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VIBRATION OF SEMI-INFINITE AND FINITE CYLINDERS

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The method of Transition Matrix (T-Matrix) is employed to obtain the characteristic equation for the vibration of finite and semi-infinite cylinders. The method is based on the Green's functions and field expansion in terms of suitable basis functions. The boundary conditions and radiation conditions at infinity are then applied, resulting in the associated scattering matrix which, in turn, leads to the characteristic equation. Numerical results are persented at the end.

Key Words : Vibration; Semi-infinite and Finite Cylinders; Transition Matrix; Green's Functions; Bessel and Hankel Functions; Spherical Harmonics

INTRODUCTION

THE analysis of vibration of finite bodies is usually very cumbersome except for simple geometries. Even the case of finite circular cylinders meets with enormous difficulties in satisfying the conditions on the flat faces and on the lateral surface simultaneously. There are some interesting previous works in this direction.[1–4] The case when the body is immersed in a fluid medium is the worst of all, since, even the choice of the Hankel functions of cylindrical type for the outer medium —though satisfies the radiation condition at infinity—violates the requirement of boundedness outside the cylinder along $r = 0$. It is in this context that the technique of T-Matrix lends a hand. This is derived in the following sections. Details of the T-Matrix method can be found in the recent publication (*See* Appendix-*B* for brief sketch).[5]

STATEMENT OF THE PROBLEM

We will first discuss the case of the free vibration of a finite cylinder of radius a and height $2h$. The equations of elastic motion are, in vector form

$$(\lambda + \mu)\,\nabla(\nabla \cdot \mathbf{D}) + \mu\nabla^2\mathbf{D} = \rho\,\frac{\partial^2\mathbf{D}}{\partial t^2} \qquad \ldots(1)$$

Here, $\mathbf{D}$ is the displacement vector, ρ is the density of the medium and λ, μ are the Lame's constants. To solve (1) we assume

$$\mathbf{D} = \nabla\Phi + \nabla\Lambda(\psi) \qquad \ldots(2)$$

and $\qquad (\nabla\psi) = 0,$

where ϕ and ψ satisfy the wave equations :

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\Phi}{\partial t^2},\ \nabla^2\psi = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}$$

$$c_1 = \{(\lambda + 2\mu/\rho\}^{1/2}\ c_2 = (\mu/\rho)^{1/2}. \qquad \ldots(3)$$

If harmonic time-dependence $e^{i\omega t}$ is assumed for all the field quantities, (3) becomes

$$(\nabla^2 + k_1^2)\,\Phi = 0,\ (\nabla^2 + k_2^2)\,\psi = 0, \qquad \ldots(4)$$

which are the Helmholtz equations. Here $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$.

Following the procedure of Varadan and Varadan,[5] we denote the solution of (2) in the new form

$$\mathbf{D} = \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2 + \boldsymbol{\psi}_3, \qquad \ldots(5)$$

the terms of which are described following equation (9).

With the origin of co-ordinates at the centre of the cylinder, the solution of the Helmholtz equations in (4) can be denoted by

$$\begin{Bmatrix} Re\,\Phi_{nm\sigma}\,(k\mathbf{R}) \\ \Phi_{nm\sigma}\,(k\mathbf{R}) \end{Bmatrix} = \xi_{nm} \begin{Bmatrix} j_n\,(kR) \\ h_n\,(kR) \end{Bmatrix} Y_{nm\sigma}\,(\theta, \phi), \qquad \ldots(6)$$

where j_n and h_n denote spherical Bessel and Hankel functions respectively.

$$\mathbf{R} = (R, \theta, \phi)$$

is the spherical polar coordinate system. Further,

$$Y_{nm\sigma}(\theta, \phi) = P_n^m\,(\cos\theta) \begin{Bmatrix} \cos m\phi,\ \sigma = 1 \\ \sin m\phi,\ \sigma = 2 \end{Bmatrix} \qquad \ldots(7)$$

are the spherical harmonics,

$$\xi_{nm} = \sqrt{\frac{4\pi\epsilon_n(n-m)!}{(2n+1)\,(n+m)}} \qquad \ldots(8)$$

and $\epsilon_0 = 1$, $\epsilon_n = 2\ (n > 0)$. In all these,

$$m = 0, 1, 2, 3, \ldots\ldots n.$$

In terms of (6) the various terms in (5) are :

$$\boldsymbol{\psi}_{1nm\sigma}\,(k_1\mathbf{R}) = \left(\frac{k_1}{k_2}\right)^{1/2} \Delta\phi_{nm\sigma}\,(k_1\mathbf{R})$$

$$\boldsymbol{\psi}_{2nm\sigma}\,(k_2\mathbf{R}) = \frac{k_2}{[n(n+1)]^{1/2}}\,\Delta\boldsymbol{\Lambda}\,\{\mathbf{R}\phi_{nm\sigma}(k_2\mathbf{R})\}$$

$$\boldsymbol{\psi}_{3nm\sigma}\,(k_2\mathbf{R}) = \frac{1}{k_2}\,\nabla\boldsymbol{\Lambda}\boldsymbol{\psi}_{2nm\sigma}(k_2\mathbf{R}) \qquad \ldots(9)$$

These are physically compressional and the two polarisations of the shear wave displacements. Their derivation is standard.

These can be denoted by $\boldsymbol{\psi}_{\tau n}$ for short. For axisymmetric problems, $m = 0$ and there being no dependence on the ϕ-coordinate, we will have only two indices

τ and n. The prefix "*Re*" in (6) denotes that the concerned solution is regular at the origin. The remaining functions automatically ensure the radiation condition at infinity.

The boundary conditions for axisymmetric, free vibration of the cylinder are :

$$\sigma_{rr} = \sigma_{rz} = 0 \quad r = a \quad |z| < h \qquad \ldots(10)$$

$$\sigma_{zz} = \sigma_{zr} = 0 \quad z = \pm h \quad 0 < r < a. \qquad \ldots(11)$$

where (r, φ, z) denote the cylindrical system.

The Representation Theorem and the Null-Field Equation

The Green's function $\vec{G}(\mathbf{R}, \mathbf{R}')$ for the elastic equation (1) for harmonic time-dependence satisfies the equation

$$\nabla\vec{\Sigma} + \rho\omega^2\vec{G} = -\mathbf{I}\delta(\mathbf{R} - \mathbf{R}') \qquad \ldots(12)$$

where the stress dyadic $\vec{\Sigma}$ is given by

$$\vec{\Sigma} = \lambda\vec{\mathbf{I}}\,\nabla\,\vec{G} + \mu(\nabla\vec{G} + \vec{G}\Delta) \qquad \ldots(13)$$

{similar to the usual stress-displacement relation viz. $\sigma_{ij}(\mathbf{u}) = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i})$}.

Here $\vec{\mathbf{I}}$ denotes the idem factor (same as the Dirac delta function) and $\vec{G}$ is the Green's function given by

$$\vec{G}(\mathbf{R},\mathbf{R}') = \frac{1}{4\pi\rho\omega^2}\left[k_2^2\vec{\mathbf{I}}\,g(k_2|\mathbf{R} - \mathbf{R}'|)\right.$$

$$\left. + \nabla\{g(k_1|\mathbf{R} - \mathbf{R}'|) - g(k_2|\mathbf{R} - \mathbf{R}')\}\nabla'\right] \qquad \ldots(14)$$

with

$$g(k\xi) = e^{ik\xi}/4\pi\xi. \qquad \ldots(15)$$

Following standard procedure (*Appendix-A*), we obtain from (1) and (12) the representation

$$\int_S \mathbf{u}_{(-)}.[\hat{n}'\boldsymbol{\Sigma}(\mathbf{R}, \mathbf{R}')] - t_{(-)}(u).\mathbf{G}(\mathbf{R}, \mathbf{R}')\ \mathrm{d}S'$$

$$= \begin{cases} \mathbf{u}^s(\mathbf{R}) \text{ for } \mathbf{R} \text{ inside } S: & \ldots(16a) \\ -\mathbf{u}^0(\mathbf{R}) \text{ for } \mathbf{R} \text{ outside } S: & \ldots(16b) \end{cases}$$

where S denotes the surface of the cylinder, $\hat{n}'$ is the unit outward normal to S, $\mathbf{u}^0(\mathbf{R})$ is the displacement due to an incident field (or applied) from a source assumed inside the cylinder, and $\mathbf{u}^s(\mathbf{R})$ is the resulting scattered field inside. The

part (16*b*) is the null field equation. Finally $\mathbf{u}_{(-)}'$ and $\vec{t}_{(-)}'$ denote the displacement and stress vectors at $\mathbf{R}'$ on S defined from inside :

$$\mathbf{t}(\mathbf{A}) = \hat{n}.\,[\lambda \mathbf{I}\nabla.\mathbf{A} + \mu(\nabla\mathbf{A} + \mathbf{A}\nabla)]$$

$$\equiv \hat{n}.\,\vec{\boldsymbol{\tau}}(\mathbf{u}),\{\vec{\boldsymbol{\tau}}(\mathbf{u})\} \equiv \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}) \qquad \ldots(17)$$

SOLUTION OF THE SCATTERED FIELD AND THE EQUATION FOR THE FREE VIBRATION

Next we solve (16) for the scattered field and incidentally also obtain the necessary equation for the free vibration of the finite cylinder. This needs a suitable representation of the various field quantities occurring in (16). For a source close enough to the centre of the cylinder, the outgoing field can be taken as

$$\mathbf{u}^0(\mathbf{R}) = \sum_{\tau}\sum_{n} a_{\tau n}\boldsymbol{\psi}_{\tau n}(\mathbf{R}). \qquad \ldots(18)$$

Here $\tau = 1$ and 3 for axisymmetry. The coefficients are assumed known for a given incident field. Next, the scattered field $\mathbf{u}^s$ for points inside S can be assumed as

$$\mathbf{u}^s(\mathbf{R}) = \sum_{\tau}\sum_{n} f_{\tau n} Re\boldsymbol{\psi}_{\tau n}(\mathbf{R}), \qquad \ldots(19)$$

[$Re.\boldsymbol{\psi}_{\tau n}$ means we take the j_n ( ) part in (6) for $\varphi_{mn\sigma}$ in (9)]

where $f_{\tau n}$ are yet to be solved for. The total displacement $\mathbf{u}_{(-)}$ on and near S can be expressed as

$$\mathbf{u}_{(-)}(\mathbf{R}) = \sum_{\tau}\sum_{n} \alpha_{\tau n}\, Re\boldsymbol{\psi}_{\tau n}(\mathbf{R}), \qquad \ldots(20)$$

where $\alpha_{\tau n}$ are yet to be found. The Green's function has the expansion[5]

$$\vec{\mathbf{G}}(\mathbf{R}, \mathbf{R}') = \left(\frac{ik_2}{\rho\omega^2}\right)\sum_{\tau}\sum_{n}\boldsymbol{\psi}_{\tau n}(\mathbf{R}_>)\, Re\boldsymbol{\psi}_{\tau n}(\mathbf{R}_<), \qquad \ldots(21)$$

where $\mathbf{R}_<$ and $\mathbf{R}_>$ denote $\mathbf{R}$ and $\mathbf{R}'$ according as $|\mathbf{R}| \gtrless |\mathbf{R}'|$. $\boldsymbol{\Sigma}$ can then be obtained from (13) and (21).

For a stress-free surface S,

$$\mathbf{t}_{(-)}'\{\mathbf{u}_{(-)}'\} = 0$$

and hence, the second term in (16) can be dropped under the integral sign there. Then we get

$$\int_S \mathbf{u}_{(-)}'\left[\hat{n}'\,\vec{\boldsymbol{\Sigma}}\,(\mathbf{R}, \mathbf{R}')\right] dS'$$

$$= \mathbf{u}^s(\mathbf{R}) \text{ for } \mathbf{R} \text{ inside } S;$$

$$= -\,\mathbf{u}^0(\mathbf{R}) \text{ for } \mathbf{R} \text{ outside } S. \qquad \ldots(22)$$

Substituting from (18) to (21) and comparing the like terms on either side here, we obtain the pair of equations

$$f_{\tau n} = i\sum_{\tau'}\sum_{n'} Q_{\tau n,\tau' n'}\,(Re,\, Ou)\,\alpha_{\tau' n'} \qquad \ldots(23)$$

and

$$a_{\tau n} = -i \sum_{\tau'} \sum_{n'} Q_{\tau n, \tau' n'}(Re, Re)\, \alpha_{\tau' n'}, \qquad \ldots(24)$$

where

$$Q_{\tau n, \tau' n'}\left[Re, \frac{Ou}{Re}\right] = \frac{k_2}{\rho \omega^2} \int_s \mathbf{t}\left(\frac{Ou}{Re} \psi_{\tau n}\right) Re \psi_{\tau' n'}\, ds \qquad \ldots(25)$$

Here

$\left(\frac{Re}{Ou}\right) \psi_{\tau n}$ means we take $\begin{pmatrix} j_n(\ ) \\ h_n(\ ) \end{pmatrix}$

as appropriate in (6) for the $\varphi_{nm\sigma}$ in (9).

Symbolically, we can write the above relations in the form

$$(f) = -a_{\tau\tau'}, \qquad \ldots(26)$$

where **T** is the 'Transition Matrix' :

$$\mathbf{T} = Q(Re, Ou)\, \{Q(Re, Re)\}^{-1}. \qquad \ldots(27)$$

Finally, for free vibrations of the finite cylinder, taking

$$a_{\tau\tau'} \equiv 0,$$

non-trivial solution of (26) can exist only if

$$\det Q(Re, Re) = 0. \qquad \ldots(28)$$

which, therefore, is our characteristic equation.

Case of Finite Cylinder Immersed in a Fluid

In the case of the finite cylinder immersed in an acoustic medium, we retain the equation (16) as it is for the cylinder part, while for the motion in the outside liquid we have the corresponding relation

$$\int_s \left[\lambda_f \left(\hat{n}' \cdot \mathbf{u}'_{f+}\right)\left(\nabla' \cdot \vec{\mathbf{G}}_f\right) - \left(\hat{n}' \cdot \mathbf{t}'_{f+}\right)\left(\hat{n}'\, \vec{\mathbf{G}}_f\right)\right] dS'$$

$$= 0 \text{ for } \mathbf{R} \text{ inside } S; \qquad \ldots(29a)$$

$$= -\mathbf{u}_f(\mathbf{R}) \text{ for } \mathbf{R} \text{ outside } S; \qquad (29b)$$

Here $\vec{\mathbf{G}}_f$ is the Green's function for the fluid. It is a solution of the equation

$$\left(\nabla^2 + k_f^2\right) \vec{\mathbf{G}}_f(\mathbf{R}_1, \mathbf{R}') = -\vec{\mathbf{I}} \delta(\mathbf{R} - \mathbf{R}') \qquad \ldots(30)$$

so that

$$\vec{\mathbf{G}}_f(\mathbf{R}, \mathbf{R}') = -\vec{\mathbf{I}} g\,(k_f\, |\, \mathbf{R} - \mathbf{R}'\, |\,) \qquad \ldots(31)$$

The displacement $\mathbf{u}_{f+}$ of the liquid near the boundary S can be expanded in the form

$$\mathbf{u}_{f+}(\mathbf{R}) = \sum_{\tau} \sum_{n} \beta_{fn}\, Re\, \psi_{fn}(\mathbf{R}),\ \mathbf{R} \text{ near } S, \qquad \ldots(32)$$

where

$$\psi_{fn} = \nabla\, \Phi(k_f \mathbf{R})$$

The Green's function $\mathbf{G}_f$ has the expansion

$$\mathbf{G}_f(\mathbf{R}, \mathbf{R}') = \left(\frac{k_f}{\rho\omega^2}\right) \sum \psi_{fn}(\mathbf{R}_>)\, Re\, \psi_{fn}(\mathbf{R}_<) \qquad \ldots(33)$$

Finally, the new boundary conditions are

$$\left.\begin{aligned} \left\{\hat{n}\cdot \mathbf{u}_{(-)}\right\}_{\text{solid}} &= \left(\hat{n}\cdot \mathbf{u}_{f+}\right)_{\text{fluid}} \\ \left\{\hat{n}\cdot \mathbf{t}_{(-)}\right\}_{\text{solid}} &= \left(\hat{n}\cdot \mathbf{t}_{f+}\right)_{\text{fluid}} \end{aligned}\right\} \text{ on } S$$

$$\text{tangential } \{\mathbf{t}_{(-)}\}_{\text{solid}} = 0 \qquad \ldots(34)$$

These ensure the continuity of the normal displacement, normal stress and pressure, and the vanishing of tangential stress on S.

Now we apply (29*a*) with the help of (20), (32), (33) and the first boundary condition in (34). This gives

$$\left\{ \sum_{\tau'} \sum_{n'} P_{n,\tau'n'}(Ou, Re)\, \alpha_{\tau'n'} - \sum_{n'} M_{n,n'}(Re, Ou)\, \beta_{fn'} \right\} = 0 \qquad \ldots(35)$$

which expresses β_{fn} in terms of $\alpha_{\tau n}$. Here,

$$P_{n,\tau'n'}(Ou, Re) = \int_S (\nabla \psi_{fn})\, (\hat{n}\cdot Re\, \psi_{\tau'n'})\, dS \qquad \ldots(36a)$$

$$M_{n,n'}(Re, Ou) = \int_S (\nabla\, Re\psi_{fn'})\, (\hat{n}\cdot \psi_{fn})\, dS \qquad \ldots(36b)$$

The solution of (35) can be written in the form

$$(\beta) = M^{-1} P(\alpha) \qquad \ldots(37)$$

In a similar manner, we apply the two relations in (16) with the help of (18) to (32) and the boundary conditions in (34). This gives

$$f_{\tau n} = i \left[\sum_{n'} Q^{(1)}_{\tau n,n'}(Re, Ou)\, \beta_{fn'} + \sum_{\tau'} \sum_{n'} R_{\tau n,\tau'n'}(Re, Ou)\, \alpha_{\tau'n'} \right]$$

and

$$a_{\tau n} = -i \left[\sum_{n'} Q^{(1)}_{\tau n,n'}(Re, Re)\, \beta_{fn'} + \sum_{\tau'} \sum_{n'} R_{\tau n,\tau'n'}(Re, Re)\, \alpha_{\tau'n'} \right],$$

$$\ldots(38)$$

where $$Q^{(2)}_{\tau n,\tau' n'}\left[Re, \frac{Ou}{Re}\right] = \frac{k_2}{\rho\omega^2}\int_s \left\{ t\left(\frac{Ou}{Re}\psi_{\tau n}\right)\cdot \hat{n}\left(\hat{n}\cdot Re\psi_{\tau' n'}\right) - \left[\frac{Ou}{Re}\psi_{\tau n}\right]\cdot \hat{n}\lambda_{\tau'}(\nabla Re\psi_{\tau' n'})\right\} dS$$

$$R_{\tau n,\tau' n'}\left[\frac{Ou}{Re}Re\right] = \frac{k_2}{\rho\omega^2}\int_s t\left[\frac{Ou}{Re}\psi_{\tau n}\right]\cdot\left[Re\psi_{\tau' n'}\right] dS_{\text{tangential}} \quad ...(39)$$

Employing (37) in (38), we obtain the vectorial (more aptly tensorial) relations

$$(f) = i\{Q^{(1)}(Re, Ou)\, M^{-1}P + R(Re, Ou)\}(\alpha)$$

$$(a) = -i\{Q^{(1)}(Re, Re)\, M^{-1}P + R(Re, Re)\}(\alpha) \quad ...(40)$$

Eliminating (α) we get the scattering coefficients to be

$$(f) = -\mathbf{T}^{(1)}(a), \quad ...(41)$$

where the Transition Matrix $\mathbf{T}^{(1)}$ is given by

$$\mathbf{T}^{(1)} \equiv \{Q^{(1)}(Re, Ou)\, M^{-1}P + R(Re, Ou)\} \times \{Q^{(1)}(Re, Re)\, M^{-1}P + R(Re, Re)\}^{-1} \quad ...(42)$$

As before, the characteristic equation for the natural oscillation of the system is obtained from the condition for non-trivial solution of (41) when $(a) \equiv 0$. This is given by

$$\det\{Q^{(1)}(Re, Re)\, M^{-1}P + R(Re, Re)\} = 0. \quad ...(43)$$

The equations (28) and (43) can be solved numerically for generating the natural modes of the free vibrations of the finite cylinder in vacuum or in contact with a fluid, respectively.

Extension to Semi-Infinite Cylinders

It is easily noted that all our previous discussions apply readily also to a semi-infinite cylinder by merely choosing S to define $z = -h$, $0 \leqslant r \leqslant a$ and $r = a$, $z \geqslant -h$. It is preferable to choose the origin inside the cylinder to enable the auxiliary assumptions governing (18) to hold good.

Numerical Example

In the end, we also give some numerical results pertaining to the free vibration of a finite cylinder in vacuum. The material is taken to be of Poisson type where $\lambda = \mu$. Fig. 1* gives the plots of the various modes relating the aspect ratio h/a

*The case of Axisymmetric Vibration with symmetry about the mid-plane $Z = 0$ is considered in these calculations.

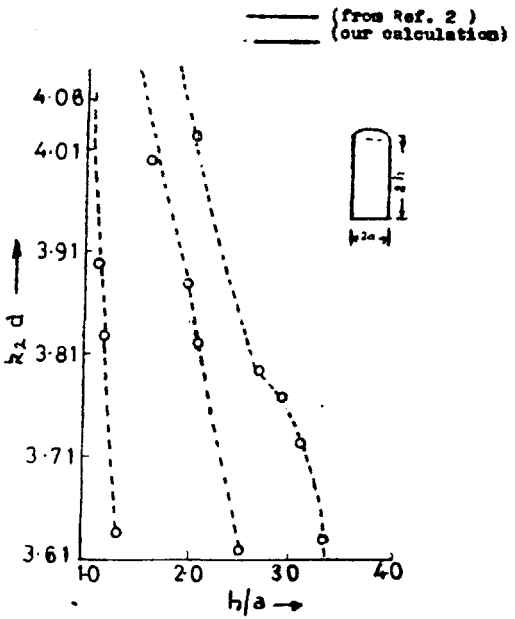

FIG 1 Solution of the eqn. (28)

—⊙—⊙— (our calculations)

(Axisymmetric case with symmetry about the mid-plane $z = 0$).

and the non-dimensional compressional wave-number $k_1 a$. The case of a solid cylinder in contact with a fluid will be treated in a subsequent paper.

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Appendix—A

Sketch of the proof of the Representation Theorem in Equation (16) :

Here we give a brief sketch of the proof of the representation theorem in equations (16, *a*, *b*) of our paper. For this we begin with the equations governing the general (total field) displacement $\mathbf{u}$ and the Green's function concerned :

$$\tau_{ij,j}(\mathbf{u}) + \rho\omega^2 u_i = F_i\delta(\mathbf{R} - \mathbf{R}_0), \tag{A-1}$$

$$\tau_{ij,j}(\mathbf{G}^k(\mathbf{R}, \mathbf{R}')) + \rho\omega^2 G_{ik}(\mathbf{R}, \mathbf{R}') = \delta_{ik}\delta(\mathbf{R} - \mathbf{R}'). \tag{A-2}$$

($\tau_{ij} \equiv$ stress tensor)

Here F_i denotes the forces generating the incident field $\mathbf{u}^0$ and $\mathbf{R}_0$ is the point in the volume V where these act. Next by multiplying the above equations by $G_{ik}(\mathbf{R}, \mathbf{R}')$ and $\mathbf{u}_i$ (i.e. using scalar products) and taking the difference, we have on integrating over the volume V,

$$\begin{aligned}
&\int_V \left[u_i' \, \tau_{ij,j}\{\mathbf{G}^k(\mathbf{R}, \mathbf{R}')\} - G_{ik}(\mathbf{R}, \mathbf{R}') \, \tau_{ij,j}(\mathbf{u}') \right] dV(\mathbf{R}') \\
&= \left[\int_V \delta_{ik}\, \delta(\mathbf{R} - \mathbf{R}') u_i(\mathbf{R}') \, dV(\mathbf{R}') \right. \\
&\left. \cdot - \int_V F_i \delta(\mathbf{R}' - \mathbf{R}_0) \, G_{ik}(\mathbf{R}, \mathbf{R}') \, dV(\mathbf{R}') \right. \\
&= \begin{cases} \left\{ \mathbf{u}_K(\mathbf{R}) - u_k^0 \, (\mathbf{R}, \mathbf{R}_0) \right\} \equiv \mathbf{u}^s(\mathbf{R}) \text{ for } \mathbf{R}, \mathbf{R}_0 \subset V \\ \left\{ 0 - \mathbf{u}_k^0 \, (\mathbf{R}, \mathbf{R}^0) \right\} \equiv - \mathbf{u}^0(\mathbf{R}) \text{ for } \mathbf{R} \not\subset V \; (\mathbf{R}_0 \subset V) \end{cases}
\end{aligned} \tag{A-3}*$$

where we have made use of the standard definition of the incident field in terms of the Green's functions, viz.,

$$\mathbf{u}_k^0 \, (\mathbf{R}, \mathbf{R}_0) = \int_V F_i \delta(\mathbf{R}' - \mathbf{R}_0) \, G_{ik}(\mathbf{R}, \mathbf{R}\mathbf{R}') \, dV \, (\mathbf{R}') \tag{A-4}$$

The volume integral on the left hand side of (A–3) can be converted into a surface integral over the surface S bounding V with the help of the Gauss' Theorem

$$\int_V (\nabla \cdot \psi) \, dV = \int_S (\psi \cdot n) \, dS \tag{A–5}$$

which then readily leads to the results given in equations (16, *a*, *b*).

---

*(Note that $u_k(\mathbf{R}) - u_k^0(\mathbf{R}, \mathbf{R}_0)$ = (Total Field—Incident Field)
= Scattered Field $\mathbf{u}^s(\mathbf{R})$
= $u_k^s(\mathbf{R})$]

Appendix—B

The Principle of the T-Matrix Method :

Here we explain the basic principle of the T-Matrix method. As in equations (16, *a*, *b*) of our paper (and *vide* Appendix 'A') we first pose the solution to the problem in the form of an integral equation :

$$\int_S \left\{ \mathbf{u}'_{(-)} \left[n' \cdot \boldsymbol{\Sigma}(\mathbf{R}_1, \mathbf{R}') \right] - \mathbf{t}'(\mathbf{u}) . \mathbf{G}(\mathbf{R}, \mathbf{R}') \right\} dS'$$

$$= \begin{cases} \mathbf{u}^s(\mathbf{R}), \ \mathbf{R} \subset V \text{ (i.e. } \mathbf{R} \text{ is inside } S) \\ -\mathbf{u}^0(\mathbf{R}), \ \mathbf{R} \not\subset V \text{ (i.e. } \mathbf{R} \text{ is outside } S) \end{cases} \qquad (B\text{-}1)$$

Here the incident field is supposed as known. Next, due to the boundary conditions of the problem we also know the surface values of *either* the displacements $\mathbf{u}'_{(-)}$ *or* the surface stresses $\mathbf{t}'_{(-)}(\mathbf{u})$ on S. In case of two media in contact we still have the corresponding continuity conditions which also serve the same purpose. Thus, to illustrate the point, taking the case of a *stress-free* solid cylinder, we assume that $\mathbf{t}'_{(-)}(\mathbf{u})$ are known on S. Hence, the above equations reduce to the following relations as given in equation (22) .

$$\int_S \mathbf{u}'_{(-)} \cdot \hat{n}' \cdot \boldsymbol{\Sigma} (\mathbf{R}, \mathbf{R}') \, dS' = \begin{cases} \mathbf{u}^s(\mathbf{R}), \ \mathbf{R} \text{ inside } S \\ -\mathbf{u}^0(\mathbf{R}), \ \mathbf{R} \text{ outside } S \end{cases} \qquad (B\text{-}2)$$

Finally, we assume suitable expansions of the unknown surface displacement $\mathbf{u}'_{(-)}$ in the integral in (B–2) which can be found by comparison with the known right hand side term (arising from the incident field) of the second relation in (B–2) above. Then, in turn, the first relation in (B–2) gives the necessary equation for the determination of the scattered field $\mathbf{u}^s(\mathbf{R})$ on the right side in terms of the now known surface displacements on the left hand side. The various expansion coefficients are thus related by the equations (23) and (24) where $(\alpha_{\tau n})$ and $(f_{\tau n})$ are associated with the unknown surface displacements and the unknown scattered field while $(a_{\tau n})$ are associated with the known incident displacements. Elimination of $(\alpha_{\tau n})$ from (23) and (24) gives rise to the T-Matrix relation (26) between the incident and the scattered fields. This, in essence, is the principle of the T-Matrix method. The elements of the matrix as given in (25) contain the parameters of the scattering body.