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ON CIRCULATION PRESERVING MFD FLOWS WITH STEADY STREAMLINES

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In this note a theoretical analysis based on the equations of Hydromagnetic circulation-preserving flows with steady streamlines has been considered. Congruence of curves normals to a family of surfaces has also been assumed. More significant results are obtained concerning the velocity and vorticity magnitude for steady vortex lines. Finally, it has been shown that present analysis is valid in the entire flow region.

Key Words : Circulation-Preserving MFD Flows; Steady Streamlines

INTRODUCTION

SURYANARAYANA[1] has shown that Bernoulli's surface exists in the case of steady incompressible hydromagnetic flows when the magnetic field is in the fixed direction. Recently, Singh and Singh[2] discussed certain properties of showing the circulation-preserving MFD flows and obtained few geometrical results of physical importance. Although the circulation-preserving property is defined in terms of a material description of fluid flow, an equivalent condition, referring to the existence of the acceleration potential, presents a differential equation for the flow velocity in the Eulerian description :

$$\frac{\partial \mathbf{v}}{\partial t} + \text{grad}\,(p + \tfrac{1}{2}\rho v^2 + \tfrac{1}{2}\mu H^2) - (\mathbf{v} \times \text{curl}\,\mathbf{v}) = 0, \qquad \ldots(1)$$

where $\mathbf{v}$ is the fluid velocity vector, p is pressure and ρ is the density and H and v are magnetic field strength and magnitude of the fluid velocity respectively. This equation can be useful for investigating the nature of the field of flow in specific classes of circulation-preserving MFD flows. The present analysis introduces an additional assumption requiring the streamlines to be steady and to form a congruence of curves normal to a family of surfaces (complex-lamellar flows with steady stream lines). Such assumptions can be used to extract from the preceding equation a simple relation, valid at each instant and along each vortex line, between flow speed and the local spacing of the normal vector of the velocity. In a section that follows, more significant results are obtained concerning the velocity and vorticity magnitude for steady vortex lines. Lastly, it has been shown that the present analysis is valid in the entire flow region.

A Preliminary Results

We consider circulation-preserving and complex-lamellar MFD flows with steady stream lines, in which flow speed need not be steady. Let the velocity and the vorticity be represented by

$$\mathbf{v} = v\,\mathbf{s} \qquad \ldots(2)$$

and

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = \omega\mathbf{b}, \qquad \ldots(3)$$

where **s** and **b** are unit vectors. Then since the flow is complex lamellar, we have

$$\omega v\mathbf{s} \,.\, \mathbf{b} = \mathbf{v} \,.\, \operatorname{curl} \mathbf{v} = 0, \qquad \ldots(4)$$

according to a well-known criterion of complex-lamellar vector fields.[3,15] Equation (4) implies the orthogonality of the steady complex-lamellar field **s** with the generally unsteady vorticity direction **b**. The vector field defined by

$$\mathbf{n} = \mathbf{b} \times \mathbf{s} \qquad \ldots(5)$$

has a unit magnitude.

The steady complex-lamellar field **s** is orthogonal to a family of steady spatial surfaces $\xi\,(x, y, z) = \text{const}$

$$\operatorname{grad} \xi = |\operatorname{grad} \xi|\,\mathbf{s} \qquad \ldots(6)$$

The vector lines of **n** and **b** are orthogonal curves on these surfaces. From $\boldsymbol{\omega} \,.\, \mathbf{s} = 0$ have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} \,.\, \mathbf{s} = 0. \qquad \ldots(7)$$

Since a circulation-preserving MFD flow possesses an acceleration potential, the acceleration formula

$$\ddot{\mathbf{x}} = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \times \mathbf{v} + \tfrac{1}{2} \operatorname{grad} v^2 \qquad \ldots(8)$$

yields

$$\operatorname{curl} \ddot{\mathbf{x}} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \operatorname{curl} (\boldsymbol{\omega} \times \mathbf{v}) = 0. \qquad \ldots(9)$$

Applying the Kelvin transformation (Stokes' theorem) to a region in a ξ-surface bounded by a pair of **n**-lines and another pair of **b**-lines, and using (7), we obtain

$$\oint \omega \times v \,.\, \mathrm{d}\mathbf{x} = \iint \operatorname{curl} (\boldsymbol{\omega} \times \mathbf{v}) \,.\, \mathbf{s}\,\mathrm{d}A = -\iint \frac{\partial \boldsymbol{\omega}}{\partial t} \,.\, \mathbf{s}\,\mathrm{d}\,A = 0 \qquad \ldots(10)$$

If the **n**-lines and **b**-lines are considered as (time dependent) coordinate lines on the ξ-surfaces, the first fundamental form of the surface is expressed by

$$\mathrm{d}s^2 = k^2(\eta, \zeta, t)\,\mathrm{d}\eta^2 + l^2(\eta, \zeta, t)\,\mathrm{d}\zeta^2, \qquad \ldots(11)$$

where η is constant along each vortex line and may vary along the orthogonal n-lines. Equation (10) implies that

$$\frac{\partial}{\partial \zeta}(\omega v k) = 0 \qquad \ldots(12)$$

We next consider a vortex-tube bounded by four intersecting vortex-sheets of which two sheets are ξ-surfaces. Since the hydromagnetic flow is circulation-preserving, the flux of vorticity is the same for all sections of the vortex tube (Helmholtz's first theorem). In the limit of infinitesimal vortex tubes, we have

$$\frac{\partial}{\partial \zeta}\left(\frac{\omega k}{|\operatorname{grad} \zeta|}\right) = 0. \qquad \ldots(13)$$

Equations (12) and (13) give us

$$\frac{\partial}{\partial \zeta}\left(v\,|\operatorname{grad} \zeta|\right) = 0 \qquad \ldots(14)$$

Thus we obtain the following preliminary result :—

Theorem 1

Consider a circulation-preserving complex-lamellar MFD flow whose field lines are along fixed direction, stream-lines are steady while the speed is not necessarily steady. At each time instant and along each (generally unsteady) vortex lines, the flow speed is inversely proportional to the local spacing of the normal surfaces of the velocity field, and the product of the magnitudes of the velocity and vorticity is inversely proportional to the spacing of the vortex lines coexisting in the same normal surface of the velocity field.

Flow Field on a Stream Surface Emanating from Steady Vortex Lines

In this section, we assume that a steady vortex line ξ exists in a flow of the type considered in the previous section and investigate the geometry of the flow within the stream surface which contains the vortex line ξ.

The steady stream lines intersecting the steady curve ξ form a steady surface Γ. Γ is a material surface and it is usually called a Lamb Surface. The vortex lines on surface Γ, being the orthogonal trajectories of steady stream lines on a steady surface, are also steady. The first fundamental form of the surface Γ may be expressed as

$$ds^2 = h^2(\xi, \zeta)\, d\xi^2 + l^2(\xi, \zeta)\, d\zeta^2, \qquad \ldots(15)$$

where the coordinate ζ is constant along each stream line. The parameter ζ introduced earlier as a coordinate on a ξ-surface may be presently be redefined so that, along the line of intersection of the ξ-surface with the Lamb surface Γ, it agrees with the coordinate ζ of equation (15). Then the two different functions l occurring in equations (11) and (13) also have the identical values along the same line of intersection. The function h in (15) is given by

$$h = |\text{ grad } \xi|^{-1} \text{ on } \Gamma. \qquad \ldots(16)$$

Applying Stokes' theorem to a region of the surface bounded by two orthogonal pairs of coordinate lines, we obtain

$$\oint \mathbf{v} \cdot d\mathbf{x} = \iint \boldsymbol{\omega} \cdot \mathbf{n} dA = 0. \qquad \ldots(17)$$

Equations (15) and (17) yield

$$\frac{\partial}{\partial \zeta}(vh) = 0 \text{ on } \Gamma. \qquad \ldots(18)$$

By equations (12) and (18), we have

$$\frac{\partial}{\partial \zeta}(\mathbf{v}^2\, \omega\, h\, k) = 0 \text{ on } \Gamma \qquad \ldots(19)$$

Substituting (16) in (13), we have

$$\frac{\partial}{\partial \zeta}(\omega h k) = 0 \text{ on } \Gamma \qquad \ldots(20)$$

Consequently, we obtain from (18)–(20) the following results :

$$\frac{\partial h}{\partial \zeta} = 0, \frac{\partial v}{\partial \zeta} = 0, \frac{\partial \omega(k)}{\partial \zeta} = 0 \text{ on } \Gamma. \qquad \ldots(21)$$

The first equation (21) implies that the vortex lines on the Lamb surface Γ are parallel curves. Hence, the stream lines on Γ are geodesics. The Frenet normal vector of a stream line on Γ coincides with the surface normal of Γ, and the binormal direction of the stream line is tangent to the intersecting vortex line.

We next consider the acceleration field

$$\ddot{\mathbf{x}} = \dot{v}\mathbf{s} + v\dot{\mathbf{s}} = \dot{v}\mathbf{s} + v^2 \mathbf{s} \cdot \text{grad } \mathbf{s}. \qquad \ldots(22)$$

Since the stream lines on Γ are geodesics, Equation (22) reduces to

$$\ddot{\mathbf{x}} = v\mathbf{s} + v^2 k_s \mathbf{n} \text{ on } \Gamma, \qquad \ldots(23)$$

where k_s[6] denotes the curvature of a stream line. Consider a region of a ξ-surface bounded by two neighbouring vortex lines situated on the opposite sides of Γ, and by a pair of intersecting $\mathbf{n}$-lines. Applying Stoke's theorem to this region, we have

$$\oint v^2(\mathbf{s} \cdot \text{grad } \mathbf{s}) \cdot d\mathbf{x} = \iint \text{curl } \mathbf{x} \cdot \mathbf{n} dA = 0. \qquad \ldots(24)$$

When the pair of vortex lines bounding the region approach each other, we obtain

$$\oint v^2 k_s \mathbf{n} \cdot d\mathbf{x} = 0. \qquad \ldots(25)$$

Consequently,

$$\frac{\partial}{\partial \zeta}(v^2 k_s k) = 0. \qquad \ldots(26)$$

Equations (26) and (21) yield

$$\frac{\partial}{\partial \zeta}\left(\frac{k_s}{\omega}\right) = 0, \quad \frac{\partial}{\partial \zeta}(k_s) = 0. \qquad \ldots(27)$$

Summarizing, we have following theorem :

Theorem 2

Consider a circulation-preserving complex-Lamellar MFD flow with stream lines. If a steady vortex line ζ exists in the flow, then the stream lines intersecting ζ form a steady Lamb surface Γ. These stream lines are geodesics on Γ and their orthogonal trajectories are (parallel) steady vortex lines. Along each vortex line of Γ and at any instant, the flow speed is constant, whereas the vorticity varies in inverse proportion to the spacing of neighbouring vortex lines coexisting in the same normal surface of the velocity field. Equivalently, the vorticity varies in proportion to k_s of stream line segments intersected by the vortex line, provided that k_s does not vanish identically along the vortex line.

Corollary

For non-vanishing Frenet curvature k_s, the vorticity vector along the vortex line has the expression

$$\boldsymbol{\omega} = f(t)\, k_s \mathbf{b}, \qquad \ldots(28)$$

where $f(t)$ is the same function for all points on the vortex lines. In particular, if $\boldsymbol{\omega}$ is steady at one point of a vortex line on Γ it is steady at all points of the vortex lines; if $\boldsymbol{\omega}$ is steady along one curved stream line on Γ, it is steady on the whole Lamb surface Γ.

Finally, on the Lamb surface Γ, the acceleration formula (22) reduces to

$$\ddot{\mathbf{x}} = \left\{\frac{\partial v}{\partial t} + \frac{1}{2}\frac{1}{h}\frac{\partial}{\partial \xi}(v^2)\right\}\mathbf{s} + v^2 k_s \mathbf{n}.$$

Since both h and v are constant along a vortex line on Γ, the s-component of the acceleration is also constant along the vortex line. The **n**-component of the acceleration varies along the line in the direct proportion to the curvature of the intersecting stream line, and therefore it either vanishes identically along the vortex line (the case $k_s = 0$) or varies in proportion to the vorticity magnitude. The surfaces of constant acceleration potential intersect the surfaces ξ = constant along the steady vortex lines on Γ.

Discussion

On the above Lamb surface, the flow speed and the vorticity magnitude are generally unsteady. In the region outside this surface, even the vortex lines are generally unsteady and do not mesh with stream lines to form neighbouring Lamb surfaces at any instant of time. Indeed, the assumptions of this kinematical

analysis may be weakened to the extent of merely requiring the hydromagnetic flow to possess steady streamlines and to be circulation-preserving and complex-lamellar in a thin layer of space that surrounds a stream sheet past a steady vortex-line. Upon this stream sheet all conclusions of the Theorem would still be valid, and nothing need to assumed concerning the motion of magnetofluid in the region of space outside the thin layer. The theorem attains a higher dimension of generality precisely because its conclusions refer a lower dimension of space. Furthermore, it follows from our results that in a circulation-preserving complex-lamellar MFD flows with both steady streamlines and steady vortex lines, the flow region is filled with a family of Lamb surfaces and the statement is valid on each surface.

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