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# COUETTE FLOW WITH POROUS WALLS OF A DILUTE SUSPENSION

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The suspension is assumed to be consisting of a solvent, which is a Newtonian fluid and small rigid spheroidal suspended particles. By using the consititutive equations for the suspension derived by Shmakov and Taran[4,5,8] and the distribution function of Scheraga,[6,7] the problem of the generalized Couette flow between two porous walls of the suspension is solved. Expression for the velocity distributions and for the viscosity of the suspension are obtained.

**Key Words : Couette Flow; Porous; Newtonian; Shmakov and Taran Suspension; Scheraga Distribution Function; Navier-Stokes Equations; Brownian Motion; Legendre Polynomial**

## INTRODUCTION

THE studies of the structure of macromolecules in dilute solution are largely by the effective viscosity dependent on flow parameters. The rheological properties of a rigid particle suspension and the effect of the Brownian motion on the rheological properties have already been discussed elsewhere.[1-3] Schmakov, Taran and Begoulev[4,5] studied the flow of the suspension between two impermeable parallel walls by using the numerical values of the distribution function of Scheraga.[6,7]

In the present work, the suspended particles are taken to be either in a complete Brownian motion ($\sigma = 0$) or nearly spherical ($\lambda \ll 1$) having Brownian motion. The flow of the suspension will be due to the uniform velocity of the upper wall together with a constant pressure gradient in the direction of the flow. Under these conditions the generalized Couette flow between two porous walls of a dilute suspension is studied.

## THE CONSTITUTIVE EQUATIONS OF THE SUSPENSION

For a suspension of spheroidal particles immersed in a viscous fluid, taking into account the rotary Brownian motion of the suspended particles, the rheological equations are given by Schmakov and Taran[8] as

$$t_{ij} = -p\delta_{ij} + 2\mu d_{ij} + \mu_1 < n_i n_j > + \mu_2 d_{km} < n_k n_m n_i n_j > + 2\mu_3 (d_{jk} < n_k n_i > + d_{ik} < n_k n_j >) \quad \ldots(1)$$

and

$$\dot{n}_i = w_{ij} n_j + \lambda (d_{ij} n_j - d_{km} n_k n_m n_i), \quad \ldots(2)$$

where $t_{ij}$ is the stress tensor, $p$ is the isotropic pressure, $\delta_{ij}$ is the Kronecker delta, $\mu$, $\mu_1$, $\mu_2$ and $\mu_3$ are rheological constants, $d_{ij}$ and $w_{ij}$ are the deformation rate tensor and the vortex tensor defined by

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ and } w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_j}\right), \qquad \ldots(3)$$

$\dot{n}_i$ are the velocity components, $\lambda = \dfrac{a^2 - b^2}{a^2 + b^2}$ where $a$ and $b$ respectively are the symmetry semiaxes and equatorial radius of the spheroidal suspended particles, $n_i$ are the components of the unit orientation vector directed along the major axis of the suspended particle and which has the cartesian components :

$$n_1 = \sin\theta\cos\phi,\ n_2 = \sin\theta\sin\phi,\ n_3 = \cos\theta, \qquad \ldots(4)$$

$n_i$ are the material derivative of $n_i$; the symbols $< >$ represent the mean particle orientation factors, where for any orientation $\alpha$ of the suspended particles they are defined by :

$$< \alpha > = \iint_{\Omega} \alpha F \, d\Omega, \qquad \ldots(5)$$

$d\Omega = \sin\theta d\theta d\phi$ is the volume element in spherical coordinates, $F$ is a distribution function given by Scheraga[7] as

$$F = \sum_{j=0}^{\infty} \lambda^j F_j = \sum_{j=0}^{\infty} \lambda^j \left[\tfrac{1}{2} \sum_{n=0}^{j} a_{n_o,j}\, P_{2n}(\cos\theta) + \sum_{n=1}^{r} \sum_{m=1}^{n} (a_{nm,j} \cos 2m\phi + b_{nm,j} \sin 2m\phi)\ P_{2n}^{2m}(\cos\phi)\right], \qquad \ldots(6)$$

where $P_{2n}(\cos\theta)$ is a Legendre polynomial of the first kind and

$$P_{2n}^{2m}(\cos\theta) = \sin\theta^{2m} \frac{d^{2m}}{(d\cos\theta)^{2m}} P_{2n}(\cos\theta).$$

The coefficient $a_{nm,j}$ and $b_{nm,j}$ are given in appendix (A).[6]

For generalized Couette flow with porous walls, the velocity distribution is given by[9]

$$u = u(y),\ v = v_0 = \text{const.} \qquad \ldots(7)$$

In this case and for nearly spherical particles the parameter $\lambda$ is very small and the distribution function $F$, given by equation (6), can be expanded in powers of $\lambda$ up to $\lambda^2$ to give

$$F = F_0 + \lambda F_1 + \lambda^2 F_2,$$

where $F_0$, $F_1$ and $F_2$ are calculated from equation (6) as

$$F_0 = \tfrac{1}{2} a_{00,0}\, P_0,$$

$$F_1 = \tfrac{1}{2}\left[a_{00,1}\, P_0 + a_{10,1}\, P_2\right] + \left[a_{11,1}\cos 2\varphi + b_{11,1}\sin 2\varphi\right] P_2^2$$

and

$$F_2 = \tfrac{1}{2}[a_{00,2}P_0 + a_{10,2}P_2 + a_{20,2}P_4] + [a_{11,2}\cos 2\phi + b_{11,2}\sin 2\phi]\,P_2^2$$

$$+ [a_{21,2}\cos 2\varphi + b_{21,2}\sin 2\varphi]\,P_4^2 + [a_{22,2}\cos 4\varphi + b_{22,2}\sin 4\varphi]\,P_4^4 .$$

From the recurrence formulas for $a_{nm,s}$ and $b_{nm,s}$ which are given in appendix (A), we obtain :

$$a_{00,0} = \frac{1}{2\pi},\ a_{00,1} = a_{10,1} = a_{00,2} = a_{11,2} = a_{21,2} = b_{11,2} = b_{21,2} = 0,$$

$$a_{11,1} = \frac{-\sigma^2}{8\pi(\sigma^2+36)},\ b_{11} = \frac{3\sigma}{4\pi(\sigma^2+36)},$$

$$a_{10,2} = \frac{-3\sigma^2}{14\pi(\sigma^2+36)},\ a_{20,2} = \frac{9\sigma^2}{140\pi(\sigma^2+36)},$$

$$b_{22,2} = \frac{-\sigma^3}{28\pi(\sigma^2+36)(\sigma^2+100)},\ a_{22,2} = \frac{\sigma^2(\sigma^2-60)}{448\pi(\sigma^2+36)(\sigma^2+100)}.$$

Also the Legendre polynomials $P_{2n}(\cos\theta)$ and $P_{2n}^{2m}(\cos\theta)$ can be written in the form :

$$P_0(\cos\theta) = 1,\ P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1),$$

$$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3),\ P_2^2(\cos\theta) = 3\sin^2\theta,$$

$$P_4^4(\cos\theta) = 105\sin^4\theta.$$

Using the values of $a_{00,0}$, $a_{00,1},\ldots a_{22,2}$ and $P_0, P_2,\ldots, P_4^4$, the distribution function $F$ can be written in the form :

$$F = \frac{1}{4\pi}\Big[1 + \lambda\left(\frac{-1}{2}\cos 2\varphi + \frac{3}{\sigma}\sin 2\varphi\right)\frac{3\sigma^2\sin^2\theta}{\sigma^2+36}$$

$$+ \lambda^2\left\{\frac{-3}{14}(3\cos^2\theta - 1) + \frac{9}{560}(35\cos^4\theta - 30\cos^2\theta + 3)\right.$$

$$\left.+\left(\left(1 - \frac{60}{\sigma^2}\right)\cos 4\varphi - \frac{16}{\sigma}\sin 4\varphi\right)\frac{15\sigma^2\sin^4\theta}{16(\sigma^2+100)}\right\}$$

$$\left(\frac{\sigma^2}{\sigma^2+36}\right) + \lambda^3\{\ldots\} + \ldots]. \qquad \ldots(8)$$

Equation (8) is exactly the same as that obtained by Saito.[10] The rotary Péclet number $\sigma$ is given by $\sigma = \left(\frac{du}{dy}\right)\Big/ D$ where $D = kT/\mu VC$ is the rotary diffusion

coefficient, $k$ is the Boltzmann constant, $T$ is the absolute temperature, $\mu_0$ is the dynamic viscosity of the solvent, $V$ is the volume of the suspended particle and $C$ is a function depending on the axes ratio $r_e = a/b$ and is given by :

$$C = \frac{4(r_e^4 - 1)\sqrt{r_e^2 - 1}}{r_e[(2r_e^2 - 1)\ln(r_e + \sqrt{r_e^2 - 1}) - r_e\sqrt{r_e^2 - 1}]}, \quad r_e > 1$$

Using equations (4), (5) and (8), the mean particle orientation factors are given by :

$$< n_1 n_2 > = \tfrac{1}{2} < \sin^2\theta \sin 2\varphi > = \int_0^{2\pi}\int_0^{\pi} F \sin 2\varphi \sin^3\theta \, d\theta d\varphi$$

$$= \frac{\sigma\lambda\sigma}{5(\sigma^2 + 36)} .$$

Similarly,

$$< n_1^2 n_2^2 > = \frac{1}{15} + \frac{\lambda^2\sigma^2}{\sigma^2 + 36}\left[\frac{3}{350} - \frac{(\sigma^2 - 60)}{42(\sigma^2 + 100)}\right]$$

and

$$< n_1^2 + n_2^2 > = \frac{2}{3} + \frac{2\lambda^2\sigma^2}{35(\sigma^2 + 36)} . \qquad ...(9)$$

The component of the stress tensor $t_{xy}$ in view of equations (1), (3), (7) and (9) takes the form

$$t_{xy} = \mu_{eff}\left(\frac{du}{dy}\right), \qquad ...(10)$$

where $\mu_{eff}$ is the effective viscosity of the suspension which is given by

$$\mu_{eff} = \mu_0(1 + \nu_1\Phi), \qquad ...(11)$$

$\mu_0$ is the viscosity of the solvent. $\Phi$ is the volume concentration of the suspended particles and $\nu_1$ is the viscosity factor

$$\nu_1 = A_1 + \frac{A_2}{\sigma^2 + 36} + \frac{A_3}{\sigma^2 + 100}, \qquad ...(12)$$

where

$$A_1 = \tilde{\mu} + \frac{\tilde{\mu}_2}{15} + \frac{2\tilde{\mu}_3}{3} + \lambda^2\left(\frac{2\tilde{\mu}_3}{35} - \frac{8\tilde{\mu}_2}{525}\right),$$

$$A_2 = \frac{6\lambda\tilde{\mu}_1}{5} - \frac{\lambda^2}{35}\left(\frac{279\,\tilde{\mu}_2}{5} + 72\,\tilde{\mu}_3\right), \quad \tilde{\mu} = \frac{\mu - \mu_0}{\mu_0\Phi},$$

$$A_2 = \frac{125}{21}\lambda^2\tilde{\mu}_2, \quad \tilde{\mu}_i = \frac{\mu_i}{\mu_0\Phi}\,(i = 2, 3) \text{ and } \tilde{\mu}_1 = \frac{\mu}{\mu_0\Phi D} .$$

For suspended particles in complete Brownian motion the viscosity factors for prolate and oblate spheroids are given by[11]

$$\nu_2 = \frac{14}{15} + \frac{r_e^2}{5}\left[\frac{1}{3(\ln 2r_e - 1.5)} + \frac{1}{(\ln 2r_e - 0.5)}\right], \; re = \frac{a}{b} > 1, \qquad ...(13)$$

and $$\nu_3 = 2.5 + \frac{32}{15\pi}\left(\frac{1}{r_s} - 1\right) - \frac{0.628(1 - r_s)}{(1 - 0.075r_s)}, \; r_s < 1, \qquad \ldots(14)$$

respectively.

## Governing Equations and Their Solution

Consider a steady flow of a dilute suspension between two parallel plates, the upper one is in uniform motion with velocity $V$ and the other is at rest with uniform injection of the same suspension at the stationary plate and a corresponding uniform suction at the moving plate. Navier-Stokes equations for the suspension reduce to[9]

$$\rho v_0 \frac{du}{dy} = \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y}, \qquad \ldots(15)$$

$$\frac{\partial t_{yy}}{\partial y} = 0. \qquad \ldots(16)$$

From equation (15), it can be seen that $\left(\frac{\partial p}{\partial x}\right)$ is a function of $y$ only i.e.

$$\frac{\partial p}{\partial y} = F_1(y) \text{ (say)} \qquad \ldots(17)$$

Similarly, from equation (16), it can be seen that $\left(\frac{\partial p}{\partial y}\right)$ is a function of $y$ only i.e.

$$\frac{\partial p}{\partial y} = F_2(y) \quad \text{(say)} \qquad \ldots(18)$$

Integrating equation (18) with respect to $y$ and then differentiating with respect to $x$ gives that $\left(\frac{\partial p}{\partial x}\right)$ is a function of $x$ only i. e.

$$\frac{\partial p}{\partial x} = F_3(x). \qquad \ldots(19)$$

Comparing equations (17) and (19), we can deduce that

$$\frac{\partial p}{\partial x} = \text{const.} \qquad \ldots(20)$$

Hence, $\left(\frac{\partial p}{\partial x}\right)$ = const. as in the case of Newtonian fluid.

From equation (18), the suspension differs form Newtonian fluid in the property that $\frac{\partial p}{\partial y} \neq 0$, i.e. the pressure changes along any cross-section perpendicular to the walls.

In view of equations (1), (10) and (15), the equation of motion of the suspension reduces to

$$\rho v_0 \frac{du}{dy} = -\frac{\partial p}{\partial x} + \frac{d}{dy}\left[\mu_{eff} \frac{du}{dy}\right]. \qquad \ldots(21)$$

Introducing the dimensionless quantities $\overset{\bullet}{u}$, $\eta$ and $\mu^{*}_{\bullet\text{eff}}$, defined by

$$u = U\overset{\bullet}{u},\ y = h\eta \text{ and } \mu_{\text{eff}} = \mu_0\, \overset{\bullet}{\mu}_{\text{eff}},$$

where $h$ is the distance between the two walls. Equation (21) written in the dimensionless form gives

$$\frac{\mathrm{d}\overset{\bullet}{u}}{\mathrm{d}\eta} = P + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\overset{\bullet}{\mu}_{\text{eff}}\frac{\mathrm{d}\overset{\bullet}{u}}{\mathrm{d}\eta}\right), \qquad \ldots(22)$$

where the dimensionless pressure gradient $P$ and the Reynolds number $R$ are given by

$$P = -\frac{h}{\rho\nu_c U}\left(\frac{\partial P}{\partial x}\right) \text{ and } R = \frac{h\nu_0\rho}{\mu_0}. \qquad \ldots(23)$$

The effective viscosity $\mu_{\text{eff}}$ which is given by equation (11) can be written in the form

$$\overset{\bullet}{\mu}_{\text{eff}} = 1 + \Phi\nu_1 \qquad \ldots(24)$$

From equation (24), we can try a solution of equation (22) for the velocity in the form

$$\overset{\bullet}{u} = \overset{\bullet}{u}_0 + \Phi\,\overset{\bullet}{u}_s, \qquad \ldots(25)$$

where $\overset{\bullet}{u}_0$ is the velocity of the solvent and $\overset{\bullet}{u}_s$ is the perturbation in the velocity due to the presence of the suspended particles. Substituting equations (24) and (25) in equation (22), the equations giving the zeroth and first perturbations of $\overset{\bullet}{u}$ with the corresponding boundary conditions are

$$\frac{\mathrm{d}\overset{\bullet}{u}_0}{\mathrm{d}\eta} = P + \frac{1}{R}\frac{\mathrm{d}^2\overset{\bullet}{u}_0}{\mathrm{d}\eta^2}, \qquad \ldots(26)$$

$$\overset{\bullet}{u}_0(0) = 0,\ \overset{\bullet}{u}_0(1) = 1, \qquad \ldots(27)$$

$$\frac{\mathrm{d}\overset{\bullet}{u}_s}{\mathrm{d}\eta} = \frac{1}{R}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\nu_1\frac{\mathrm{d}\overset{\bullet}{u}_0}{\mathrm{d}\eta}\right) + \frac{\mathrm{d}^2\overset{\bullet}{u}_s}{\mathrm{d}\eta^2}\right], \qquad \ldots(28)$$

$$\overset{\bullet}{u}_s(0) = 0,\ \overset{\bullet}{u}_s(1) = 0, \qquad \ldots(29)$$

equation (26) subject to the boundary conditions (27) has the solution

$$\overset{\bullet}{u}_0 = \frac{1}{R}(P - C_1) + P\eta + C_2 e^{R\eta}, \qquad \ldots(30)$$

where the constants of integration $C_1$ and $C_2$ are given by

$$C_2 = \frac{1 - P}{e^R - 1} \text{ and } C_1 = P + RC_2. \qquad \ldots(31)$$

In view of (30), equation (28) can be integrated once with respect to $\eta$ to give

$$\frac{d\overset{*}{u}_s}{d\eta} - R\overset{*}{u}_s = C_3 - \left(A_1 + \frac{A_2}{36 + \sigma_0^2} + \frac{A_3}{100 + \sigma_0^2}\right)(RC_2 e^{R\eta} + P) \quad \ldots(32)$$

$C_3$ is the constant of integration and

$$\sigma_0 = \frac{1}{D}\frac{du_0}{dy} = B_1 e^{R\eta} + B_2, \quad \ldots(33)$$

where

$$B_1 = \frac{URC_2}{hD} \text{ and } B_2 = \frac{UP}{hD}.$$

In view of (33), equation (32) can be integrated to give

$$\overset{*}{u}_s = \frac{1}{R}(K_2 - C_3) - e^{R\eta}\Big\{C_4 + K_1\eta + \ln[(F_1 + F_2 e^{R\eta} + F_3 e^{2R\eta})^{K_3}$$

$$(F_{11} + F_2 e^{R\eta} + F_3 e^{2R\eta})^{K4}] + m_1 \tan^{-1}\left(\frac{h_3 + e^{R\eta}}{h_1}\right)$$

$$+ m_2 \tan^{-1}\left(\frac{h_3 + e^{R\eta}}{h_{21}}\right)\Big\} \quad \ldots(34)$$

where the constants of intgeration $C_3$ and $C_4$ are determined by using equation (29) as

$$C_4 = \frac{1}{(e^R - 1)}(-K_1 e^R + \ln[(F_1 + F_2 + F_3)^{K_3}(F_{11} + F_2 + F_3)^{K4}]$$

$$+ m_1 \tan^{-2}\left(\frac{1 + h_3}{h_2}\right) + m_2 \tan^{-1}\left(\frac{1 + h_3}{h_{21}}\right)$$

$$- e^R\{\ln[(F_1 + F_2 e^R + F_3 e^{2R})^{K3}(F_{11} + F_2 e^R + F_3 e^{2R})^{K4}]$$

$$+ m_1 \tan^{-1}\left(\frac{h_3 + e^R}{h_2}\right) + m_2 \tan^{-1}\left(\frac{h_3 + e^R}{h_{21}}\right)\Big\}\Big) \quad \ldots(35)$$

$$C_3 = K_2 - R\{C_4 + \ln[(F_1 + F_2 + F_3)^{K3}(F_{11} + F_2 + F_4)^{K4})$$

$$+ m_1 \tan^{-1}\left(\frac{1 + h_3}{h_2}\right) + m_2 \tan^{-1}\left(\frac{1 + h_3}{h_{21}}\right)\Big\}, \quad \ldots(36)$$

and

$$F_1 = 36 + \frac{P^2U^2}{D^2h^2}, \; F_{11} = 100 + \frac{P^2U^2}{D^2h^2}, \; F_2 = \frac{2RPC_2U^2}{D^2h^2},$$

$$F_3 = \frac{R^2C_2^2U^2}{D^2h^2},$$

$$g_1 = \frac{P}{F_1}, \; g_{11} = \frac{P}{F_1}, \; g_2 = \frac{F_1RC_2 - F_2P}{F_1^2}, \; g_{21} = \frac{F_{11}RC_2 - F_2P}{F_{11}^2},$$

$$g_3 = \frac{1}{F_1}\left[\frac{F_2}{F_1}(F_2P - RF_1C_2) - F_3P\right],\ g_{31} = \frac{1}{F_{11}}\left[\frac{F_2}{F_{11}}(F_2P - RF_{11}C_2) - F_3P\right],\ g_4 = \frac{F_3}{F_1^2}(F_2P - F_1RC_2),\ g_{41} = \frac{F_3}{F_{11}^2}(F_2P - F_{11}RC_2),$$

$$h_1 = \frac{2F_3g_3}{g_4} - F_2,\ h_{11} = \frac{2F_3g_{31}}{g_{41}} - F_2,\ h_2^2 = \frac{F_1}{F_3} - \frac{F_2^2}{4F_3^2},\ h_3 = \frac{F_2}{2F_3},$$

$$h_{21}^2 = \frac{F_{11}}{F_3} - \frac{F_2^2}{4F_3^2},\ K_1 = ARC_2 + A_2g_2 + A_3g_{21},\ K_2 = A_1P + A_2g_1 + A_3g_{11},\ K_3 = \frac{A_2g_{21}}{2RF_3},\ K_4 = \frac{A_3g_{41}}{2RF_3},$$

$$m_1 = \frac{A_2g_4h_1}{2Rh_2F_3^2},\ m_2 = \frac{A_3g_{41}h_{11}}{2Rh_{21}F_3^2}.$$

Using equations (30) and (34–36), the variation of $\overset{*}{u}_0$ and $\overset{*}{u}$ with $\eta$ is displayed in Figs. 1, 2 and 3 with $\Phi = 0.02$, $U/h = 25$ sec$^{-1}$, $T = 295$ °K, $\mu_0 = 0.001$ N-sec $M^2$ and $r_{eff} = 3\sqrt{(ab)^2} = 10^{-7}\,M$ for different values of $r_e$, $P$ and $R$.

## Complete Brownian Motion

For the case of complete Brownian motion, the viscosity factor $\nu_2$ is used for prolate spheroids. The effective viscosity in the dimensionless form $\overset{*}{\mu}_{eff}$ will take the form

$$\overset{*}{\mu}_{eff} = 1 + \Phi\nu_2 \qquad \ldots(37)$$

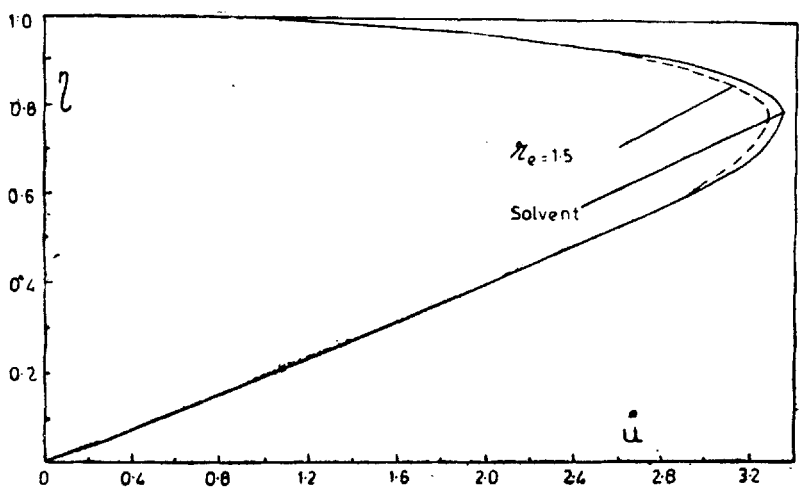


FIG 1 The variation of $\overset{*}{u}$ with $\eta$ for the suspension with $r_e = 1.5$ (dashed line) and for the solvent with $P = 5$ and $R = 9$.

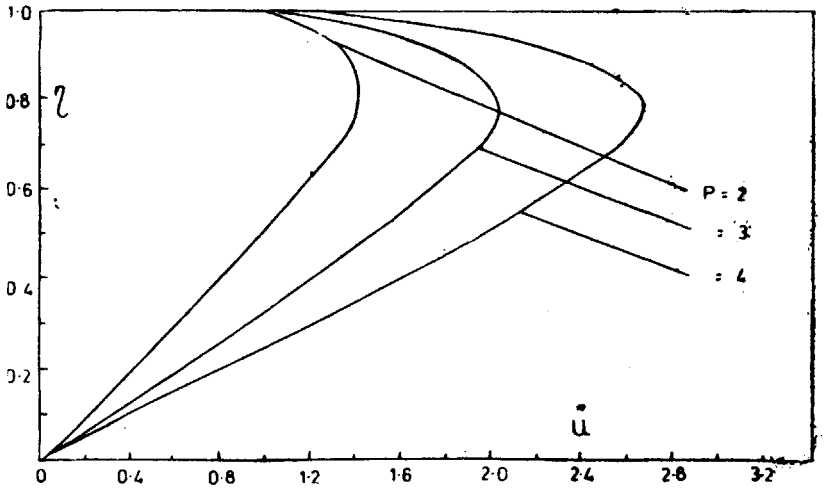


FIG 2 The variation of $\overset{*}{u}$ with $\eta$ for $P = 2, 3, 4$; $r_e = 1.5$ and $R = 9$.

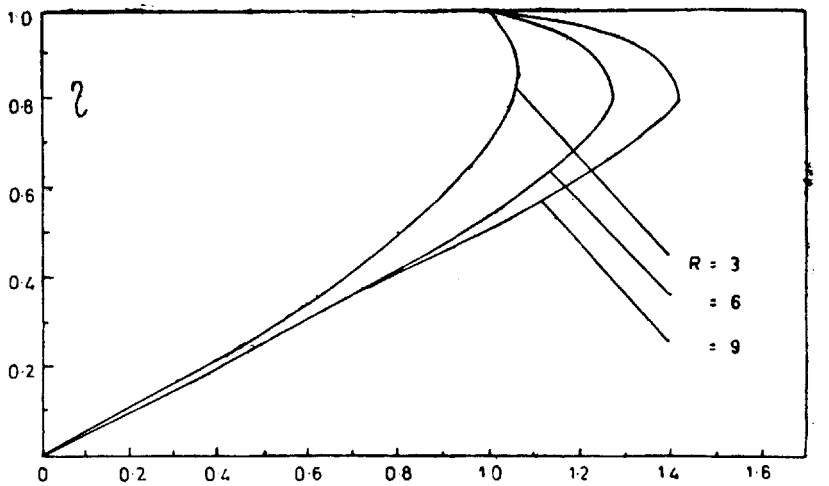


FIG 3 The variation of $\overset{*}{u}$ with $\eta$ for $R = 3, 6, 9$; $r_e = 1.5$ and $P$ 2.

As before, we try a solution of equation (22) for the velocity in the form

$$\overset{*}{u} = \overset{*}{u}_0 + \Phi \overset{*}{u}_s \qquad \ldots(38)$$

Substituting equations (37) and (38) in equation (22), the equations giving the zeroth perturbation with the corresponding boundary conditions are the same as equations (26) and (27). Consequently, the variation of $\overset{*}{u}_0$ with $\eta$ will be given by equation

(30). On the other hand, the equations representing the first perturbation $\overset{*}{u}_s$ with the corresponding boundary conditions are

$$\frac{d\overset{*}{u}_s}{d\eta} = \frac{1}{R}\left(\nu_2 \frac{d^2\overset{*}{u}_0}{d\eta^2} + \frac{d^2\overset{*}{u}_s}{d\eta^2}\right), \qquad ...(39)$$

$$\overset{*}{u}_s(0) = 0, \quad \overset{*}{u}_s(1) = 0 \qquad ...(40)$$

In view of equation (30), the solution of equation (39) subject to the boundary conditions (40) is

$$\overset{*}{u}_s = \frac{1}{R}(\nu_2 P - C_5) - e^{R\eta}(C_6 + \nu_2 R C_2 \eta), \qquad ...(41)$$

where

$$C_6 = \frac{\nu_2 R C_2 e^R}{1 - e^R} \text{ and } C_5 = \nu_2 P - RC_6.$$

By using equations (30) and (41), the variation of $\overset{*}{u}_0$ and $\overset{*}{u}$ with $\eta$ is displayed in Figs. 4, 5 and 6 with $\Phi = 0.01$ for different values of $r_e$, $P$ and $R$.

## Effective Viscosity of the Suspension and the Shear Stress

For nearly spherical particles, equations (30–34) can be used to calculate the rotary Péclet number $\sigma$ as

$$\sigma = \frac{1}{D}\frac{du}{dy} = \frac{U}{hD}\frac{d\overset{*}{u}}{d\eta} = \frac{U}{hD}\left(\frac{d\overset{*}{u}}{d\eta} + \Phi \frac{d\overset{*}{u}_s}{d\eta}\right), \qquad ...(42)$$

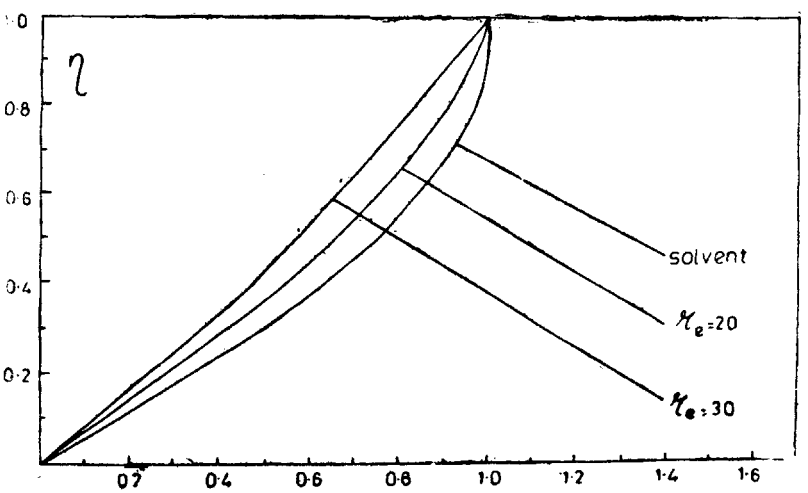


FIG 4 The variation of $\overset{*}{u}$ with $\eta$ for $P = 3$, $R = 1$ and $r_e = 20, 30$ as compared to one of the solvents which is Newtonian fluid.

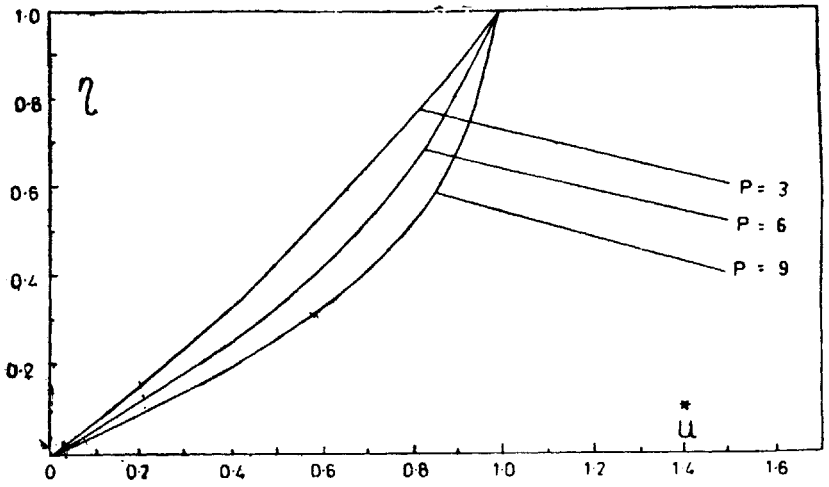


FIG 5 The variation of $\overset{*}{u}$ with $\eta$ for $P = 3, 6, 9$ and $r_e = 30$, $R = 1$.

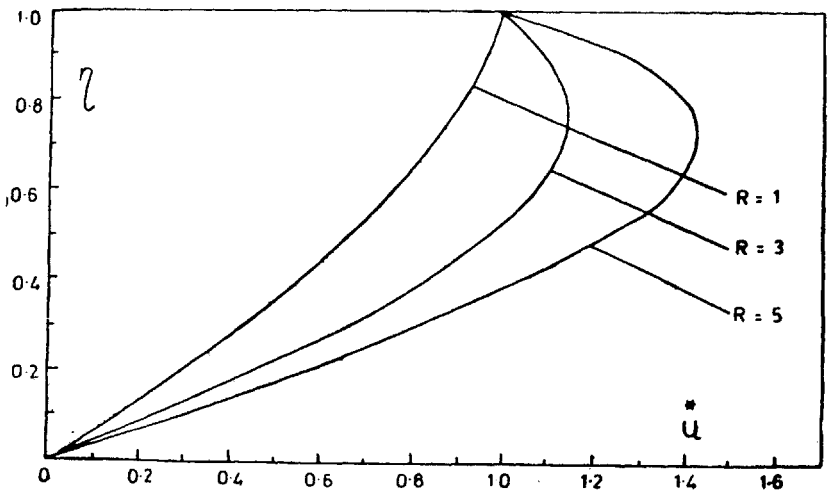


FIG 6 The variation of $\overset{*}{u}$ with $\eta$ for $R = 1, 3, 5$ and $r_e = 20$, $P = 3$.

but

$$\frac{d\overset{*}{u}_0}{d\eta} = P + RC_2e^{R\eta}, \qquad \ldots(43)$$

$$\frac{d\overset{*}{u}_s}{d\eta} = K_2 - Re^{R\eta}\{C_4 + K_1\eta + \ln[(F_1 + F_2e^{R\eta} + F_3e^{2R\eta})K_3$$

$$(F_{11} + F_2 e^{R\eta} + F_3 e^{2R\eta})K_4] + m_1 \tan^{-1}\left(\frac{h_3 + e^{R\eta}}{h_2}\right) + m_2 \tan^{-1}\left(\frac{h_3 + e^{R\eta}}{h_{21}}\right)\Big\} - (RC\, e^{R\eta} + P)\Big[A_1 + \frac{A_2}{36 + (B_1 e^{R\eta} + B_2)^2} + \frac{A_3}{100 + (B_1 e^{R\eta} + B_2)^2} \quad \ldots(44)$$

Equations (12) and (42–44) are used to calculate the variation of the viscosity factor $\nu_1$ with the parameters of the flow $r_e$, $P$ and $R$.

The shear stress for the suspension $\tau_{ws}$ calculated at the lower wall is defined by :

$$\tau_{ws} = \left[\mu_{eff}\frac{du}{dy}\right]_{y=0}, \quad \ldots(45)$$

Writing equation (45) in the dimensionless form then using equations (24) and (25) give

$$\tau_{ws}\Big/\frac{\mu_0 U}{h} = \left[(1 + \nu_1 \Phi)\left(\frac{d\overset{*}{u}_0}{d\eta} + \Phi\frac{d\overset{*}{u}_s}{d\eta}\right)\right]_{\eta=0};$$

neglecting terms of order $\Phi^2$, the above equation reduces to

$$\tau_{ws}\Big/\frac{\mu_0 U}{h} = \left[\frac{d\overset{*}{\mu}_0}{d\eta} + \Phi\left\{\nu_1\frac{d\overset{*}{u}_0}{d\eta} = \frac{d\overset{*}{u}_s}{d\eta}\right\}\right]_{\eta=0}, \quad \ldots(46)$$

Also, the shear stress for the solvent $\tau_{w0}$ calculated at the lower wall is defined by :

$$\tau_{w0} = \mu_0\left(\frac{du_0}{dy}\right)_{y=0} = \frac{\mu_0 U}{h}\left(\frac{d\overset{*}{u}_0}{d\eta}\right)_{\eta=0} \quad \ldots(47)$$

equations (46) and (47) give

$$\tau_{ws}/\tau_{w0} = 1 + \Phi\left\{\nu_1 + \left(\frac{d\overset{*}{u}_s}{d\eta}\right)\Big/\left(\frac{d\overset{*}{u}_0}{d\eta}\right)\right\}_{\eta=0} \quad \ldots(48)$$

The values of $\nu_1$ and $(\tau_{ws}/\tau_{wo})$, calculated at the lower wall, are reported in Table I with $\Phi = 0.02$, $U/h = 25\ \text{sec}^{-1}$, $T = 295\ ^\circ\text{K}$, $\mu_0 = 0.001(\text{N.sec/M}^2)$ and $r_{eff} = 10^{-7}$ M for different values of $r_e$, $P$ and $R$.

TABLE I

| $r_e$ | $P$ | $R$ | $\nu_1$ | $\tau_{ws}/\tau_{wo}$ |
|---|---|---|---|---|
| 1.1 | 5 | 9 | 2.5067 | 1.0498 |
| 1.5 | 5 | 9 | 2.6276 | 1.0522 |
| 2.0 | 5 | 9 | 2.8973 | 1.0575 |
| 1.5 | 1 | 6 | 2.6297 | 1.0526 |
| 1.5 | 3 | 6 | 2.6290 | 1.0499 |
| 1.5 | 5 | 6 | 2.6276 | 1.0494 |
| 2.0 | 4 | 3 | 2.9026 | 1.0413 |
| 2.0 | 4 | 6 | 2.9011 | 1.0547 |
| 2.0 | 4 | 9 | 2.9009 | 1.0576 |

## Results and Discussion

Many special cases can be deduced from this work, for example, the flow between two impermeable walls, fixed walls[5] and without pressure gradient is obtained by taking $v_0 = 0$, $U = 0$ and $P = 0$ respectively. Also the results representing the flow of a Newtonian fluid are easily obtained by taking the viscosity factors $\nu_1 = \nu_2 = 0$.

From Figs (1–6) and Table I, we can deduce that the velocity profiles at any section of the channel increase with the decrease of $r_e$ and increase of $P$ and $R$, in all cases they are less than the corresponding profiles representing the flow of the solvent (Newtonian fluid). The Reynolds number $R$ has a dominant effect on the flow.

The results obtained in this work can be applied to:

(*a*) Suspension of spherical particles[12] with $r_e = 1$ and $\nu_3 = 2.5$.

(*b*) The expression for the effective viscosity given by Hinch and Leal[3] (equation (45)) for which $\mu_{eff}$ depends on $\sigma$ and a small parameter $\epsilon$ where $r_e = 1 + \epsilon$.

(*c*) The calculated value of the viscosity factor $\nu$ given by Anezurowski and Mason[13] (equation (4)) with $r_e = 14.2$.

*For Nearly Spherical Particles*:

(*a*) The pressure gradient, for the suspension of nearly spherical particles, in the direction of the flow is constant as in the case of a Newtonian fluid.

(*b*) The suspension differs from a Newtonian fluid in the property that the pressure changes ccross any cross section perpendicular to the walls.

(*c*) To the best of the author's knowledge paragraph[5] of this paper gives for the first time, variation of the dimensionless effective viscosity $\overset{*}{\mu}_{eff}$ with the parameters of the flow $r_e$, $P$ and $R$ across any cross section of the channel. All the previous works [1–8, 13] give the effective viscosity $\mu_{eff}$ as a function depending on the rotary Péclet number $\sigma$ and the axes ratio $r_e$.

(*d*) The viscosity factor $\nu_1$ calculated at the lower wall, increases with the increase of $r_e$ and the decrease of $P$ and $R$. As reported in,[3] the effect of particle non-sphericity on $\nu_1$ is very weak.

(*e*) The ratio of the shear stresses $\tau_{ws}/\tau_{wo}$, calculated at the lower wall, increases with the increase of both $r_e$ and $R$ and the decrease of $P$.

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## Appendix A

The recurrence formulas for $a_{nm,j}$ and $b_{nm,j}$ are :

$$\frac{n(2n+1)}{\sigma}\,a_{n0,j} = \frac{-1}{4}\Bigg[-\frac{(2n-3)(2n-2)(2n-1)(2n)(2n+1)}{(4n-3)(4n-1)}\,b_{n-1,1,j-1} + \frac{3(2n-1)(2n)(2n+1)(2n+2)}{(4n-1)(4n+3)}\,b_{n,1,j-1} + \frac{2n(2n+1)(2n+2)(2n+3)(2n+4)}{(4n+3)(4n+5)}\,b_{n+1,1,j-1}\Bigg] \quad \ldots(A)$$

$$b_{n0,j} = 0, \quad \ldots(B)$$

$$\frac{2n(2n+1)}{\sigma}\,a_{nm,j} + mb_{nm,j} = \frac{-1}{4}\Bigg[\frac{2n+1}{(4n-3)(4n-1)}\,b_{n-1,m-1,j-1} - \frac{3}{(4n-1)(4n+3)}\,b_{n,m-1,j-1} - \frac{2n}{(4n+3)(4n+5)}\,b_{n+1,m-1,j-1} - \frac{(2n-2m-3)(2n-2m-2)(2n-2m-1)(2n-2m)(2n+1)}{(4n-3)(4n-1)}\,b_{n-1,m+1,j-1} + \frac{3(2n-2m-1)(2n-2m)(2n+2m+1)(2n+2m+2)}{(4n-1)(4n+3)}\,b_{n,m+1,j-1} + \frac{2n(2n+2m+1)(2n+2m+2)(2n+2m+3)(2n+2m+4)}{(4n+3)(4n+5)}\,b_{n+1,m+1,j-1}\Bigg] \quad (m \neq 0), \quad \ldots(C)$$

$$-ma_{nm,j} + \frac{2n(2n+1)}{\sigma}\,b_{nm,j} = \frac{1}{4}\Bigg[\frac{2n+1}{(4n-3)(4n-1)}\,a_{n-1,m-1,j-1} - \frac{3}{(4n-1)(4n+3)}\,a_{n,m-1,j-1} - \frac{2n}{(4n+3)(4n+5)}\,a_{n+1,m-1,j-1} - \frac{(2n-2m-3)(2n-2m-2)(2n-2m-1)(2n-2m)(2n+1)}{(4n-3)(4n-1)}\,a_{n-1,m+1,j-1} + \frac{3(2n-2m-1)(2n-2m)(2n+2m+1)(2n+2m+2)}{(4n-1)(4n+3)}\,a_{n,m+1,j-1}$$

$$+\frac{2n(2n+2m+1)(2n+2m+2)(2n+2m+3)(2n+2m+4)}{(4n+3)(4n+5)}a_{n+1,m+1,j-1}\Bigg]$$

$$(m \neq 0), \quad \ldots(D)$$

where $\sigma'$ is the rotary Peclet number. In computing $a_{nm,j}$ and $b_{nm,j}$, Scheraga[6] noted the following relations :

1—All terms with negative indexes are zero.

2—$a_{nm,0} = 0$ and $a_{00,j} = 0$, except $a_{00,0} = 1/2\pi$.

3—$b_{nm,0} = b_{n0,j} = b_{0n,j} = 0$.

4—If $m$ is even and $j$ is odd, $a_{nm,j} = 0$, and $b_{nm,j} = 0$.

5—If $m$ is odd and $j$ is even, $a_{nm,j} = 0$, and $b_{nm,j} = 0$.

6—If $n > j$, or if $m > j$, $a_{nmj} = 0$ and $b_{nmj} = 0$.

7—In this problem, terms where $m > n$ need not be computed.