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EIGENFUNCTION EXPANSION METHOD TO THE SOLUTION OF LONGITUDINAL VIBRATION OF A CIRCULAR CYLINDER COUPLED WITH A THERMAL FIELD AND A MAGNETIC FIELD

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The basic equations of the problems of longitudinal vibration of a cylinder coupled with a thermal field and a magnetic field have been written in the form of a vector matrix differential equation and solved by the eigenfunction expansion method. The effect of the coupling is determined for a given intensity of the magnetic field and wave number. The results are also compared with the existing literature in absence of the magnetic field.

Key Words : Eigenfunction; Longitudinal Vibration; Circular Cylinder; Thermal and Magnetic Fields

INTRODUCTION

THE problem of interaction of the elastic field coupled with a thermal field is generally very complex is nature, because it relates two separately and independently developed sciences. Such problems become unwiedly when the magnetic field is also coupled with the thermoelastic field.

Although a bulk of papers on magnetothermoelastic interactions have been published,[1,2] but still the researches in this field are in the formative state. From the last decade some researchers became interested in this field because of its wide applicability in the field of Engineering and Technology.

Suhubi[3] considered the problem of longitudinal vibration of a circular cylinder coupled with a thermal field. Suhubi[4] also considered the problem of small torsional oscillation of a circular cylinder with finite electric conductivity in a constant axial magnetic field.

In the present study we have dealt with the longitudinal vibration of a circular cylinder under prescribed conditions of the thermal and the magnetic fields. Das and Bhakta[5] have established the solution of a homogeneous vector matrix differential equation when the operator is of the linear second order. The coupled equations of the three fields viz., elastic, thermal and magnetic in the present problem are written in the form of a vector-matrix differential equation and solved by the method of eigenfunction expansion.

Basic Equations

We consider the cylinder to be elastic, perfectly conducting and the magnetic vector is parallel to its axis. In agreement with Paria[1] the equations of magnetothermoelasticity for a perfect conductor have the form

$$(\lambda + 2\mu)\,\text{grad div}\,\mathbf{u} - \mu\,\text{rot rot}\,\mathbf{u} - \gamma\,\text{grad}\,\theta + \mu_e\,[\text{rot rot}\,(\mathbf{u} \times \mathbf{H})] \times \mathbf{H} = \rho\frac{d^2\mathbf{u}}{dt^2}$$

and

$$\mathbf{K}_1\,\text{div grad}\,\theta - \frac{\nu T}{\rho c}\frac{\partial}{\partial t}\,\text{div}\,\mathbf{u} = \frac{d\,\theta}{dt},$$

where

$$\mathbf{E} = -\frac{\mu_e}{c}\left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}\right) \qquad \ldots (1)$$

and

$$\mathbf{h} = \text{rot}\,(\mathbf{u} \times \mathbf{H}),$$

where λ, μ are the Lame' constants, $\mathbf{u}$ the displacement vector, $\gamma = \alpha_T(3\lambda + 2\mu)$, α_T is the thermal expansion coefficient, θ is the differential temperature distribution, ρ is the mass density of the medium, $K_1 = \dfrac{K}{\rho c}$ is the diffusivity, K is the thermal conductivity, c is the specific heat per unit mass at constant strain, T is the reference temperature, E is the electric field vector, $\mathbf{H}$ is the magnetic field vector, $\mathbf{h}$ is the perturbation of the magnetic field, μ_e is the magnetic permeability.

Reducing the equation (1) in cylindrical polar coordinates (r, ϕ, z), z being the axis of the cylinder, and assuming $u_\phi = 0$ and u_r, u_z, θ are function of r and z only, we get

$$(\lambda + 2\mu + \mu_e H^2)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) + (\mu + \mu_e H^2)\frac{\partial^2 u_r}{\partial z^2} + (\lambda + \mu) \times \frac{\partial^2 u_z}{\partial r \partial z} - \frac{\gamma \partial \theta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \qquad \ldots(2)$$

$$(\lambda + 2\mu)\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) - \frac{\mu}{r}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) - \gamma\frac{\partial \theta}{\partial z} - \mu\left(\frac{\partial^2 u_r}{\partial z \partial r} - \frac{\partial^2 u_z}{\partial r^2}\right) = \rho\frac{\partial^2 u_z}{\partial t^2} \qquad \ldots(3)$$

and

$$K_1\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2}\right) - \frac{\gamma T}{\rho c}\frac{\partial \Delta}{\partial t} = \frac{\partial \theta}{\partial t}, \qquad \ldots(4)$$

where $\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$, $\mathbf{H} = (0, 0, H)$ and for simplicity we assumed $\mu_e \simeq 1$.

Solution of the Equations

Let us assume that all the quantities are simple harmonic functions of z and t and

$$(u_r, u_z, \theta) = [U(r), iW(r), \theta] \exp [i(qz + pt)]. \quad ...(5)$$

Substituting (5) in (2)–(4), we get

$$\frac{d^2U}{d\zeta^2} + \frac{1}{\zeta}\frac{dU}{d\zeta} + \left(\omega^2 - \frac{1}{\beta^2} - \frac{1}{\zeta^2}\right)U - \left(1 - \frac{1}{\beta^2}\right)\frac{dW}{d\zeta} - \frac{\gamma}{pq(C_1^2 + a_0^2)}\frac{d\theta}{d\zeta} = 0 \quad ...(6)$$

$$\frac{d^2W}{d\zeta^2} + \frac{1}{\zeta}\frac{dW}{d\zeta} + [\omega^2(\beta_0^2 + \alpha_0^2) - \beta_0^2]W + (\beta_0^2 - 1)\left(\frac{dU}{d\zeta} + \frac{U}{\zeta}\right) - \frac{\gamma}{\mu q}\theta = 0 \quad ...(7)$$

$$\frac{d^2\theta}{d\zeta^2} + \frac{1}{\zeta}\frac{d\theta}{d\zeta} - \left(1 + \frac{ip}{K_1 q^2}\right)\theta - \frac{ip\gamma T}{K_1 \rho c q}\left(\frac{dU}{d\zeta} + \frac{U}{\zeta}\right) + \frac{ip\gamma T}{K_1 q \rho c}W = 0 \quad ...(8)$$

The equations (6)–(8) can be written as

$$L(U) = a_{11}U + a_{12}\frac{dW}{d\zeta} + a_{13}\frac{d\theta}{d\zeta},$$

$$L\left(\frac{dW}{d\zeta}\right) = a_{21}U + a_{22}\frac{dW}{d\zeta} + a_{23}\frac{d\theta}{d\zeta} \quad ...(9)$$

and

$$L\left(\frac{d\theta}{d\zeta}\right) = a_{31}U + a_{32}\frac{dW}{d\zeta} + a_{33}\frac{d\theta}{d\zeta},$$

where

$$a_{11} = \left(\frac{1}{\beta^2} - \omega^2\right), a_{12} = \left(1 - \frac{1}{\beta^2}\right), a_{13} = \frac{\gamma}{\rho q(C_1^2 + a_0^2)},$$

$$a_{21} = (1 - \beta_0^2)\left(\frac{1}{\beta^2} - \omega^2\right), a_{22} = \frac{\beta^2 + \beta_0^2 - 1}{\beta^2} - \omega^2(\beta_0^2 + \alpha_0^2),$$

$$a_{23} = \frac{\gamma}{\rho q}\frac{1}{C_2^2\beta^2}, a_{31} = \frac{i\omega\sqrt{C_1^2 + a_1^2}\gamma T}{K_1\rho c}\left(\frac{1}{\beta^2} - \omega^2\right),$$

$$a_{32} = -\frac{i\omega\sqrt{C_1^2 + a_0^2}\gamma T}{K_1\rho c\beta^2},$$

$$a_{33} = \frac{i\omega\gamma^2 T}{K_1\rho^2 qc\sqrt{C_1^2 + a_0^2}} + \frac{i\omega\sqrt{C_1^2 + a_0^2}}{K_1 q} + 1,$$

$$\beta^2 = \frac{C_1^2 + a_0^2}{C_2^2 + a_0^2},\ C_1^2 = \frac{\lambda + 2\mu}{\rho},\ C_2^2 = \frac{\mu}{\rho},$$

$$a_0^2 = \frac{H^2}{\rho},\ \alpha_0^2 = \frac{a_0^2}{C_2^2},\ \beta_0^2 = \frac{C_1^2}{C_2^2},\ \zeta = qr,$$

$$\omega^2 = \frac{p^2}{q^2(C_1^2 + a_0^2)} \qquad \ldots(9a)$$

and

$$L \equiv \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta} - \frac{1}{\zeta^2}. \qquad \ldots(9b)$$

We now write the equations (9) in the form of a vector-matrix differential equation as

$$L\mathbf{v}\ (\zeta) = \mathbf{A}\,\mathbf{v}\,(\zeta). \qquad \ldots(10)$$

In order to solve (10) we substitute

$$\mathbf{v}(\zeta) = \mathbf{X}J_1\,(\alpha\zeta)\ (\textit{see Appendix}) \qquad \ldots(11)$$

where α is a scalar. This leads to the algebraic eigenvalue problem

$$\mathbf{AX} = \bar{\lambda}\,\mathbf{X}, \bar{\lambda} = -\alpha^2. \qquad \ldots(12)$$

The characteristic equation corresponding to the matrix **A** is given by

$$\bar{\lambda}^3 - (D + F)\,\bar{\lambda}^2 + (G + E + DF - AC)\,\bar{\lambda} - (DE - AB - GM) = 0, \qquad \ldots(13)$$

where

$$A = (1 - \beta_0^2)\left(\frac{1}{\beta^2} - \omega^2\right),\ B = C\left(i + \frac{i\omega\sqrt{C_1^2 + a_0^2}}{K_1 q}\right) + i\omega\Gamma$$

$$C = \left(1 - \frac{1}{\beta^2}\right),\ D = C + \frac{\beta_0^2}{\beta^2} - \omega^2\,(\beta_0^2 + \alpha_0^2),\ E = N\left(\frac{1}{\beta^2} - \omega^2\right)$$

$$F = i\omega\Gamma + \left(\frac{1}{\beta^2} - \omega^2\right) + N,\ G = i\omega\Gamma\left(\frac{1 - \beta_0^2}{\beta^2} + 1 + \alpha_0^2\right)$$

$$M = \omega^2\beta^2 - 1,\ N = \left(1 + \frac{i\omega\sqrt{C_1^2 + a_0^2}}{Kq}\right)$$

and

$$\Gamma = \frac{\gamma^2 T}{K_1\rho^2 qc\sqrt{C_1^2 + a_0^2}}.$$

Γ is called the magnetothermoelastic coupling factor.

The eigenvectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ corresponding to the eigenvalues λ_1, λ_2, λ_3 (which are roots of (13)) can be calculated as

$$\mathbf{X}_j = \left\{ \begin{array}{l} a_{12}(a_{33} - \lambda_j) - a_{32}\, a_{13} \\ a_{13}a_{31} - (a_{11} - \lambda_j)\,(a_{33} - \lambda_j) \\ (a_{11} - \lambda_j)\, a_{32} - a_{31}a_{12} \end{array} \right\} = \left\{ \begin{array}{l} x_{1j} \\ x_{2j} \\ x_{3j} \end{array} \right\} \text{(say)} \quad \ldots(14)$$

$$j = 1, 2, 3$$

This leads to the solution of (10) as

$$\mathbf{v}(\zeta) = A\mathbf{X}_1 J_1(\alpha_1\zeta) + B\mathbf{X}_2 J_1(\alpha_2\zeta) + C\mathbf{X}_3 J_1(\alpha_3\zeta), \quad \ldots(15)$$

where $\alpha_j^2 = -\lambda_j$, $j = 1, 2, 3$ and A, B, C are arbitrary constants.

The components of the space vector can now be written as

$$\begin{aligned} U &= Ax_{11}J_1(\alpha_1\zeta) + Bx_{12}J_1(\alpha_2\zeta) + Cx_{13}J_1(\alpha_3\zeta) \\ \frac{dW}{d\zeta} &= Ax_{21}J_1(\alpha_1\zeta) + Bx_{22}J_1(\alpha_2\zeta) + Cx_{23}J_1(\alpha_3\zeta) \quad \ldots(16) \\ \frac{d\theta}{d\zeta} &= Ax_{31}J_1(\alpha_1\zeta) + Bx_{32}J_1(\alpha_2\zeta) + Cx_{33}J_1(\alpha_3\zeta) \end{aligned}$$

Frequency Equation

We assume that the lateral surface of the cylinder is kept at the ambient temperature. In this case the boundary conditions are

$$\tau_{rr} + T_{rr} = \tau_{rz} + T_{rz} = \theta = 0, \text{ for } \zeta = qa = u \text{ (say)}, \quad \ldots(17)$$

where a is the radius of the cylinder, τ_{rr}, τ_{rz} are elastic stresses and T_{rr}, T_{rz} are the corresponding Maxwell's stresses for the material. We assume that the Maxwell's stress in the vacuum are negligible. The stresses are given by[2]

$$\tau_{ij} = 2\mu e_{ij} + (\lambda\Delta - \gamma\theta)\,\delta_{ij}, \Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

and

$$T_{ij} = \mu_e(H_i h_j + H_j h_i - \delta_{ij}\, \mathbf{h}.\mathbf{H}), \quad \ldots(17a)$$

where $\mathbf{h}$ is given by the equation (1).

Thus using equations (16), (17), (17a) we determine the temperature, elastic stresses and the Maxwell's stresses. We now use the boundary conditions (17) and obtain three homogeneous linear equations in A, B, C. In order that a set of non-trivial solution must exist, we get the following condition, which is also the frequency equation, must be satisfied

$$\det(b_{ij}) = 0, \; i, j = 1, 2, 3, \quad \ldots(18)$$

where

$$b_{11} = \left[\left\{(\mu\beta_0 + H^2)\ x_{11}\alpha_1 + \lambda\frac{x_{21}}{\alpha_1}\right\} J_0(\alpha_1 u) - 2\mu x_{11}\alpha_1 \frac{J_1(\alpha_1 u)}{\alpha_1 u}\right],$$

$$b_{12} = \left[\left\{(\mu\beta_0^2 + H^2)\ x_{12}\alpha_2 + \lambda\frac{x_{22}}{\alpha_2}\right\} J_0(\alpha_2 u) - 2\mu x_{12}\alpha_2 \frac{J_1(\alpha_2 u)}{\alpha_2 u}\right],$$

$$b_{13} = \left[\left\{(\mu\beta_0^2 + H^2)\ x_{13}\alpha_3 + \lambda\frac{x_{23}}{\alpha_3}\right\} J_0(\alpha_3 u) - 2\mu x_{13}\alpha_3 \frac{J_1(\alpha_3 u)}{\alpha_3 u}\right],$$

$$b_{21} = (x_{11} + x_{21})\, J_1(\alpha_1 u),\ b_{22} = (x_{12} + x_{22})\, J_1(\alpha_2 u),$$

$$b_{23} = (x_{13} + x_{23}) J_1(\alpha_3 u),\ b_{31} = \frac{x_{13}}{\alpha_1} J_0(\alpha_1 u)$$

and

$$b_{32} = \frac{x_{32}}{\alpha_2} J_0(\alpha_2 u),\ b_{33} = \frac{x_{33}}{\alpha_3} J_0(\alpha_3 u).$$

It is extremely difficult to obtain roots of this transcendental equation. However, if the radius of the cylinder and the coupling factor are assumed to be small, we can get an approximate expression for ω, and for velocity of propagation.

Expanding the Bessel's function and omitting terms of higher order than the first in u, we find the following expression :

$$\begin{aligned}&\alpha_1^2 x_{11}(F_2 x_{22} x_{33} - F_3 x_{23} x_{32}) + \alpha_2^2 x_{12}(F_3 x_{23} x_{31} - F_1 x_{21} x_{33})\\ &\quad + \alpha_3^2 x_{13}(F_1 x_{21} x_{32} - F_2 x_{22} x_{31}) - (F_4 x_{21} x_{22} x_{33})\\ &\quad + F_5 x_{22} x_{23} x_{31} + F_6 x_{21} x_{23} x_{32}) = 0, \qquad \ldots(19)\end{aligned}$$

where $F_1 = -(\lambda + \mu + H^2)\,\alpha_1^2 + \lambda,\ F_4 = \lambda(\alpha_2^2 - \alpha_1^2),$

$F_2 = -(\lambda + \mu + H^2)\,\alpha_2^2 + \lambda,\ F_5 = \lambda(\alpha_3^2 - \alpha_2^2)$

and $F_3 = -(\lambda + \mu + H^2)\,\alpha_3^2 + \lambda,\ F_6 = \lambda(\alpha_1^2 - \alpha_3^2).$

To find ω directly from this equation is still very difficult. Instead, assuming Γ, α_0^2 to be small quantities, we shall find approximate solution to this equation in the following special cases.

Case I

If we put $\alpha_0^2 = 0$ and $\Gamma = 0$ (i.e. in absence of the magnetic field and the thermal field), the equation (19) reduces to

$$\begin{aligned}&\left(1 - \frac{1}{\beta_0^2}\right)\alpha_3^2\,\{\alpha_1^2\mu(\beta_0^2 - 1) - \lambda\} + \left(\omega_0^2 + 1 - \frac{1}{\beta_0^2} - \beta_0^2\omega_0^2\right)\\ &\qquad \times [\,2\lambda\alpha_1^2 - \alpha_3^2\{\mu(\beta_0^2 - 1)\,\alpha_1^2 + \lambda\}] = 0, \qquad \ldots(20)\end{aligned}$$

where $\beta_0^2 = [\beta^2]_{\alpha=\Gamma=0},\quad \omega_0^2 = [\omega^2]_{\alpha=\Gamma=0}$

$\alpha_1^2 = 1 - \omega_0^2$ and $\alpha_3^2 = \beta_0^2\omega_0^2 - 1.$

The solution of the equation (20) is well known[3] and is given by

$$\omega_0^2 = \left[\frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)(\lambda + \mu)}\right]. \qquad \ldots(21)$$

Case II

If we put $\alpha_0^2 = 0$ (i.e., in the absence of magnetic field) in the equation (19), we obtain the frequency equation which is the same as the equation (4.1) of Suhubi[3] in another form. The form (4.1) of Suhubi[3] can be obtained from our equation (18) by the use of the characteristic equation (13).

Case III

We now apply numerical methods in the frequency equation (19) to determine the change in the frequency ω when both the thermal field and the magnetic field are prescribed. As such, we consider the material to be Aluminium for which the data (in *MKS* units) considered are as follows :

$\lambda = 0.4137E + 11$ N/m^2 $\qquad \mu = 0.2750E + 11$ N/m^2

$\rho = 0.2700E + 04$ Kg/m^3 $\qquad K_1 = 0.8418E - 04$ m^2/s

$c = 0.896E + 03 J$/Kg °C $\qquad T_0 = 0.293E + 03$ °C

$q = 0.1E + 10$ $\qquad \alpha_T = 0.24E - 04$ °C

$\alpha = 0.1E - 01$ A^2/N

We apply the Newton-Raphson method to estimate the change in the frequency ω.

Since there is no first order change of ω when thermal field is introduced (in absence of magnetic field), *vide* Suhubi,[3] it is instructive to choose the initial value of ω from (21) as

$$\omega_0 = \sqrt{\left[\frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)(\lambda + \mu)}\right]} = 0.86024360.$$

The computations have been made in Burroughs 6700 Computer of the Regional Computer Centre, Calcutta, and the stable value of ω has been obtained as

$$\omega = (0.61555365) + i(0.0032861 9) \qquad \ldots(22)$$

Thus the velocity of propagation of the wave and damping factor can be obtained from (22) by using this value of ω in (5) and (9*a*) for the above data.

CONCLUSION

The method of elimination for the solution of (6)–(8) is perhaps, not at all practicable. Instead, the present problem could have been solved by choosing the general solution of (6)-(8) as

$$U = A_1\nu_1 J_1(\nu_1\zeta) + A_2\nu_2 J_1(\nu_2\zeta) + A_3\nu_3 J_1(\nu_3\zeta),$$

$$W = A_1 B_1 J_0(\nu_1\zeta) + A_2 B_2 J_0(\nu_2\zeta) + A_3 B_3 J_0(\nu_3\zeta) \qquad \ldots(23)$$

and $$\theta = A_1 B'_1 J_0(\nu_1\zeta) + A_2 B'_2 J_0(\nu_2\zeta) + A_3 B'_3 J_0(\nu_3\zeta),$$

where A_1, A_2, A_3 are the arbitrary constants. Substituting (23) in (6)–(8), we may finally obtain nine equations for the determination of B_1, B_2, B_3, B_1', B_2', B_3' and the characteristic equation for the values of ν_1, ν_2, ν_3. But the process requires very extensive algebra.

In fact, the above inspection (23) has led us to write the equations (6)–(8) in the form (10), and transform the problem to an algebraic eigenvalue problem. Thus we have avoided a very extensive algebra to solve the problem.

The advantage of this approach lies in its conceptual simplicity and its flexibility. In this problem the eigenvalues are distinct, but in case, the eigenvalues are repeated—the detailed theory along with applications can be found.[5]

Thus a broad class of problems[6,7,8,9] where a number of interacting fields provides a system of simultaneous equations, can be solved by this method.

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Appendix

Theory

Let the vector-matrix differential equation be of the form

$$L\mathbf{X} = t^2 \mathbf{A}\mathbf{X} \qquad \ldots(24)$$

where L is the linear second order differential operator

$$L = t^2 \frac{d^2}{dt^2} + tp(t)\frac{d}{dt} + q(t).$$

$\mathbf{A}$ is an $n \times n$ constant matrix and $p(t)$, $q(t)$ are two real valued continuous functions on [0,1]. The initial conditions are assumed as

$$\mathbf{X}(1) = \mathbf{a} \text{ and } \mathbf{X}'(1) = \mathbf{b}, \qquad \ldots(25)$$

where $\mathbf{X}$, $\mathbf{a}$, $\mathbf{b}$ are n–vectors.

Assume that $\mathbf{X}(t) = \mathbf{X}(\lambda)\,\omega(t, \lambda)$ is a solution of the equation (1), where λ is a scalar, $\mathbf{X}$ is an n-vector independent of t and $\omega(t,\lambda)$ is the nontrivial solution of the scalar differential equation

$$t^2 \frac{d^2y}{dt^2} + tp(t) \frac{dy}{dt} + q(t)\, y = t^2\lambda y,$$

i.e. $Ly = \lambda t^2 y$ ...(26)

Applying the operator L on $\mathbf{X}$, we get

$$L\mathbf{X} = L(\mathbf{X}\omega) = \mathbf{X}L\omega = \mathbf{X}(t^2\lambda\omega) = \lambda t^2\mathbf{X}\omega.$$

Thus the equation (1) becomes

$$\lambda t^2\, \mathbf{X}\omega = t^2\mathbf{A}(\mathbf{X}\omega) = t^2(\mathbf{A}\mathbf{X})\omega$$

or $$t^2(\lambda\mathbf{X} - \mathbf{A}\mathbf{X})\omega = 0$$

Since $\omega(t,\lambda)$ is non-trivial and $(\lambda\mathbf{X} - \mathbf{A}\mathbf{X})$ is independent of t, it follows that

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$$

This gives rise to the algebraic eigenvalue problem where λ is the eigenvalue of the matrix $\mathbf{A}$ and $\mathbf{X}$ is the corresponding eigenvector.

Distinct Eigenvalues

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the distinct eigenvalues of the matrix $\mathbf{A}$ and let $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n$ be the corresponding eigenvectors. Then $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n$ are linearly independent and so they form a basis of the space Γ^n, where Γ is the field of complex numbers. We can find scalars $a_1, a_2 \ldots_n$ and $b_1, b_2 \ldots b_n$ such that $\mathbf{a} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + \ldots + a_n\mathbf{X}_n$ and

$$\mathbf{b} = b_1\mathbf{X}_1 + b_2\mathbf{X}_2 + \ldots + b_n\mathbf{X}_n$$

Let $u(t,\lambda_i)$ and $v(t,\lambda_i)$ denote two linearly independent solutions of the differential equation

$$Ly = \lambda_i t^2 y$$

with initial conditions

$$u(1, \lambda_i) = 1,\ u'(1, \lambda_i) = 0$$

and

$$v(1, \lambda_i) = 0,\ v'(1, \lambda_i) = 1.$$

Now let $$\mathbf{X}(t) = \sum_{i=1}^{n} \mathbf{X}_i\{a_i u(t,\lambda_i) + b_i v(t, \lambda_i)\}.$$ ...(27)

Then clearly, $\mathbf{X}(t)$ is a solution of the differential equation (1) and

$$\mathbf{X}(1) = \sum_{i=1}^{n} \mathbf{X}_i\{a_i u(1, \lambda_i) + b_i v(1, \lambda_i)\} = \sum_{i=1}^{n} a_i\mathbf{X}_i = \mathbf{a}$$

$$\mathbf{X}'(1) = \sum_{i=1}^{n} \mathbf{X}_i\{a_i u'(1, \lambda_i) + b_i v'(1, \lambda_i)\} = \sum_{i=1}^{n} b_i\mathbf{X}_i = \mathbf{b}$$

Therefore $\mathbf{x}(t)$, given by (27), is the unique solution of the system (24) and (25).