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A TRANSVERSELY ISOTROPIC HALF-SPACE INDENTED BY A PARABOLIC CONCAVE PUNCH

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The present paper deals with an elastic contact problem for a transversely isotropic half-space indented by a parabolic concave rigid punch. The problem is reduced into dual integral equations and the rigorous solution of the problem has been derived.

Key Words : Transversely Isotropic Half-Space; Parabolic Concave Punch; Elastic Contact; Dual Integral Equations

INTRODUCTION

THE punch problem for a semi-infinite medium has been considered by Harding and Sneddon,[1] Sneddon,[2] and Payne.[3] Using a representation of axisymmetric harmonic functions in terms of plane harmonic function, Gubenko and Mossakovskii[4] solved the punch problem by obtaining a pair of integral equations which they combined into one. Olesiak[5] attempted to solve the Boussinesq contact problem for the annulus by reducing it to a series of dual integral equations. This method, though appears to converge satisfactorily, is extremely cumbersome even for the flat punch. Subsequently, Dhawan[6] analysed a transversely isotropic half-space problem indented by a flat annular rigid stamp. Recently, Lal[7] dealt with an elastic contact problem for a neo-Hookean half-space indented by a annular rigid stamp of arbitrary profile. The present paper is concerned with an elastic contact problem of a transversely isotropic half-space indented by a parabolic concave punch. The rigorous solution of the punch problem has been derived.

BASIC EQUATIONS OF ELASTICITY

Let us consider (r, θ, z) to be cylindrical coordinates of a point where z-axis coincides with the material axis of symmetry. For a symmetric deformation, the non-vanishing components of displacement and stress are denoted by (u_r, u_z) and $(\sigma_{rr}, \sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rz})$ respectively. The displacements and stresses which satisfy the equilibrium equations are expressed in terms of two potential functions $\phi_1(r, z_1)$ and $\phi_2(r, z_2)$ and they are harmonic in their respective (r, θ, z) space. Following Green and Zerna,[8] we have

$$u_r(r, z) = \frac{\partial \phi_1}{\partial r} + \frac{\partial \phi_2}{\partial r}, \qquad \ldots(1)$$

$$u_z(r, z) = \frac{k_1}{\sqrt{\nu_1}} \frac{\partial \phi_1}{\partial z_1} + \frac{k_2}{\sqrt{\nu_2}} \frac{\partial \phi_2}{\partial z_2}, \qquad \ldots(2)$$

$$\sigma_{zz}(r, z) = a_{44}\left[(1 + k_1)\frac{\partial^2\phi_1}{\partial z_1^2} + (1 + k_2)\frac{\partial^2\phi_2}{\partial z_2^2}\right] \quad ...(3)$$

and

$$\sigma_{rz}(r, z) = a_{44}\left[\frac{(1 + k_1)}{\sqrt{v_1}}\frac{\partial^2\phi_1}{\partial r\partial z_1} + \frac{(1 + k_2)}{\sqrt{v_2}}\frac{\partial^2\phi_2}{\partial r\partial z_2}\right]. \quad ...(4)$$

The other components may be similarly expressed. The dimensionless parameters v_α and k_α ($\alpha = 1, 2$) are functions of elastic constants a_{ij} i.e.; v_1 and v_2 are the roots of the equation:

$$a_{44}a_{11}v^2 + [a_{13}(2a_{44} + a_{13}) - a_{11}a_{33}]\, v + a_{33}a_{44} = 0, \quad ...(5)$$

and k_1, k_2 are defined as

$$\left.\begin{aligned} k_\alpha &= \frac{(a_{11}v_\alpha - a_{44})}{(a_{13} + a_{44})} \\ z_\alpha &= \frac{z}{\sqrt{v_\alpha}};\ \alpha = 1, 2 \end{aligned}\right\}. \quad ...(6)$$

Formulation of the Problem

Consider an infinite elastic half-space occupying the region $z > 0$ and the axisymmetric concave face of a parabolic rigid punch is indented at $z = 0$. We assume that the interface between punch and half-space is frictionless, while the rest of the surface is free from traction. Thus the boundary conditions are :

$$\sigma_{zz}(r, 0) = 0; \quad r > a, \quad ...(7)$$

$$u_z(r, 0) = k_0 - g(r); \quad 0 \leqslant r \leqslant a \quad ...(8)$$

and

$$\sigma_{rz}(r, 0) = 0; \quad r \geqslant 0, \quad ...(9)$$

where

$$g(r) = u\{1 - (r/a)^2\}, \quad ...(10)$$

and k_0 is the depth of the punch.

In addition to this, all stresses and displacements must tends to zero as $r \to \infty$. The appropriate potential functions satisfying the above boundary condition are written as :

$$\phi_1(r, z_1) = \int_0^\infty \xi^{-2}A(\xi) \exp(-\xi z_1)\, J_0(\xi r)\, d\xi \quad ...(11)$$

and

$$\phi_2(r, z_2) = \int_0^\infty \xi^{-2}B(\xi) \exp(-\xi z_2)\, J_0(\xi r)\, d\xi. \quad ...(12)$$

From equations (4), (11) and (12), we find that the boundary condition (9) is satisfied if

$$(1 + k_1)\sqrt{v_1}\, A(\xi) = -(1 + k_2)\sqrt{v_2}\, B(\xi). \quad ...(13)$$

The boundary conditions (7) and (8) are reduced to the following dual integral equations :

$$\left.\begin{aligned} &\int_0^\infty \xi B(\xi)\, J_0(\xi r)\, d\xi = 0; \quad r > a, \\ &\int_0^\infty B(\xi)\, J_0(\xi r)\, d\xi = -p\{k_0 - g(r)\}; \quad 0 \leqslant r \leqslant a \end{aligned}\right\}, \qquad \ldots(14)$$

where

$$p = \frac{\sqrt{v_2}}{(1+k_2)}(\sqrt{v_1} - \sqrt{v_2})\, a_{44}. \qquad \ldots(15)$$

Solution of the Integral Equations and Discussions

For solving the equation (14), the Abel transform method is used.[9] Then the equation (14) may be rewritten as follows :

$$\int_0^\infty B(\xi)\cos(\xi x)\, d\xi = p\left\{a_0 + x \int_0^\infty \frac{ug'(t)}{\sqrt{(x^2 - t^2)}}\, dt\right\} H(b - x), \qquad \ldots(16)$$

where $H(x)$ is Heaviside's step function and a_0 is an unknown constant. Therefore, the function $B(\xi)$ is obtained by Fourier inverse transform. Now from (16) and (10) and using Fourier inverse transform, we have

$$B(\xi) = \frac{2p}{\pi}[(a_0 - 2u)\,\xi^{-1}\sin(\xi a) - 4ua^2\xi^{-3}\{\xi a\cos(\xi a) - \sin(\xi a)\}]. \qquad \ldots(17)$$

Substituting equation (17) into equation (14), $u_z(r, 0)$ and $\sigma_{zz}(r, 0)$ are written as

$$u_z(r, 0) = -2/\pi\left[a_0\left\{\begin{matrix} \pi/2, & 0 \leqslant r \leqslant a \\ \sin^{-1}(a/r), & r \geqslant a \end{matrix}\right\}\right.$$

$$\left. - \frac{2u}{a^2}\int_0^a \frac{x^2}{\sqrt{(r^2 - x^2)}} H(r - x)\, dx\right] \qquad \ldots(18)$$

and

$$\sigma_{zz}(r, 0) = \frac{2p}{\pi}\left\{\frac{a_0 - 2u}{\sqrt{(a^2 - r^2)}} - \frac{4u}{a^2}\sqrt{(a^2 - r^2)}\right\} H(a - r). \qquad \ldots(19)$$

Since, in the case of parabolic punch

$$u_z(r, 0) = k_0 - u\left\{1 - \left(\frac{r}{a}\right)^2\right\}; 0 \leqslant r \leqslant a, \qquad \ldots(20)$$

therefore, equation (18) must concide with equation (20) in $0 \leqslant r \leqslant a$. Hence,

$$a_0 = u - k_0; \qquad \ldots(21)$$

then the normal displacement $u_z(r, 0)$ on the free surface $r \geqslant a$ is

$$u_z(r, 0) = \frac{2}{\pi}\left[\left\{k_0 - u\left(-\frac{r^2}{a^2}\right)\right\} \sin^{-1}\left(\frac{a}{r}\right) - \left(\frac{ru}{a}\right)\left\{1 - \frac{a^2}{r^2}\right\}^{1/2}\right], \quad r \geqslant a, \qquad \ldots(22)$$

and the normal stress $\sigma_{zz}(r, 0)$ in the contact region $0 \leqslant r \leqslant a$ is

$$\sigma_{zz}(r, 0) = -(2p/a\pi)\,[\{k_0 - 3u + 4u(r/a)^2\}\{1 - (r/a)^2\}^{1/2}], \quad 0 \leqslant r \leqslant a. \qquad \ldots(23)$$

When the entire punch surface makes contact with the half-space, then the condition is $\sigma_{zz}(r, 0) < 0$ in $0 < r < a$. Therefore, from equation (23), we obtain

$$k_0 \geqslant 3u, \qquad \ldots(24)$$

and the indenting load

$$P = -2\pi \int_0^a \sigma_{zz}(r, 0)\, r\,dr = 4ap\left(k_0 - \frac{u}{3}\right), \left(\geqslant P_0 = \frac{32}{3} pau\right), \qquad \ldots(25)$$

where the critical load P_0 means the minimum indented load for the entire face to contact. If $P \geqslant P_0$, P increases linearly with an increase of the indenting depth k_0. If $P < P_0$ or $k_0 < 3u$, $\sigma_{zz}(r, 0)$ becomes tensile in the neighbourhood of $r = 0$. Additionally, if $u < 0$ and $k > -u$, the punch face is convex and the stress $\sigma_{zz}(r, 0)$ is always compressible in the contact region $0 < r < a$. If $u < 0$ and $k_0 < -u$, the stress $\sigma_{zz}(r, 0)$ is tensile in the neighbourhood of $r = a$.

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