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TORSIONAL VIBRATION OF NON-HOMOGENEOUS CONE WITH SPHERICAL CAPS

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The frequency equation corresponding to the torsional vibration of a non-homogeneous cone with spherical caps for which the rigidity varies as the *n*th power of the distance from the vertex of the cone has been obtained in the paper.

Key Words : Torsional Vibration; Non-homogeneous Cone; Spherical Cap

INTRODUCTION

THE problem of equilibrium and vibration of non-homogeneous bodies has been studied by many investigators like Olszak and Urbanowski,[1,2] Chatterjee,[3] Mukherjee[4] and Bhanja[5] for various types of non-homogeneity. Mukherjee[4] studied the torsional vibration of a non-homogeneous cone with spherical caps for which the rigidity varies as the square of the distance from the vertex of the cone and density is constant. Maiti[6] discussed the same problem as that of Mukherjee[4] and solved the frequency equation correctly since unfortunately Mukherjee obtained erroneous results for the displacements, stresses, periods etc.

In this paper, the author has studied the torsional vibration of a non-homogeneous cone with spherical caps for which the rigidity varies as the *n*th power of the distance from the vertex of the cone and the density is constant. Also as the explanation given by Maiti[6] seems to be not proper it was thought worthwhile to discuss the problem of Maiti as a particular case.

FUNDAMENTAL EQUATIONS

We take vertex of the cone as the origin, the axis of the cone coinciding with the axis of the co-ordinate system. We use spherical polar co-ordinates (r, θ, φ). The stress equations of motion, *vide* Love[7] are then given by

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\widehat{rr} + \frac{1}{r}\frac{\partial}{\partial \theta}\widehat{r\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\widehat{r\varphi} + \frac{1}{r}\left[2\widehat{rr} - \widehat{\theta\theta} - \widehat{\varphi\varphi} + \widehat{r\theta}\cot\theta\right] \\
&\qquad = \rho\,\frac{\partial^2 u_a}{\partial t^2}, \\
&\frac{\partial}{\partial r}\widehat{r\theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\widehat{\theta\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\widehat{\theta\varphi} + \frac{1}{r}\left[(\widehat{\theta\theta} - \widehat{\varphi\varphi})\cot\theta + 3\widehat{r\theta}\right] \\
&\qquad = \rho\,\frac{\partial^2 u\theta}{\partial \theta^2}
\end{aligned}\right\} \ldots(1)$$

(equation cont.)

and

$$\left.\begin{aligned}\frac{\partial}{\partial r}\widehat{r\varphi}+\frac{1}{r}\frac{\partial}{\partial\theta}\widehat{\theta\varphi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\widehat{\varphi\varphi}+\frac{1}{r}[3\widehat{r\varphi}+2\widehat{\theta\varphi}\cot\theta]\\ =\rho\frac{\partial^2 u\varphi}{\partial t^2},\end{aligned}\right]$$

where u_r, $u\theta$, $u\varphi$ are the Components of displacement in the directions or (r, θ, ϕ) respectively, 't' is the time and ρ is the density.

Solution of the Problem

For the torsional vibration of a non-homogeneous cone we assume,

$$\left.\begin{aligned}&u_r=0,\quad u\theta=0\\ &u\varphi=f(r)\sin\theta\, e^{ipt}\\ \text{and}\quad &\mu=\mu_0 r^n,\quad \rho=\rho_0,\end{aligned}\right\}\qquad\ldots(2)$$

where μ_0, ρ_0 being constants.

All the stress-components except $\widehat{r\varphi}$ are zero.

Now

$$\widehat{r\varphi}=\mu_0 r^n\left[f'(r)-\frac{f(r)}{r}\right]\sin\theta e^{ipt}.\qquad\ldots(3)$$

Since

$$\widehat{\theta\varphi}=\mu_0 r^{n-1}\left[\frac{\partial u\varphi}{\partial\theta}-u\varphi\cot\theta\right]=0,$$

the conical surface is free from stress.

Then the two equations of motion are satisfied identically and the remaining one reduces to

$$\frac{d^2 f}{dr^2}+\frac{n+2}{r}\frac{df}{dr}+\left[-(n+2)+\frac{\rho_0 p^2}{\mu_0}\frac{1}{r^{n-2}}\right]\frac{f}{r^2}=0\qquad\ldots(4)$$

When $n\neq 2$, applying the transformation,

$$\left.\begin{aligned}z&=\frac{2}{2-n}\sqrt{\frac{\rho_0 p^2}{\mu_0}}\cdot\ r^{2-n/2}\\ f&=vr^{-(n+1)/2}\end{aligned}\right\}\qquad\ldots(5)$$

we get from (4),

$$\frac{d^2 v}{dz^2}+\frac{1}{z}\frac{dv}{dz}+\left(1-\frac{v^2}{z^2}\right)v=0,\qquad\ldots(6)$$

where

$$\nu = \frac{n+3}{2-n}. \qquad \ldots(7)$$

Solution of equation (6) is given by

$$v = AJ_\nu(z) + BY_\nu(z), \qquad \ldots(8)$$

where A and B are arbitrary constants.

Therefore,

$$f = r^{-(n+1)/2}\,[AJ_\nu(qr^{(2-n)/2}) + BY_\nu(qr^{(2-n)/2})] \qquad \ldots(9)$$

and
$$u_\varphi = r^{-(n+1)/2}\,[AJ_\nu(qr^{(2-n)/2}) + BY_\nu(qr^{(2-n)/2})]\sin\theta e^{ipt}, \qquad \ldots(10)$$

where

$$q = \frac{}{2-n}\sqrt{\frac{\rho_0 p^2}{\mu_0}}.$$

Now

$$\widehat{r\varphi} = \frac{\mu_0}{2} r^{(n-3)/2}\,[-(n+3)\{AJ_\nu(qr^{(2-n)/2}) + BY_\nu(qr^{(2-n)/2})\}$$
$$+ (2-n)\{A\nu J_\nu(qr^{(2-n)/2}) - Aqr^{(2-n)/2} J_{\nu+1}(qr^{(2-n)/2})$$
$$+ B\nu Y_\nu(qr^{(2-n)/2}) - Bqr^{(2-n)/2} \times Y_{\nu+1}(qr^{(2-n)/2})\}]\sin\theta e^{ipt} \qquad \ldots(11)$$

Case I : *End Caps Fixed*

The boundary conditions are,

$$u_\varphi = 0 \quad \text{on } r = a \text{ and } r = b, \qquad \ldots(12)$$

where a and b are the radii of the spherical caps.

Therefore, from (10) we get,

$$\left.\begin{aligned} AJ_\nu(qa^{(2-n)/2}) + BY_\nu(qa^{(2-n)/2}) &= 0 \\ AJ_\nu(qb^{(2-n)/2} + BY_\nu(qb^{(2-n)/2}) &= 0 \end{aligned}\right\} \qquad \ldots(13)$$

Eliminating A and B from (13), we get the frequency equation as,

$$J_\nu(qa^{(2-n)/2})\,Y_\nu(qb^{(2-n)/2}) - J_\nu(qb^{(2-n)/2})\,Y_\nu(qa^{(2-n)/2}) = 0 \qquad \ldots(14)$$

Numerical Calculations

Eqn. (14) reduces to the form

$$J_\nu(x_0)\,Y_\nu(k'x_0) - J_\nu(k'x_0)\,Y_\nu(x_0) = 0, \qquad \ldots(14a)$$

where

$$\left.\begin{aligned} x_0 &= qa^{(2-n)/2} \\ k' &= \left(\frac{b}{a}\right)^{(2-n)/2} \end{aligned}\right\} \qquad \ldots(14b)$$

The first six roots of the equation (14[a]) for different values of K^1 and different values of ν have been tabulated below (Jahnke and Emde[7])

K^1	$x_0^{(1)}$	$x_0^{(2)}$	$x_0^{(3)}$	$x_0^{(4)}$	$x_0^{(5)}$	$x_0^{(6)}$	ν
1,2	15,7014	31,4126	47,1217	62,8302	78,5385	94,2467	$\nu = 0$
1,5	6,2702(−0,3)	12,5598	18,8451	25,1294	31,4133	37,6969	whnn
2,0	3,1230(−6,2)	6,2734(−0,2)	9,4182	12,5614	15,7040	18,8462	$n = 0$
1,2	15,7080	31,4159	47,1239	62,8319	78,5398	94,2478	$\nu = 1/2$
1,5	6,2832	12,5664	18,8496	25,1327	31,4159	37,6991	when
2,0	3,1416	6,2832	9,4248	12,5664	15,7080	18,8496	$n = -4/3$
1,2	15,7277	31,4259	47,1305	62,8368	78,5438	94,2511	$\nu = 1$
1,5	6,3218(+0,9)	12,5861	18,8628	25,1427	31,4239	37,7057	when
2,0	3,1966(+15,7)	6,3124(−0,5)	9,4445	12,5812	15,7199	18,8595	$n = -1/2$
1,2	15,7607	31,4424	47,1416	62,8451	78,5504	94,2566	$\nu = 3/2$
1,5	6,3858(+2,2)	12,6190	18,8848	25,1592	31,4371	37,7168	when
2,0	3,2866(+33,6)	6,3607(+1,0)	9,4772(+0,1)	12,6059	15,7397	18,8760	$n = 0$
1,2	15,8066	31,4656	47,1570	62,8567	78,5597	94,2644	$\nu = 2$
1,5	6,4742(+3,9)	12,6648(+0,1)	18,9156	25,1823	31,4556	37,7322	when
2,0	3,4063(+45,6)	6,4277(−1,4)	9,5228(+0,2)	12,6404	15,7673	18,8991	$n = 1/3$
1,2	15,8655(+0,2)	31,4953	47,1769	62,8716	78,5716	94,2743	$\nu = 5/2$
1,5	6,5860(+6,8)	12,7235(+0,2)	18,9551	25,2121	31,4795	37,7521	when
2,0	3,5514(+49,6)	6,5130(+1,6)	9,5813(+0,2)	12,6846	15,8029	18,9288	$n = 4/7$

Case II : *One End Cap Fixed, The Other End Free*

The boundary conditions are,

$$\left.\begin{aligned} u\phi &= 0 \quad \text{on } r = a \\ \text{and} \qquad r\varphi &= 0 \quad \text{on } r = b. \end{aligned}\right\} \qquad \ldots(15)$$

Therefore from (10), (11) and (15) we get,

$$AJ_\nu(qa^{(2-n)/2}) + BY_\nu(qa^{(2-n)/2}) = 0 \qquad \ldots(16)$$

and

$$\begin{aligned} & A\,[-(n+3)\,J_\nu(qb^{(2-n)/2}) + (2-n)\,\nu J_\nu(qb^{(2-n)/2}) \\ & \quad -(2-n)\,qb^{(2-n)/2}\,J_{\nu+1}(qb^{(2-n)/2})] \\ & + B\,[-(n+3)\,Y_\nu(qb^{(2-n)/2}) + (2-n)\,\nu Y_\nu(qb^{(2-n)/2}) \\ & \quad -(2-n)\,qb^{(2-n)/2}\,Y_{\nu+1}(qb^{(2-n)/2})] = 0 \end{aligned} \qquad \ldots(17)$$

Eliminating A and B from (16) and (17) we get the frequency equation as,

$$\begin{aligned} & J_\nu(qa^{(2-n)/2})\,\{(n+3)\,Y_\nu(qb^{(2-n)/2}) + (2-n)\,qb^{(2-n)/2}\,Y_{\nu+1}(qb^{(2-n)/2}) \\ & \quad -(2-n)\,\nu Y_\nu(qb^{(2-n)/2})\} \\ & - Y_\nu(qa^{(2-n)/2})\,\{(n+3)\,J_\nu(qb^{(2-n)/2}) + (2-n)\,qb^{(2-n)/2}\,J_{\nu+1}(qb^{(2-n)/2}) \\ & \quad -(2-n)\,\nu J_\nu(qb^{(2-n)/2})\} = 0 \end{aligned} \qquad \ldots(18)$$

Numerical Calculations

If $n = -3$, equation (18) reduces to the form,

$$J_0(x_0') Y_1(k^{\parallel} x_0') - Y_0(x_0) J_1(k^{\parallel} x_0') = 0, \qquad \ldots(18a)$$

where

$$\left.\begin{aligned} x_0^1 &= qa^{5/2} \\ k^{\parallel} &= \left(\frac{b}{a}\right)^{5/2} \end{aligned}\right\} \qquad \ldots(18b)$$

The first roots of the equation (18*a*) for different values of $k^{\parallel}$ have been tabulated below [Bogert[8]]

$k^{\parallel}$	1.05	1.25	1.5	1.75	2	3	4
x_0'	31.10	6.003	2.89	1.866	1.361	0.6256	0.3935
$k^{\parallel}$	5	6	7	8	9	10	20
x_0'	0.2824	0.2181	0.1765	0.1476	0.1276	0.1103	0.0465

Case III : *Both End Caps Being Free*

The boundary condition are,

$$\widehat{r\varphi} = 0 \quad \text{on } r = a \text{ and } r = b. \qquad \ldots(19)$$

Therefore, from (11) and (19) we get,

$$\begin{aligned} A\,[&-(n+3)\,J_\nu(qa^{(2-n)/2}) + (2-n)\,\nu J_\nu(qa^{(2-n)/2}) \\ &-(2-n)\,qa^{(2-n)/2}\,J_{\nu+1}(qa^{(2-n)/2})] \\ +\,B\,[&-(n+3)\,Y_\nu(qa^{(2-n)/2}) + (2-n)\,\nu Y_\nu(qa^{(2-n)/2}) \\ &-(2-n)\,qa^{(2-n)/2}\,Y_{\nu+1}(qa^{(2-n)/2})] = 0 \end{aligned} \qquad \ldots(20)$$

and

$$\begin{aligned} A\,[&-(n+3)\,J_\nu(qb^{(2-n)/2}) + (2-n)\,\nu J_\nu(qb^{(2-n)/2}) \\ &-(2-n)\,qb^{(2-n)/2}\,J_{\nu+1}(qb^{(2-n)/2})] \\ +\,B\,[&-(n+3)\,Y_\nu(qb^{2-n/2}) + (2-n)\,\nu Y_\nu(qb^{2-n/2}) \\ &-(2-n)\,qb^{(2-n)/2}\,Y_{\nu+1}(qb^{(2-n)/2})] = 0 \end{aligned} \qquad \ldots(21)$$

Eliminating A and B from (20) and (21) we get the frequency equation as,

$$\begin{aligned} &[(n+3)\,J_\nu(qa^{(2-n)/2}) + (2-n)\,qa^{(2-n)/2}\,J_{\nu+1}(qa^{(2-n)/2}) \\ &\quad -(2-n)\,\nu J_\nu(qa^{(2-n)/2})] \\ &\times [(n+3)\,Y_\nu(qb^{(2-n)/2}) + (2-n)\,qb^{(2-n)/2}\,Y_{\nu+1}(qb^{(2-n)/2}) \\ &\quad -(2-n)\,\nu Y_\nu(qb^{(2-n)/2})] \\ &-[(n+3)\,Y_\nu(qa^{(2-n)/2}) + (2-n)\,qa^{2-n/2}\,Y_{\nu+1}(qa^{(2-n)/2}) \\ &\quad -(2-n)\,\nu Y_\nu(qa^{(2-n)/2})] \end{aligned}$$

$$\times [(n+3)\, J_\nu(qb^{(2-n)/2}) + (2-n)\, qb^{(2-n)/2}\, J_{\nu+1}(qb^{(2-n)/2}) - (2-n)\, \nu J_\nu(qb^{2-n/2})] = 0 \quad \ldots(22)$$

Numerical Calculations

If $n = -3$, equation (22) reduces to the form

$$J_1(x_0')\, Y_1(k^{\|}x_0') - Y_1(x_0')\, J_1(k^{\|}x_0') = 0, \quad \ldots(22a)$$

where x_0' and $k^{\|}$ are given by (18*b*).

The first six roots of the equation (22*a*) for different values of $k^{\|}$ have been tablated below [Jahnke and Emde[7]].

$k^{\|}$	$x_0'^{(1)}$	$x_0'^{(2)}$	$x_2'^{(3)}$	$x_0'^{(4)}$	$x_0'^{(5)}$	$x_0'^{(6)}$
1,2	15,7277	31,4259	47,1305	62,8368	78,5438	94,2511
1,5	6,3218(+0,9)	12,5861	18,8628	25,1427	31,4239	37,7057
2,0	3,1966(+15,7)	6,3124(+0,5)	9,4445	12,5812	15,7199	18,8595

Particular Case

Let us now discuss the case when $n = 2$ as a particular case.[6]

The differential equation (4) reduces to

$$\frac{d^2f}{dr^2} + \frac{4}{r}\frac{df}{dr} + \left[-4 + \frac{\rho_0 p^2}{\mu_0}\right]\frac{f}{r^2} = 0 \quad \ldots(23)$$

on substitution $f = vr^{-3/2}$ the equation (23) further reduces to,

$$r^2\frac{d^2v}{dr^2} + r\frac{dv}{dr} + \lambda^2 v = 0, \quad \ldots(24)$$

where

$$\lambda^2 = \frac{\rho_0 p^2}{\mu_0} - \frac{25}{4} \quad \ldots(25)$$

The solution of equation (24) is

$$v = C\cos(\lambda \log r) + D\sin(\lambda \log r), \quad \ldots(26)$$

where C and D are constants. Thus, we get

$$f = r^{-3/2}[C\cos(\lambda \log r) + D\sin(\lambda \log r)] \quad \ldots(27)$$

Hence,

$$u\phi = r^{-3/2}[C\cos(\lambda \log r) + D\sin(\lambda \log r)]\sin\theta e^{ipt} \quad \ldots(28)$$

and

$$\widehat{r\phi} = \mu_0 r^{-1/2}[-C\{\tfrac{5}{2}\cos(\lambda \log r) + \lambda \sin(\lambda \log r)\} + D\{\lambda\cos(\lambda \log r) - \tfrac{5}{2}\sin(\lambda \log r)\}]\sin\theta e^{ipt} \quad \ldots(29)$$

Case I : *End Caps Fixed*

The boundary conditions are given by eq. (12). Therefore, from (28) we get,

$$\left.\begin{aligned} &C \cos(\lambda \log a) + D \sin(\lambda \log a) = 0 \\ \text{and} \quad &C \cos(\lambda \log b) + D \sin(\lambda \log b) = 0 \end{aligned}\right\} \quad \ldots(30)$$

Eliminating the constants C and D we get,

$$\lambda = \frac{n\pi}{k}, \; n = 0, \pm 1, \pm 2, \ldots\ldots$$

where $k = \log(a/b)$, a real constant.

This implies that

$$p = \sqrt{\frac{\mu_0}{\rho_0}\left(\frac{25}{4} + \frac{n^2\pi^2}{k^2}\right)}, \text{ a real quantity.}$$

Case II : *One End Cap Fixed, The Other End Free*

The boundary conditions are given by equation (15). Therefore, from (28) and (29) we get,

$$C \cos(\lambda \log a) + D \sin(\lambda \log a) = 0$$

and

$$-C\{\tfrac{5}{2}\cos(\lambda \log b) + \lambda \sin(\lambda \log b)\} + D\{\lambda \cos(\lambda \log b) - \tfrac{5}{2}\sin(\lambda \log b)\} = 0$$

Eliminating C and D we get,

$$\tan \lambda k = -\frac{2\lambda}{5},$$

where

$$k = \log\left(\frac{a}{b}\right).$$

This is the frequency equation which gives denumerable infinite number of complex roots of λ and the corresponding periods (real) can be found out.

Case III : *Both End Caps Free*

The boundary conditions are given by equation (19). Using (19), we get from (29),

$$-C\{\tfrac{5}{2}\cos(\lambda \log a) + \lambda \sin(\lambda \log a)\} + D\{\lambda \cos(\lambda \log a) - \tfrac{5}{2}\sin(\lambda \log a)\} = 0$$

and

$$-C\{\tfrac{5}{2}\cos(\lambda \log b) + \lambda \sin(\lambda \log b)\} + D\{\lambda \cos(\lambda \log b) - \tfrac{5}{2}\sin(\lambda \log b)\} = 0$$

Eliminating the constants C and D we get,

$$\lambda = \frac{n\pi}{k}, \; n = 0, \pm 1, \pm 2, \ldots\ldots$$

This implies that,

$$p = \sqrt{\frac{\mu_0}{\rho_0}\left(\frac{25}{4} + \frac{n^2\pi^2}{k^2}\right)}$$

which is the same as that obtained in the Case I.

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