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TORSIONAL OSCILLATION OF A NONHOMOGENEOUS SEMIINFINITE CIRCULAR CYLINDER BY AN IMPULSIVE TWIST APPLIED AT ONE END

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The paper contains the solution of the torsional oscillation of a semiinfinite nonhomogeneous circular cylinder acted on by an impulsive force applied at one end. The nonhomogeneity is chosen in the form

$$\mu = \mu_0 (1 + k z)^2, \quad \rho = \rho_0 (1 + kz)^2.$$

Key Words: Torsional Oscillation: Nonhomogeneous Semiinfinite Circular Cylinder; Impulsive Twist; Laplace Transform

INTRODUCTION

SEVERAL attempts have been made to solve the problem of torsional oscillation in a circular cylinder by Love,[1] Kolsky,[2] and Davies.[3] In their solution they assume the displacement component as a simple harmonic function of the axial coordinate z and time t. Transient torsional vibration of an infinite circular bar were studied by Jones and Campbell.[4] Campbell used the Laplace transform method to solve the problem for an isotropic homogeneous cylinder. Mitra[5] considered the problem of torsional oscillation of an isotropic semiinfinite circular cylinder acted on by an impulsive twist at one end. Chatterjee[6] extended the problem of torsional oscillation to the case of a cylinder of transversely isotropic material. Banerjee[7] further extended the problem to the case of a cylinder of nonhomogeneous transversely isotropic material in which the elastic constants vary exponentially with the length of the cylinder.

In the present problem an attempt has been made to solve the problem of torsional oscillation of a semiinfinite solid circular cylinder of nonhomogeneous material with one end of the cylinder is acted on by an impulsive twist. The non homogeneity is taken in the form

$$\mu = \mu_0 (1 + kz)^2, \quad \rho = \rho_0 (1 + kz)^2,$$

where μ_0, ρ_0, k are constants and z the axial coordinate. The above expressions for the modulus of rigidity and density are taken in view of the realistic Earth model B of Bullen.[8] For instance, using the values $k = 0.00047$, $\mu_0 = 0.625$ and $\rho_0 = 3.22$ for a depth of 33km, we obtain the density and rigidity by employing the above forms in close agreement (within less than 4 per cent) with the data of

Bullen at subsequent depths. Solution is derived by the use of Laplace transform method. Explicit formula are given for the stress component, particle displacement and particle velocity in integral form. These formula are used to compute the resulting particle displacement for a particular type of stress distribution throughout the end surface. The displacement for the associate homogeneous case has been derived and compared graphically with the nonhomogeneous case.

Formulation of the Problem

We consider a semiinfinite circular cylinder of radius b. The origin is taken at the centre of one end of the cylinder, z-axis is directed along the axis of the cylinder. We use cylindrical polar coordinates (n, θ, z) to specify the points of the cylinder. The surface of the cylinder $n = b$ is free from stress and the disturbance is produced by an impulsive twist applied at its end surface $z = 0$.

It is assumed that the stress is distributed with radial symmetry so that the displacement is tangential and independent of the angular co-ordinate θ. Then the displacement components are given by

$$u_r = u_z = 0, \quad u_\theta = u_\theta(r, z, t). \qquad \ldots(1)$$

For this type of displacement the only nonvanishing stress components are

$$\tau_{r\theta} = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \tau_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} \qquad \ldots(2)$$

We assume the modulus of rigidity and density of material in the form

$$\mu = \mu_0(1 + kz)^2, \quad \rho = \rho_0(1 + kz)^2, \qquad \ldots(3)$$

where μ_0, ρ_0, k are constants, then the stress components take the form

$$\tau_{r\theta} = \mu_0(1 + kz) \left(\frac{\partial \upsilon}{\partial r} - \frac{\upsilon}{r} \right)$$

and

$$\tau_{\theta z} = \mu_0 \left\{ (1 + kz) \frac{\partial \upsilon}{\partial z} - k\upsilon \right\},$$

where

$$\upsilon = (1 + kz)\, u_\theta. \qquad \ldots(4)$$

The only nonvanishing equation of motion then reduces to

$$\frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r} \frac{\partial \upsilon}{\partial r} - \frac{\upsilon}{r^2} + \frac{\partial^2 \upsilon}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \upsilon}{\partial t^2},$$

where

$$c^2 = \frac{\mu_0}{\rho_0}. \qquad \ldots(5)$$

The initial and boundary conditions are

$$u_\theta = \frac{\partial u_\theta}{\partial t} = 0, \qquad t = 0, \qquad 0 \leqslant z < \infty$$

$$\tau_{n\theta} = 0 \quad r = b, \qquad t \geqslant 0$$

$$\tau_{\theta z} = f(r)\ \delta(t), \qquad z = 0, \qquad 0 \leqslant r < a, \qquad t \geqslant 0$$

$$u_\theta/r \text{ is finite} : r = 0,\ t \geqslant 0$$

$$U\theta \text{ is finite} ; z \to \infty, \quad t \geqslant 0, \qquad \ldots(6)$$

where $\delta(t)$ is the Dirac delta function.

Solution of the Problem

Using the method of separation of variable we assume the solution of the equation (5) in the form

$$\upsilon = R(r)\ Z(z). \qquad \ldots(7)$$

Substituting of (7) into (5) it yields

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2}\right) = -\frac{1}{Z}\left(\frac{\partial^2 z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}\right) = -\lambda^2,$$

where λ is constant, which then splits up into two equations namely

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{1}{r^2}\right)R = 0 \qquad \ldots(8)$$

and

$$\frac{\partial^2 Z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 Z}{\partial t^2} - \lambda^2 Z = 0. \qquad \ldots(9)$$

The solution of (8) is

$$R = R_0 = C_1 r + C_2/r$$

and $$R = C_3 J_1(\lambda r) + C_4 Y_1(\lambda r) \qquad \text{for } \lambda > 0,$$

where C_1, C_2, C_3, C_4 are constants and J_1, Y_1 are Bessel functions of the first and second kinds respectively.

Also from the fourth boundary condition of (6) we get

$$C_2 = C_4 = 0.$$

$$R = R_0 = C_1 r \qquad \text{for } \lambda = 0.$$

$$R = C_3 J_1(\lambda r) \qquad \text{for } \lambda > 0. \qquad \ldots(10)$$

Again applying the second boundary condition of (6) we get

$$J_2(\lambda b) = 0. \qquad \ldots(11)$$

The above equation gives a series of values of λ say $\lambda_1, \lambda_2, \lambda_n$, where $\lambda_n = \alpha_n/b$, $n = 1, 2, 3$; α_n being the zeroes of $J_2(x)$. Now to solve equation (9) we introduce the Laplace transform $\bar{Z}$ of Z defined by the relation

$$\bar{Z} = \int_0^\infty e^{-pt} Z(z, t)\, dt.$$

Multiplying both sides of (9) by e^{-pt} and on integration with respect to t from 0 to ∞ we get

$$\frac{d^2\bar{Z}}{dz^2} = \nu^2 \bar{Z},$$

where $$\nu^2 = \lambda^2 + \frac{p^2}{c^2}. \qquad \ldots(12)$$

The soultion of the above equation satisfying fifth boundary condition of (6) is

$$\bar{Z} = C_5 e^{-\nu z} \qquad \ldots(13)$$

If $\bar{\upsilon}$ be the Iaplace transform of υ then

$$\bar{\upsilon} = A_0 r e^{-pz/c} + \sum_{n=1}^{\infty} AnJ_1(\lambda_n r)\, e^{-\nu_n z} \qquad \ldots(14)$$

where

$$\nu_n^2 = \lambda_n^2 + \frac{p^2}{c^2}; \qquad \ldots(15)$$

also $A_0 = C_1C_5$ and A_n is the value of C_3C_5 corresponding to n th root of equation (11).

Taking Laplace transform of second equation of (4)

$$\bar{\tau}_{\theta z} = -A_0 r\mu_0 \left\{k + (1 + kz)\frac{p}{c}\right\} e^{-pz/c}$$

$$- \sum_{n=1}^{\infty} A_n\mu_0 J_1(\lambda_n r)\,(k + \nu_n)\, e^{-2hz} \qquad \ldots(16)$$

The third boundary condition of (6) reduces to

$$\bar{\tau}_{\theta z} = f(r) \qquad \text{on } z = 0. \qquad \ldots(17)$$

From (16) and (17)

$$A_0\mu_0 r\left(k + \frac{p}{c}\right) + \sum_{n=1}^{\infty} A_n\mu_0 J_1(\lambda_n r)\,(k + \nu_n) = -f(r) \qquad \ldots(18)$$

Using the orthogonality relation of Bessel function we have from (18)

$$A_0 = -\frac{2T}{\pi\mu_0 b^4}\frac{1}{(k + p/c)}, \qquad \ldots(19)$$

where

$$T = \int_0^b 2\pi r^2 f(r)\, dr \qquad \ldots(20)$$

and

$$A_n = -\frac{D_n}{k + \nu_n}, \qquad \ldots(21)$$

where

$$D_n = -\frac{2\int_0^b rJ_1(\lambda_n r) f(r)\, dr}{b^2\mu_0\{J_1(\lambda_n b)\}^2}. \qquad \ldots(22)$$

From (14), (19) and (22), the expression for υ takes the form

$$\upsilon = -\frac{2rcT}{\pi\mu_0 b^4}\frac{1}{(p + kc)} e^{-pz/c} - \sum_{n=1}^{\infty} D_n J_1(\lambda_n r) \int_0^{\infty} \exp - (\propto (K + \nu_n)) \exp(-\nu_n z)\, d\propto$$

Taking inverse transform of υ we get the expression for υ as

$$\upsilon = -\frac{2rcT}{\pi\mu_0 b^4} \exp(-KC(t - z/c))\ H(t - z/c) - \sum_{n=1}^{\infty} D_n J_1(\lambda_n r)\,[C e^{-k(ct-z)} - C\lambda_n e^{kz} \int_0^{\sqrt{c^2t^2-z^2}} \exp(-k\sqrt{c^2t^2 - y^2}) \times J_1(\lambda_n y)\, dy \qquad \ldots(23)$$

From the third relation of (4) and (23)

$$\upsilon_\theta = -\frac{2rCT}{\pi\mu_0 b^4(1 + kz)} \exp(-KC(t - Z/c))\ H(t - zc) - \frac{1}{1 + kz}\sum_{n=1}^{\infty} D_n J_1(\lambda_n r) \perp c \exp(-kc(t - z/c)) - C\lambda_n e^{kz} \times \int_0^{\sqrt{c^2t^2-z^2}} \exp(-k\sqrt{c^2t^2 - y^2})\, J_1(\lambda_n y))\, dy \qquad \ldots(24)$$

Form the second relation of (4) and (23) we get

$$\tau_{\theta z} = -\frac{2k^2rcz}{\pi b^4}\exp(-kc(t-z/c) - \sum_{n=1}^{\infty}\mu_0 k^2 cz(J_1(\lambda_n r))\chi.$$

$$\exp(-k(ct-z)) + \sum_{n=1}^{\infty}\mu_0 D_n J_1(\lambda_n r)\, C\lambda_n e^{kz}$$

$$\times\Bigg[\int_0^{\sqrt{c^2t^2-z^2}}(\exp(-k\sqrt{c^2t^2-y^2}))\,J_1(\lambda_n y)$$

$$\times\left\{1+kz+k\sqrt{c^2t^2-y^2}\right\}\frac{dy}{\sqrt{c^2t^2-y^2}}$$

$$+\int_0^{\sqrt{c^2t^2-z^2}}\exp(-k\sqrt{e^2t^2-y^2}\times(1+kz)$$

$$\times\frac{\sqrt{c^2t^2-y^2}}{y}\,J_2(\lambda_n y)\,dy \qquad \ldots(25)$$

Using (23) in the first equation of (4),

$$\tau_{r\theta} = \mu_0(1+kz)\sum_{n=1}^{\infty}D_n\lambda_n J_2(\lambda_n r)\Bigg[C\exp(-k(ct-z))$$

$$-C\lambda_n e^{kz}\int_0^{\sqrt{c^2t^2-z^2}}\exp(-k\sqrt{c^2t^2-y^2})\,J_1(\lambda_n y)\,dy\Bigg] \qquad \ldots(26)$$

Also

$$\frac{du_\theta}{dt} = \frac{1}{1+kz}\Bigg[\frac{2rkc^2T}{\pi\mu_0 b^4}\exp(-k(ct-z)) + H(t-z/c)$$

$$-\sum_{n=1}^{\infty}D_n J_1(\lambda_n r)\Bigg[-kc^2\exp(-k(ct-z)) + c^2\lambda_n^2 te^{kz}$$

$$\times\int_0^{\sqrt{c^2t^2-z^2}}\exp(-k\sqrt{c^2t^2-y^2})\frac{J_1(\lambda_n y)}{y}\,dy. \qquad \ldots(27)$$

Associated Homogeneous Case

By substitution, $K = 0$ in (24) to (26) we get,

$$v_\theta = -\frac{2rCT}{\pi\mu_0 b^4} - \sum_{n=1}^{\infty} CD_n J_1(\lambda_n r) J_1(\lambda_n \sqrt{c^2t^2 - z^2}) \qquad \ldots(28)$$

The result is in complete agreement of the result obtained by Mitra[6]

$$\tau_{\theta z} = \sum_{n=1}^{\infty} \mu_0 D_n J_1(\lambda_n r) \int_0^{\sqrt{c^2t^2 - z^2}} \left[\frac{J_1(\lambda_n y)}{\sqrt{c^2t^2 - y^2}} + \frac{\sqrt{c^2t^2 - y^2}}{y} J_2(\lambda_n y) \right] dy \qquad \ldots(29)$$

$$\tau_{r\theta} = \mu_0 C \sum_{n=1}^{\infty} D_n \lambda_n J_2(\lambda_n r) \left[1 - 2 \sum_{k=0}^{\infty} J_2(\lambda_n \sqrt{c^2t^2 - z^2}) \right] \qquad \ldots(30)$$

$$\frac{du_\theta}{dt} = - \sum_{n=1}^{\infty} D_n J_1(\lambda_n r) C^2 \lambda_n^2 t \int_0^{\sqrt{c^2t^2 - z^2}} \frac{J_1(\lambda_n y)}{y} dy \qquad \ldots(31)$$

Particular Case

When $\quad f(r) = Q(b^2 - r^2)_r^{\lambda}$

and Q is a constant, then,

$$T = \frac{\pi Q b^{2\lambda+4}}{(\lambda + 1)(\lambda + 2)} \qquad \ldots(32)$$

and

$$D_n = \frac{2^{\lambda+1} Q\lambda!\, b^{\lambda+2} J^{\lambda+2}(\lambda_n b)}{\lambda_n^{\lambda+1} b^2 \mu_0 \{J_1(\lambda_n b)\}^2}. \qquad \ldots(33)$$

Numerical Results and Conclusion

To obtain the displacement numerically we take

$$\lambda = 1, \quad \frac{ct}{b} = \sqrt{2}, \frac{z}{b} = 1{\cdot}0, kb = 1{\cdot}0$$

The distribution of $P = -\frac{\mu_0 u_\theta}{4c\theta b^3}$ at different values of $\frac{r}{b}$ are compared graphically with that of associated homogeneous case in Fig. 1. The homogenous medium is a

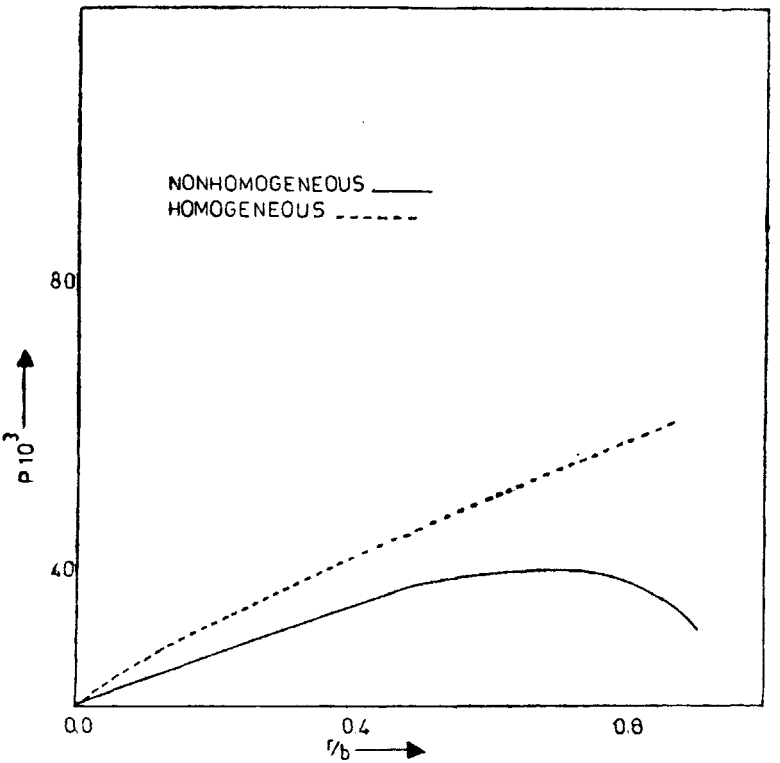

FIG 1

hypothetical case where as seismological observation render support to nonhomogeneity in density and rigidity. In order to check our calculation we recover the previously solved homogeneous case by setting $k = 0$. As is evident from our results in the figure the displacement increases very slowly with the radius in contrast to the homogeneous case where it rises sharply.

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