of codimension one in $X(\tau)$. The fact that P is of classical type implies that $d_i \leqslant 2$ and minuscule implies that $d_i = 1$.

Definition 4: A *non-trivial admissible pair* (τ, φ) in $W/W(P)$ is a pair of elements (τ, φ) in $W/W(P)$ such that $X(\varphi)$ is a Schubert divisor in $X(\tau)$ of the form $X(\varphi_i)$ as above with $d_i = 2$. A *trivial admissible pair* in $W/W(P)$ is just a pair of elements of the form (w, w), $w \in W/W(P)$. A *standard tableau of length m on* G/P (resp. $X(\tau)$) is a set of m admissible pairs $(\tau_1, \varphi_1), (\tau_2, \varphi_2), \ldots (\tau_m, \varphi_m)$ such that

$$\begin{cases} \tau_1 \geqslant \varphi_1 \geqslant \tau_2 \geqslant \varphi_2 \geqslant \ldots\ldots \geqslant \tau_m \geqslant \varphi_m \\ \text{(resp.) } \tau \geqslant \tau_1 \geqslant \varphi_1 \geqslant \tau_2 \geqslant \varphi_2 \geqslant \ldots\ldots \geqslant \tau_m \geqslant \varphi_m \end{cases}$$

Now we have the following:

Theorem 5: (A) There exists a canonical basis $\{p(\tau, \varphi)\}$ of $H^0(G/P, L)$ (L is the ample generator of Pic G/P) indexed by the admissible pairs in $W/W(P)$ such that

(i) weight of $p(\tau, \varphi) = -\frac{1}{2}(\tau(\theta) + \varphi(\theta))$

(ii) $p(\tau, \varphi)/X(\lambda) \not\equiv 0 \Leftrightarrow \lambda \geqslant \tau$.

(B) Standard monomials of length m on $X(\tau)$ form a basis of $H^0(X(\tau)\, L^m)$. In particular, standard monomials of length m on G/P form a basis of $H^0(G/P, L^m)$.

The above theorem can be generalized to the case of the flag variety G/B, as follows:—

Let G be a classical group i.e. a simply connected group of type A, B, C or D. Let $\theta_1, \ldots, \theta_l$ be the fundamental weights *chosen in some order and fixed in the sequel.* Let $P_1, \ldots\ldots, P_l$ be the corresponding maximal parabolic subgroups. Note that these are all of classical type. Then we have canonical morphisms $G/B \to G/B_i$ and this gives an immersion

$$\frac{G}{B} \hookrightarrow \frac{G}{P_1} x \ldots\ldots x \frac{G}{P_l}.$$

Let L_i be the ample generator of Pic G/P_i. We denote by the same L_i its pull-back to G/B. We have Pic $G/B \simeq Z^l$ and L_i generate Pic G/P. The Schubert varieties $X(\tau)$ in G/B are indexed by $\tau \in W$.

Definition 6 : A *standard tableau T of type* $m = (m_1, \ldots\ldots, m_l)$ $(m_i \geqslant 0)$ on $X(\tau)$, $\tau \in W$ is the following :

$$T = (T_1 \ldots\ldots T_l)$$

where

$$\begin{cases} T_i = (t_1^i, \ldots\ldots, t_{m_i}^i), \text{ with} \\ T_k^i = (\tau_k^i, \varphi_k^i \text{ admissible pair in } \frac{W}{W}(P_i). \end{cases}$$

Besides, there exist elements $\bar{\tau}_k^i, \bar{\varphi}_k^i$ in W whose cannonical images in $W/W(P_i)$ are respectively $\bar{\tau}_k^i$ and $\bar{\varphi}_k^i$. Further, they have the crucial property

$$\tau \geqslant \bar{\tau}_1^1 \geqslant \bar{\varphi}_1^1 \geqslant \bar{\tau}_2^1 \geqslant \bar{\varphi}_2^1 \geqslant \ldots \geqslant \bar{\tau}_m^1 \geqslant \bar{\varphi}_m^1 > \bar{\tau}_1^2 \geqslant \bar{\varphi}_1^2 \geqslant \ldots$$

We define a *standard monomial of type m* on $X(\tau)$ associated to T as follows :

$$p(T) = \prod_{1 \leqslant k < l} \prod_{1 < j < m_i} p(\tau_j^k, \varphi_j^k).$$

We see that $p(T)$ defines an element of $H^0(X(\tau), L^m)$, where $L^m = L_1^{m_1} \otimes L_2^{m_2} \otimes \ldots \otimes L_l^{m_l}$.

Theorem 7 : Standard monomials of type m on $X(\tau)$, form a basis of $H^0(X(\tau), L^m)$.

The above theorem says, in particular, that standard monomials of type m on G/B, form a basis of $H^0(G/B, L^m)$. Thanks to a theorem of Borel-Weil, one knows that when the base field is of characteristic zero, every finite dimensional irreducible G module is of the form $H^0(G/B, L^m)$ for some m. Hence, the above theorem gives, in particular, cannonical bases for finite dimensional irreducible G-modules for a classical group (base field being of characteristic zero).

The proofs of the above Theorem 5 and 7 are rather long. A self-contained treatment is given in $G/P - V$.[11] A proof of Theorem 5 given in $G/P - IV$.[9]

Applications

The following result follows without much difficulty from Theorem 7.[11,15] We use the notations as in Theorem 7.

Theorem 8 : Let $X(\tau)$ be a Schubert variety in G/P, G a classical group. Let $\tau_i \in W/W(P_i)$ be the image of τ in $W/W(P_i)$, $1 \leqslant i \leqslant l$. Then the ideal sheaf of $X(\tau)$

in G/B is generated by all the elements of the form $p(\lambda, \mu)$, where (λ, μ) is an admissible pair in $W/W(P_i)$ with $\tau_i \not\geqslant \lambda$.

This theorem leads to the determination of the singular locus of a Schubert variety in G/B by applying the Jacobian criterion.[12] For example, if $G = (n)SL$, one deduces that a Schubert variety $X(\tau)$ in G/B is smooth if and only if the number of reflections s_α (α a positive root) such that $\tau \geqslant s_\alpha$, is equal to dim $X(\tau)$.

Determinental varieties, variety of complexes and varieties whose coordinate rings are the invariant rings occurring in classical invariant theory can be considered as nice open subsets of Schubert varieties associated to classical groups.[10,15] These examples provide many applications for Theorem 5, 7 and 8 and provide motivations for them.

Let now L be a line bundle on G/B (G a classical group) of the from $L = L_1^{m_1} \otimes \cdots \otimes L_l^{m_l}$, $m = (m_1, \ldots, m_l)$, $m_i \geqslant 0$. Then one deduces as a consequence of standard monomial theory that $H^i(X(\tau), L) = 0$, $i > 0$. This result has now been proved for an arbitrary semi-simple algebraic group G by V Mehta and Ramanathan.[13] A Ramanathan has also shown[17] that Schubert varieties in G/B are *arithmetically Cohen-Macaulay* (G any semi-simple, simply-connected, algebraic group). Many particular cases of this result had earlier been deduced as consequences of standard monomial theory, especially the case when $X(\tau)$ is a Schubert variety in G/P, where P is of classical type.[1] Standard monomial theory gives something more, namely that a canonical *degeneration* Y of $X(\tau)$ is itself Cohen-Macauly. The homogeneous coordinate ring of Y can be defined purely combinatorially (e.g. when $G = SL(n)$ and $X(\tau) = G/P$-Grassmannian, this coordinate ring is just the Stanley-Reisner ring associated to the poset $W/W(P)$ [3] and it gives interesting connections between standard monomial theory and combinatories of the Weyl group like *shellability*. A complete generalization of Ref. 1 has not still been worked out for the case of G/B,G a classical group (for $G = SL(n)$, cf.[5] It seems to be of interest to carry this out.

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