© *Proc. Indian natn. Sci. Acad.*, **52**, A, No. 2, 1986, pp 497–501.
Printed in India.

TORSIONAL VIBRATIONS OF CIRCULAR CYLINDERS WITH RIGID BOUNDARIES

Y BHUJANGA RAO*

Department of Mathematics, Regional Engineering College, Warangal-500 004, India

(*Received 4 February 1985*)

The problem of torsional vibrations of a circular cylinder, of finite or infinite length, with rigid boundaries is solved and the phase velocity is evaluated in different cases.

Key words : Torsional Vibrations Circular Cylinder; Rigid Boundaries; Phase Velocity

IN the classical analysis of vibrational problems, the curved boundary of a circular cylinder of an isotropic homogeneous elastic material is taken to be stress-free; which is a natural state of a body. But now-a-days it is of common occurrence that a cover or a shield is provided as a protection. Therefore, the particles on the curved boundary are constrained from moving. Such a boundary condition may be mathematically accounted for by taking it as a rigid boundary. The torsional vibrations of (*a*) an infinitely long solid circular cylinder, (*b*) a hollow cylinder of infinite length and (*c*) a hollow circular cylinder of finite length of a homogeneous isotropic material are examined and solved. In each case, the phase velocity is seen to be smaller than its counterpart of classical analysis. This may be due to the assumption of rigid boundary. As regards nodal cylinder, the pattern is the same as in classical analysis but are of smaller size.

FORMATION AND SOLUTION OF PROBLEMS

Let z-axis and origin, of a cylindrical co-ordinate system (r, θ, z), be along the axis and at any point on the axis of an isotropic elastic homogeneous infinite cylinder. In the case of torsional vibrations, the displacement u is $(0, u_\theta, 0)$ where,

$$u_\theta = u(r) \exp \{i(\gamma z + pt)\}, \qquad \ldots(1)$$

where $\frac{2\pi}{\gamma}$ is the wavelength and $\left(\frac{p}{2\pi}\right)$ is the frequency. Substituting in the equation of motion

$$(\lambda + \mu) \nabla (\nabla \cdot \bar{u}) + \mu \nabla^2 \bar{u} = \rho \frac{\partial^2 \bar{u}}{\partial t^2} \qquad \ldots(2)$$

and solving, we get

$$u_\theta = AJ_1(\alpha r) \exp \{i(\gamma z + pt)\}.$$

* Present address : Head, Department of Mathematics, A.V. College, Domalaguda, Hyderabad.

In the case of solid cylinders,

$$u_\theta = [AJ_1(\alpha r) + BY_1(\alpha r)] \exp i(\gamma z + pt);$$

in the case of a hollow cylinder, where $\alpha^2 = \frac{\rho p^2}{\mu} - \gamma^2$.

(A) Solid Cylinder :

The boundary condition is $u_\theta = 0$ when $r = a$, where a is the radius of the cylinder which in turn gives

$$J_1(\alpha a) = 0. \qquad \ldots(3)$$

The first five non-zero roots of (3) as given by Abramovitz and Stegun[1] are 3.83171, 7.01559, 10.17347, 13.32369, 16.47063.

In the classical analysis, the curved boundary is assumed to be stress-free. In this case, the first five non-zero roots, of the frequency equations are 5.13562, 8.41724, 11.61984, 14.79595, 17.95982. The roots are, in order, greater than that of the previous case. Hence, the phase velocity, c, in the case of clamped boundary is smaller than that of stress-free boundary. The zero root of the frequency equation refers to a possible state of stress in the case of stress-free boundary,[2] whereas, it is not so in other case. Because $C = \frac{c}{c_2}$ depends always on wavelength γa, therefore all the modes of vibrations are dispersive. The phase velocity becomes infinite for

$$a\gamma \to 0 \text{ and } c_2 \text{ as } a\gamma \to \infty.$$

The group velocity is seen to be $\left(\frac{c_2^2}{c}\right)$ and hence, it starts from zero for short wavelength and tends asymptotically to c_2 for large wave lengths. Corresponding to higher mode the cylinder divides into as many nodal cyliders as the order of zero minus 1.

In the following, corresponding to first mode of each case (excluding the non-dispersive case of stress-free boundary) for comparison u_θ and $\sigma_{r\theta}$ are given in Fig. 1.

Notice that u_θ changes its sign in the case of free boundary between 0.6 and 0.8 whereas it remains positive through-out, starting from zero on the axis and ending with zero on the boundary. $\sigma_{r\theta}$ remains negative throughout the cylinder starting from zero on the axis and ending with zero on the curved boundary. (In the case of stress-free boundary). In other case also, it starts from zero on the axis; remains positive and maximum occurs around $\frac{r}{a} = 0.8$.

(B) Hollow Infinite Circular Cylinder

Let the inner and outer boundaries be $r = b$ and $r = a$ respectively.

Case 1. Both boundaries be clamped.

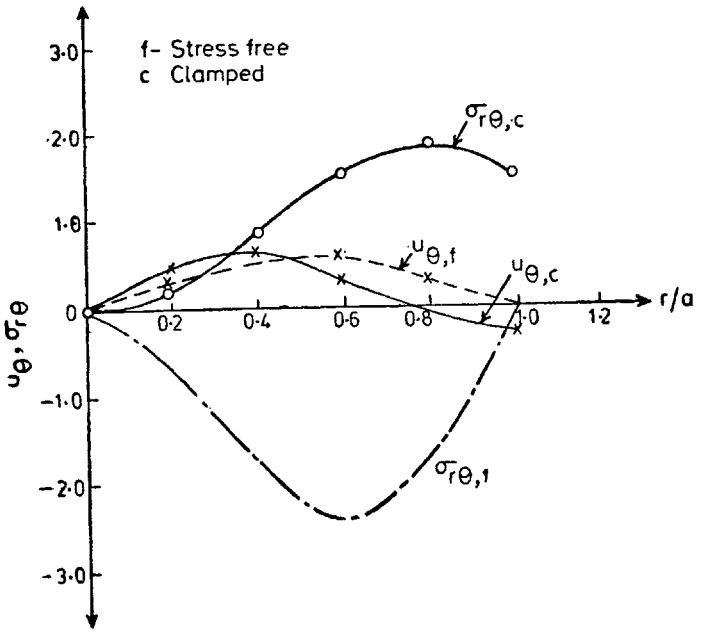

FIG 1

In this case, the frequency equation is :

$$J_1(\alpha a)\, Y_1(\alpha b) - J_1(\alpha b)\, Y_1(\alpha a) = 0.$$

The roots are given by Abramowitz and Stegun.[1]

If c is the phase velocity and $C = \dfrac{c}{c_2}$, then

$$C^2 = 1 + \left(\frac{\alpha a}{\sqrt{a}}\right)^2.$$

Because αa depends on $\lambda = \dfrac{a}{b}$, C is same for the cylinders having the same ratio of outer to inner radii.

$\sigma_{r\theta}$ is given by

$$\sigma_{r\theta} = -\frac{A/ua}{Y_1(\alpha b)} \left[J_2(\alpha r)\, Y_1(\alpha b) - J_1(\alpha b)\, Y_2(\alpha r)\right] \exp\{i(\gamma z + pt)\}.$$

The displacement u_θ is given by

$$u_\theta = \frac{A \exp\{i(\gamma z + pt)\}}{Y_1(\alpha b)} \left[J_1(\alpha r)\, Y_1(\alpha b) - J_1(\alpha b)\, Y_1(\alpha r)\right]$$

and $\dfrac{r}{a}$ varies from λ to 1.

TABLE I

Displacement $u_\theta = D$, *Stress* $\sigma_{r\theta} = S$ $\lambda^{-1} = 0.4$

$\frac{r}{b}$	Both Edges Clamped		Outer Clamped Inner Free		Outer Free and Inner Clamped	
	D	*S*	*D*	*S*	*D*	*S*
	0	29.02500*	0.47276	— 0.00488	0	0.47954
1.00	0	— 1.74904	—3.60930	— 0.37740	0	—1.87128
	6.22530	15.23530*	0.55390	— 0.25051	0.10675	0.29801
1.25	— 0.32442	— 0.40286	—3.28233	8.73954	—0.38046	—0.77608
	9.93537	2.34780*	0.54584	— 0.52917	0.19487	0.19661
1.50	— 0.29983	+ 1.00420	—0.99830	+11.41652	—0.49982	+0.39390
	10.64964	— 8.85797*	0.47049	— 0.02445	0.27118	0.12877
1.75	— 0.00138	1.33421	1.55541	+ 7.48054	—0.37972	+1.21413
	8.68164	—16.57550*	0.33912	— 0.77896	0.33737	0.07742
2.00	0.25531	0.36603	2.75658	— 0.54480	—0.06109	1.37404
	4.99895	—20.30290*	0.01786	— 0.77308	0.39392	0.03632
2.25	0.24261	— 0.74414	2.01161	+ 6.98097	+0.24214	0.87068
	0	—18.87410*	0	— 0.68011	0.44276	0
2.50	0	— 1.10936	0	— 8.68090	+0.38768	0.00000

TABLE II

Displacement $= D = u_\theta$, *Stresses* $= \sigma_{r\theta} = S$, $\lambda^{-1} = 0.6$

$\frac{r}{q}$	Both Edges Clamped		Outer Clamped Inner Free		Outer Free Inner Clamped	
	D	*S*	*D*	*S*	*D*	*S*
1	0	—2.80406	+1.94156	0	0	2.17394
	0	—3.25883	+0.50999	0	0	2.10048
1.1	—0.20396	—2.05348	2.04185	—1.78493	0.21025	1.76833
	—0.26721	—1.45925	0.42846	—2.42423	0.18508	1.29727
1.2	—0.46186	—0.96017	1.95981	—3.31321	0.39011	1.40724
	—0.30072	1.29199	0.14285	—3.57660	0.27388	0.03876
1.3	—0.51690	+0.20434	1.70451	—4.49653	0.55331	1.07271
	—0.09454	2.82929	—0.18814	—2.83470	0.24100	—1.14018
1.4	—0.47987	+1.24410	1.33677	—5.27286	0.68944	0.75932
	+0.17148	+2.05070	—0.40495	—0.79415	0.09940	—1.74732
1.5	—0.34498	1.96843	0.86604	—5.60389	0.80160	0.46303
	+0.28266	—0.27876	—0.40050	+1.44530	—0.07220	—1.57131
1.6	0	2.18264	0	—5.15221	0.93219	0
	0	—2.52775	0	+2.97556	—0.23480	0

Case (2) : Outer boundary is clamped and the inner is stress-free :

The frequency equation is given by

$$J_1(\alpha a)\, Y_2(\alpha b) - J_2(\alpha b)\, Y_1(\alpha a) = 0.$$

For $\lambda^{-1} = 0.4$, the first two modes of vibrations corresponding to $\alpha b = 1.664$ and 3.482 : For $\lambda^{-1} = 0.6$ the first two modes or vibrations correspond to $\alpha b = 3.129$ and 7.410.

Case (3) : Inner boundary is clamped and the outer is stress-free :
The frequency equation is

$$J_2(\alpha a)\, Y_1(\alpha b) - J_1(\alpha b)\, Y_2(\alpha a) = 0.$$

For $\lambda^{-1} = 0.4$, the first two modes of vibrations correspond to $\alpha b = 0.554$ and 3.060 and for $\lambda^{-1} = 0.6$, the first two modes of vibrations correspond to $\alpha b = 1.687$ and 6.906.

The expressions for phase and group velocities are similar to the earlier case. Nodal cylinders exist. The displacement u_θ and stress $\sigma_{r\theta}$ are tabulated in the following, corresponding to the three cases and corresponding to the two modes.

Table I corresponds to $\lambda = 2.5$ and Table II corresponds to $\lambda = 5/3$. In the above tables, for a given value of $\frac{r}{a}$, the first row corresponds to first mode and the second row corresponds to the second mode. The results of the first case of $\lambda = 2.5$ are too big in magnitude. This is because of $Y_1(\alpha b)$ in the denominator which is very close to zero.

(*C*) *Hollow Finite Circular Cylinder* :

The cylinder is bound by $r = a$ and $r = b$ and $z = 0$ and $z = h$.

Proceeding as in the above case, the displacement u_θ and stresses $\sigma_{r\theta}$ and $\sigma_{\theta z}$ are given by

$$u_\theta = [A_1 J_1(\alpha r) + A_2 Y_1(\alpha r)]\,[B_1 \cos nz + B_2 \sin nz]\, e^{ipt}$$

$$\sigma_{\theta z} = \mu[A_1 J_1(\alpha r) + A_2 Y_1(\alpha r)]\,[-B_1 n \sin nz + B_2 n \cos nz]\, e^{ipt}$$

$$\sigma_{\theta r} = -\mu\alpha[A_1 J_2(\alpha r) + A_2 Y_2(\alpha r)]\,[B_1 \cos nz + B_2 \sin nz]\, e^{ipt}$$

The analysis is exactly similar to the case given above and hence no further details are given.

ACKNOWLEDGEMENTS

I wish to express my thanks to Dr K S Sarma, Department of Mathematics, Regional Engineering College, Warangal for his help in preparing this note.

REFERENCES

1. M Abramowitz and E A Stegun *Handbook of Mathematical Functions* Dover Publications (1972) 409, 415
2. H Kolsky *Stress Waves in Solids* Dover (1963)