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LINE CRACK IN AN ELASTIC PLATE SUBJECT TO VARYING SHEAR LOAD IN NONLOCAL ELASTICITY

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The problem of a line crack in an elastic plate subject to varying shear load along the surfaces of the crack using nonlocal theory of elasticity is solved. The stress intensity factors for varying shear load and uniform shear load are calculated and the corresponding results of the classical theory are obtained as particular cases.

Key Words : Line Crack; Elastic Plate; Shear Load; Nonlocal Elasticity

INTRODUCTION

INTEREST in crack problems in the Mathematical Theory of Elasticity is revived in recent years as a result of experimental discovery that at high or low temperatures many structural elements composed of commonly used materials that display plastic properties in standard tensile tests fail by a quasibrittle process. By this it is meant that failure occurs by the propagation of cracks and that although there is a plastic zone, it is of limited extent and concentrated at the crack tip.

A systematic approach to nonlocal elasticity was presented by Eringen and Edelen,[1] through both the balance laws and thermodynamic and nonlocal variational principles. A few problems that have been studied recently using the nonlocal theory of elasticity include the stress concentration at the tip of a crack,[2] line crack subject to shear,[3] line crack subject to antiplane shear,[4] and line crack subject to varying internal pressure.[6]

In this paper, an attempt has been made to solve the problem of a line crack in an elastic plate subject to varying shear load along the length of the crack using the nonlocal theory of elasticity. The resulting dual integral equations are completely solved. Removing the singularity that is present in the stress distribution, the stress intensity factor is calculated. When the shearing load is uniform and when the theory is 'Local', the corresponding expressions for the stress intensity factor are obtained as particular cases.

CRACK UNDER VARYING SHEAR LOAD

Consider a plate in x_1x_2-plane ($x_1 = x$, $x_2 = y$) weakened by a line crack of length $2c$ along the x-axis which is subjected to varying shear stress $p(x)$ along the surfaces of the crack. Using the notation of Eringen,[3] for this plane strain problem the equation of equilibrium takes the form

$$f_{R\alpha}(|\hat{x}' - \hat{x}|)\,\sigma_{kl,k}(x', y')\,dx'\,dy' - \int\!\!\!\!\!\!-_{-c}^{c} \alpha(|\hat{x}' - \hat{x}|)\,[\sigma_{21}(x', 0)]\,dx' = 0, \quad \ldots(1)$$

where the integral with slash is over the two-dimensional infinite space excluding the crack line ($|x| < c$, $y > 0$). A bold-face bracket indicates a jump at the crack line.

Under the varying shear load on the unopened surfaces of the crack, the displacement field possesses the following symmetry regulations :

$$u(x, -y) = -u(x, y); \quad v(x, -y) = v(x, y) \quad \ldots(2)$$

The general solution of (1) subject to the conditions (2) is given by[3]

$$u = -(2\pi)^{-1/2} \int_{-\infty}^{\infty} \xi^{-1}\left[|\xi|A(\xi) + \left(|\xi|y - \frac{\lambda + 3\mu}{\lambda + \mu}\right)B(\xi)\right] \exp(-|\xi|y - i\xi x)\,d\xi,$$

$$v = (2\pi)^{-1/2} \int_{-\infty}^{\infty} i\,[A(\xi) + yB(\xi)] \exp(-|\xi|y - i\xi x)\,d\xi, \quad \ldots(3)$$

where $A(\xi)$ and $B(\xi)$ are to be calculated from the following boundary conditions.

$$\left.\begin{aligned} &\sigma_{yy}(x, 0) = 0 \quad \text{for all } x \\ &t_{yx}(x, 0) = p(x) \quad |x| < c \\ &u(x, 0) = 0 \quad |x| > c \end{aligned}\right\} \text{ at } y = 0$$

$$u = v = 0 \text{ as } y \to \infty \quad \ldots(4)$$

On account of symmetry $u(x, y) = u(-x, y)$, the displacement field may be put in the form

$$u(x, y) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\lambda + \mu}{\mu} \int_0^{\infty} A(\xi)\left(\frac{\lambda + 2\mu}{\lambda + \mu} - \xi y\right) \exp(-\xi y)\cos(\xi x)\,d\xi, \quad \ldots(5)$$

and

$$v(x, y) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\lambda + \mu}{\mu} \int_0^{\infty} A(\xi)\left(\frac{\mu}{\lambda + \mu} + \xi y\right)\exp(-\xi y)\sin(\xi x)\,d\xi,$$

where $$B(\xi) = \frac{\lambda + \mu}{\mu}|\xi|A(\xi).$$

The nonlocal stress field is given by

$$t_{xx}(x, y = \int_0^{\infty}\int_{-\infty}^{\infty} \sigma_{xx}(x', y')\,[\alpha(|x' - x|, |y' - y|) - \alpha(|x' - x|, |y' + y|)]\,dx'dy',$$

$$t_{yy}(x, y) = \int_0^\infty \int_{-\infty}^\infty \sigma_{yy}(x', y') \, [\alpha(|x' - x|, |y' - y|) - \alpha(|x' - x|, |y' + y|)] \, dx'dy'$$

and

$$t_{xy}(x, y) = \int_0^\infty \int_{-\infty}^\infty \sigma_{xy}(x', y') \, [\alpha(|x' - x|, |y' - y|) + \alpha(|x' - x|, |y' + y|)] \, dx'dy', \quad \ldots(6)$$

where α is taken as,

$$\alpha(|\hat{x}' - \hat{x}|) = \frac{1}{\pi}\left(\frac{\beta}{a}\right)^2 \exp\left\{-\left(\frac{\beta}{a}\right)^2 [x_1' - x_1)^2 + (x_2' - x_2)^2]\right\},$$

β being a real constant and a being the lattice parameter.[3]

Evaluating the integrals,

$$t_{xx} = -\left(\frac{2}{\pi}\right)^{1/2} (\lambda + \mu) \int_0^\infty A(\xi) \sin(\xi x) \left\{e^{-\xi y}\left(2\xi + \frac{\xi^2}{2m}(\xi - 2my)\right) \times \left[1 - \phi\left(\frac{\xi - 2my}{2\sqrt{m}}\right)\right] - e^{\xi y}\left[2\xi + \frac{\xi^2}{2m}(\xi + 2my)\right] \times \left[1 - \phi\left(\frac{\xi + 2my}{2\sqrt{m}}\right)\right]\right\} d\xi,$$

$$t_{yy} = \left(\frac{2}{\pi}\right)^{1/2} (\lambda + \mu) \int_0^\infty \xi^2 A(\xi) \sin(\xi x) \left\{e^{-\xi y}\left[\frac{\xi - 2my}{2m}\right] \times \left[1 - \phi\left(\frac{\xi - 2my}{2\sqrt{m}}\right)\right] - e^{\xi y}\left[\frac{\xi + 2my}{2m}\right] \times \left[1 - \phi\left(\frac{\xi + 2my}{2\sqrt{m}}\right)\right]\right\} d\xi$$

and

$$t_{xy} = -\left(\frac{2}{\pi}\right)^{1/2} (\lambda + \mu) \int_0^\infty A(\xi) \cos(\xi x) \left\{e^{-\xi y}\left[1 + \frac{\xi}{2m}(\xi - 2my)\right] \times \left[1 - \phi\left(\frac{\xi - 2my}{2\sqrt{m}}\right) + e^{\xi y}\left[1 + \frac{\xi}{2m}(\xi + 2my)\right] \times \left[1 - \phi\left(\frac{\xi + 2my}{2\sqrt{m}}\right)\right] - 2\xi (\pi m)^{-1/2} \exp\left[-my^2 - \frac{\xi^2}{4m}\right]\right\} d\xi, \quad \ldots(7)$$

where $m = (\beta/a)^2$ and $\phi(z)$ is the error function defined by

$$\phi(z) = 2\pi^{-1/2} \int_0^z \exp(-t^2) \, dt.$$

By introducing $x/c \equiv z$, $\xi c, \equiv \zeta$, $A(\xi) = \zeta^{-1/2}C(\zeta)$, $T(x) \equiv p(x)c^2/2(\lambda + \mu)$

$$K(\zeta, m) \equiv \left[1 + \frac{\zeta^2}{2mc^2}\right]\left[1 - \phi\left(\frac{\zeta}{2c\sqrt{m}}\right)\right] - \frac{\zeta}{c\sqrt{(\pi m)}} \exp\left(-\frac{\zeta^2}{4mc^2}\right) \qquad \ldots(8)$$

the boundary conditions $(4)_2$ and $(4)_3$ may be expressed as

$$\int_0^\infty \zeta^{1/2}C(\zeta)\, K(\zeta, m) \cos(\zeta z)\, d\zeta = -\left(\frac{\pi}{2}\right)^{1/2} T(z), \quad 0 < z < 1$$

$$\int_0^\infty \zeta^{-1/2}C(\zeta) \cos(\zeta z)\, d\zeta = 0 \qquad z > 1 \qquad \ldots(9)$$

Solving these integral equations we determine the unknown function $C(\zeta)$.

Solution of Dual Integral Equations

By introducing $\cos(\zeta z) = (\pi\zeta z/2)^{1/2} J_{-1/2}(\zeta z)$, where $J_\nu(z)$ is the Bessel's function of the first kind of order ν, we write (9) as

$$\int_0^\infty \zeta C(\zeta)\,[1 + k(\epsilon\zeta)]\, J_{-1/2}(\zeta z)\, d\zeta = -z^{-1/2}T(z), \quad 0 < z < 1$$

and

$$\int_0^\infty C(\zeta)\, J_{-1/2}(\zeta z)\, d\zeta = 0, \qquad z > 1, \qquad \ldots(10)$$

where the Kernel function $k(\epsilon z)$ is given by

$$k(\epsilon\zeta) = K(\zeta, m) - 1$$

$$= 2(\epsilon\zeta)^2\,[1 - \phi(\epsilon\zeta)] - \phi(\epsilon\zeta) - 2\pi^{-1/2}(\epsilon\zeta) \exp[-(\epsilon\zeta)^2] \qquad \ldots(11)$$

and

$$\epsilon = \frac{1}{2c\sqrt{m}} = \frac{a}{2\beta c}. \qquad \ldots(12)$$

The solution of the dual integral equations (10) is not readily available. However, it is possible to reduce this problem to that of solving the Fredholm equation of the second kind.[8]

$$h(z) + \int_0^1 h(u)\, L(z, u)\, du = -\frac{z^{1/2}}{\sqrt{\pi}} \int_0^z \frac{T(u)}{\sqrt{(z^2 - u^2)}}\, du \qquad \ldots(13)$$

for the function $h(u)$, where

$$L(z, u) = (zu)^{1/2} \int_0^\infty tk(\epsilon t)\, J_0(zt)\, J_0(ut)\, dt \qquad \ldots(14)$$

Once eq. (13) is solved, then $C(\zeta)$ is calculated by

$$C(\zeta) = (2\zeta)^{1/2} \int_0^1 t^{1/2} J_0(\zeta t)\, h(t)\, dt \qquad \ldots(15)$$

For $\epsilon = 0$, $k(\epsilon t) = 0$, and hence the dual integral equations (10) reduces to the integral equations in the classical elasticity[7] and $h(z)$ takes the value

$$h_0(z) = -\frac{z^{1/2}}{\sqrt{\pi}} \int_0^z \frac{T(u)}{\sqrt{(z^2 - u^2)}}\, du \qquad \ldots(16)$$

Hence, $C_0(\zeta)$ is given by

$$C_0(\zeta) = -\sqrt{\frac{2}{\pi}}\, \zeta^{1/2} \int_0^1 t q(t)\, J_0(\zeta t)\, dt,$$

where $$q(t) = \int_0^t \frac{T(u)}{\sqrt{(t^2 - u^2)}}\, du, \qquad \ldots(17)$$

or $$A_0(\xi) = -\sqrt{\frac{\pi}{2}} \int_0^1 f(t)\, J_0(\xi c t)\, dt, \qquad \ldots(18)$$

where we have introduced $f(t) = \frac{2t}{\pi}\, q(t).$ $\qquad \ldots(19)$

Since ϵ is very small number, $k(\epsilon\zeta)$ is very small compared to unity and hence, it is sufficient to take the solution of (10) as

$$A(\xi) = -\sqrt{\frac{\pi}{2}} \int_0^1 f(t)\, J_0(\xi c t)\, dt \qquad \ldots(20)$$

to calculate the stress field.

If we take the shear load to be uniform, say $p(x) = p_0$, then

$$q(t) = T_0 \frac{\pi}{2}, \qquad \ldots(21)$$

where $$T_0 = \frac{p_0 c^2}{2(\lambda + \mu)}.$$

Hence, $A(\xi)$ takes the form

$$A(\xi) = -\sqrt{\frac{\pi}{2}}\, T_0 \frac{J_1(\xi c)}{(\xi c)} \qquad \ldots(22)$$

which tallys with the classical result.[7]

Substituting (20) in $(7)_3$, we obtain

$$t_{zy}(x, 0) = (\lambda + \mu) \int_0^1 f(t) \{ \int_0^\infty \xi J_0(\xi ct) \cos(\xi x) [1 + k(\epsilon\xi c)] \, d\xi \} \, dt \quad ...(23)$$

Due to the complexity of the integrand, we resort to an approximation in which the terms upto second degree in the expansion of $k(\epsilon\xi c)$ only are retained. Thus

$$k(\epsilon\xi c) = -\frac{4}{\sqrt{\pi}} (\epsilon\xi c) + 2\epsilon^2\xi^2c^2. \quad ...(24)$$

Using the results

$$\int_0^\infty J_0(\xi t) \sin(\xi x) \, d\xi = \frac{H(x-t)}{\sqrt{(x^2 - t^2)}}, \int_0^\infty J_0(\xi t) \cos(\xi x) \, d\xi$$

$$= \frac{H(t-x)}{\sqrt{(t^2 - x^2)}},$$

where H is Heavy side unit function, we see that (23) is equivalent to

$$t_{zy}(x, 0) = \frac{(\lambda + \mu)}{c} \int_0^c f\left(\frac{r}{c}\right) \left\{ \frac{d}{dx} \frac{H(x-r)}{\sqrt{(x^2 - r^2)}} + \frac{4}{\sqrt{\pi}} (\epsilon c) \frac{d^2}{dx^2} \right.$$

$$\left. \times \frac{H(r-x)}{\sqrt{(r^2 - x^2)}} - 2\epsilon^2c^2 \frac{d^3}{dx^3} \frac{H(x-r)}{\sqrt{(x^2 - r^2)}} \right\} dr. \quad ...(25)$$

For $x > c$, at the outer tip of the crack, (25) takes the form

$$t_{zy}(x, 0) = \frac{(\lambda + \mu)}{c} \left[\frac{d}{dx} \int_0^c \frac{f\left(\frac{r}{c}\right)}{\sqrt{(x^2 - r^2)}} \, dr \right.$$

$$\left. - 2\epsilon^2c^2 \frac{d^3}{dx^3} \int_0^c \frac{f\left(\frac{r}{c}\right)}{\sqrt{(x^2 - r^2)}} \, dr \right]. \quad ...(26)$$

On simplification, (26) reduces to

$$t_{zy}(x, 0) = \frac{2(\lambda + \mu)}{\pi c^2} \left\{ -\frac{x}{\sqrt{(x^2 - c^2)}} q(1) + \int_0^c \frac{x}{\sqrt{(x^2 - r^2)}} q'\left(\frac{r}{c}\right) dr \right.$$

$$\left. - 2\epsilon^2c^2 \left[-q(1) \frac{3xc^2}{(x^2 - c^2)^{5/2}} + 3x \int_0^c \frac{r^2}{(x^2 - r^2)^{5/2}} q'\left(\frac{r}{c}\right) dr \right] \right\}$$

...(27)

For $\epsilon = 0$, the local shear stress at the outer tip of the crack is given by

$$t_{xy}(x, 0) = \frac{2(\lambda + \mu)}{\pi c^2} \left\{ - \frac{x}{\sqrt{(x^2 - c^2)}} q(1) + \int_0^c \frac{x}{\sqrt{(x^2 - r^2)}} q'\left(\frac{r}{c}\right) dr \right\} \quad ...(28)$$

Hence, the local stress intensity factor for varying shear load I is given by

$$I = \frac{\sqrt{c}}{\pi} \int_0^1 \frac{p(u)}{\sqrt{(1 - u^2)}} du. \quad ...(29)$$

If the shear load is uniform, from (21) the local stress intensity factor I_0 is given by

$$I_0 = \frac{1}{2} p_0 \sqrt{c} \quad ...(30)$$

which tallys with the result of classical elasticity.[7]

The nonlocal stress intensity factor for varying shear load I' is calculated by removing the singularity $\frac{1}{(x - c)^{5/2}}$ and is given by

$$I' = \epsilon^2 \frac{3}{2\sqrt{(2\pi)}} c^2 \sqrt{c} \int_0^1 \frac{p(u)}{\sqrt{(1 - u^2)}} du \quad ...(31)$$

If the shear load is uniform then the nonlocal stress intensity factor for uniform shear load I_0' is given by

$$I_0' = \epsilon^2 \frac{3}{4\sqrt{2}} p_0 c^2 \sqrt{c} \quad ...(32)$$

The stress intensity factors in various cases are summarised as follows :

(i) Local stress intensity factor for varying shear load

$$I = \frac{\sqrt{c}}{\pi} \int_0^1 \frac{p(u)}{\sqrt{(1 - u^2)}} du$$

(ii) Local stress intensity factor for uniform shear load

$$I_0 = \frac{1}{2} p_0 \sqrt{c}$$

(iii) Nonlocal stress intensity factor for varying shear load

$$I' = \epsilon^2 \frac{3}{2\sqrt{(2\pi)}} c^2 \sqrt{c} \int_0^1 \frac{p(u)}{\sqrt{(1 - u^2)}} du$$

(iv) Nonlocal stress intensity factor for uniform shear load

$$I_0' = \epsilon^2 \frac{3}{4\sqrt{2}} p_0 c^2 \sqrt{c}$$

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