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CLOSED FORM SOLUTION TO STEADY MAGNETO-HYDRODYNAMIC FLOW OF INVISCID INCOMPRESSIBLE FLUID BETWEEN A PARALLEL FLAT WALL AND A CURVED WALL WITH A CONTINUOUS SUCTION THROUGH THE FORMER WALL*

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Three different models of two dimensional steady parallel incompressible non-viscous magneto-hydrodynamic flow between a flat wall and a stream lined curved wall, coupled with a continuous suction through the former wall, are studied. The boundary conditions are not necessarily the same for all the case-studies. However, for each model of flow the velocity and pressure distributions are determined alongwith the equation of the required curved boundary surface.

Key Words : Non-viscous MHD Flow; Inviscid Incompressible Fluid; Flat/ Curved Wall; Two-dimensional Motion

INTRODUCTION

MAGNETOHYDRODYNAMIC flow between two parallel flat walls with a constant suction through one of them has been studied by many workers. N Bhaskara Reddy and D Bathaih[1] worked on the magneto-hydrodynamic flow of a viscous, incompressible, slightly conducting fluid between a parallel flat wall and a long wavy wall coupled with a constant suction through the former wall.

A few fundamental problems of magnetohydrodynamics without suction have been showed in textbook.[2] In this paper an attempt has been made to develop a model of magnetohydrodynamic flow of inviscid incompressible fluid between a flat wall and a curved surface where, at the same time, there will be a continuous suction through the flat wall.

In the presence of a magnetic field perpendicular to the flat wall, the two-dimensional motion of the fluid is governed by the eqns :

$$\rho\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right)=-\frac{\partial P^*}{\partial X}-\sigma\mu_e^2H_0^2U, \qquad \ldots(1)$$

$$\rho\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right)=-\frac{\partial P^*}{\partial Y} \qquad \ldots(2)$$

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and $$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \qquad \ldots(3)$$

where ρ is the fluid density, U and V the velocity components along the $X-$ and $Y-$ axes of co-ordinate system, P^* the fluid pressure, σ the electrical conductivity, μ_e the magnetic permeability and H_0 the intensity of the magnetic field neglecting the induced magnetic field.

Formulation of the Problem

Such a two-dimensional flow is irrotational or rotational having the curved boundary wall as stream lined surface. At the same time the present analysis aims at obtaining closed form solutions to the fluid flow. Keeping this in view, let us investigate whether it is possible without the sacrifice of physical significance to ensure three different models of MHD flow having respectively :

$$(i) \quad \frac{\partial U}{\partial X} = 0 \qquad \text{or equivalently} \qquad \frac{\partial V}{\partial Y} = 0, \qquad \ldots(4)$$

i.e., $U = U(Y)$ or equivalently $V = V(X)$ with a constant suction through the former flat wall,

$$(ii) \quad \frac{\partial U}{\partial X} = 0 \text{ and } \frac{\partial V}{\partial Y} = 0 \qquad i.e., \; U = U(Y) \qquad \ldots(5)$$

and $V = V(X)$ together with a continuous suction through the flat wall,

$$(iii) \quad \frac{\partial U}{\partial Y} = 0, \; \frac{\partial V}{\partial X} = 0 \qquad \ldots(6)$$

or equivalently $U = U(X)$ and $V = V(Y)$, coupled with a continuous suction through the flat wall, and if possible, then what kind of boundary conditions should be prescribed; the consequent pressure gradients or pressure function, and the type of curved boundary wall required are to be determined in each of the above cases. For each possible flow, the equation of the curved streamlined boundary wall has to be determined; this curved wall can be employed in a practical situation.

Equations of Motion and Closed Form Solutions to the Typical MHD Flow

Case 1 : In view of (4), eqns (1) to (3) get simplified to

$$\rho V \frac{\partial U}{\partial Y} = -\frac{\partial P^*}{\partial X} - \sigma \mu_e^2 H_0^2 U, \qquad \ldots(7)$$

$$0 = \frac{\partial P^*}{\partial X} \qquad \ldots(8)$$

and $$0 = \frac{\partial U}{\partial X} = \frac{\partial V}{\partial Y}, \qquad \ldots(9)$$

where the origin of the two dimensional co-ordinate system lies on the curved surface; the X-axis is a line parallel to the flat wall and passing through the origin; the Y-axis is perpendicular to the X-axis, and passing through the origin.

The fluid passes through the system of a flat wall represented by the equation

$$Y = d^* \qquad \ldots(10)$$

and the streamlined curved wall whose equation is yet to be found out depending on the nature of flow and the boundary conditions. Thus the system through which the fluid flows consists of a flat parallel wall and a curved wall, and is of length b^* equal to that of the former wall, and has inlet and outlet of different cross sections. The origin is chosen to lie on the inlet. Let the following boundary conditions be set forth :

$$U = 0, V = V_0 = \text{constant suction velocity, on the flat wall } Y = \mathrm{d}^*. \quad \ldots(11)$$

The condition (11), as a consequence of eqn (9) reveals that the Y-component of the fluid velocity is always constant in this case i.e.,

$$V = V_0 \qquad \ldots(12)$$

On introducing the non-dimentional parameters,

$$u = \frac{U}{V_0}, \quad P = \frac{P^* h^3}{V_0^2}, \quad v = \frac{V}{V_0}$$

$$x = \frac{X}{h}, \quad \eta = \frac{Y}{h}, \quad \mathrm{d} = \frac{\mathrm{d}^*}{h}$$

$$b = \frac{b^*}{h}, \quad u_0 = \frac{U_0}{V_0} \qquad \ldots(13)$$

the eqns (7) to (9) combined with (12) turn out in dimensionless form :

$$R \frac{\mathrm{d}u}{\mathrm{d}\eta} + Mu = -\frac{\partial P}{\partial x} = C \text{ (constant)}, \qquad \ldots(14)$$

where $R = \rho h^3$ = suction parameter, and $M = \dfrac{\sigma \mu_e H_0^2 h^4}{V_0}$ = magnetic parameter and by virtue of eqns (7) and (8) the pressure gradient becomes a constant; the corresponding non-dimensional boundary conditions become

$$u = 0, \quad v = v_0 = 1 \text{ on the flat boundary surface } \eta = \mathrm{d}; \qquad \ldots(15)$$

$u = u_0$, $v = V_0 = 1$ at the origin lying on the curved wall, so that the solutions to (14) yield

$$u = \frac{C}{M}\left\{1 - \exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right)\right\}, \; u_{\max} = u_0 = \frac{C}{M}\left(1 - \exp\left(\frac{M}{R}\,\mathrm{d}\right)\right) \qquad \ldots(16)$$

$$\text{so that } u = u_0 \frac{1 - \exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right)}{1 - \exp\left(\frac{M}{R}\mathrm{d}\right)}, \quad v = 1; \quad \eta < \mathrm{d} \qquad \ldots(17)$$

So as to make the curved wall the streamlined surface, the tangent at any point of which gives the direction of the fluid velocity,

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{V}{U} \qquad \ldots(18)$$

which, because of (12) and (13), gives in dimensionless form :

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \frac{1}{u}$$

and by the use of (15) and (17) one can have

$$x = \frac{u_0\left[\eta + \frac{R}{M}\left\{\exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right) - \exp\left(\frac{M}{R}\mathrm{d}\right)\right\}\right]}{1 - \exp\left(\frac{M}{R}\mathrm{d}\right)}, \qquad \ldots(19)$$

which is the equation of the required curved boundary wall. Since, for this curved surface as x increases η increases, the Y co-ordinate (η) of the highest point of this wall that lies at the outlet satisfies the transcendental eqn.

$$b = \frac{u_0\left[\eta + \frac{R}{M}\left\{\exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right) - \exp\left(\frac{M}{R}\mathrm{d}\right)\right\}\right]}{1 - \exp\left(\frac{M}{R}\mathrm{d}\right)}$$

Now the vorticity ξ and the stream function ψ for the pattern of flow are expressed as

$$\xi = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial \eta}\right) = \frac{C}{R}\exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right), \qquad \ldots(20)$$

$$\mathrm{d}\psi = -\frac{u_0}{1 - \exp\left(\frac{M}{R}\mathrm{d}\right)}\left\{1 - \exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right\}\mathrm{d}\eta + \mathrm{d}x\right.$$

$$\text{or} \qquad \psi = \frac{-u_0}{1 - \exp\left(\frac{M}{R}\mathrm{d}\right)}\left\{\eta + \frac{R}{M}\exp\left(\frac{M}{R}(\mathrm{d} - \eta)\right\} + x, \qquad \ldots(21)\right.$$

which shows that the fluid motion is rotational.

If P_0 pnd P_b be the pressures at the inlet and outlet respectively, the constant pressure gradient from (14) after being combined with (16) gives

$$P_b = P_0 - \frac{u_0 M b}{1 - \exp\left(\frac{M}{R}\mathrm{d}\right)} \qquad \ldots(22)$$

Case II

This flow pattern evolves $U = U(Y)$ and $V = V(X)$ and as such eqns (1) to (3) in conjunction with the boundary conditions (13) lead to

$$Rv \frac{du}{d\eta} + Mu = -\frac{\partial P}{\partial x}, \qquad \ldots(23)$$

$$Ru \frac{dv}{dx} = -\frac{\partial P}{\partial \eta} \qquad \ldots(24)$$

and $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \eta} = 0$$

which can be combined to

$$v \frac{d^2u}{d\eta^2} + \frac{M}{R}\frac{du}{d\eta} = u\frac{d^2v}{dx^2} = -\frac{1}{R}\frac{\partial^2 P}{\partial x \partial \eta} \qquad \ldots(25)$$

which divulges that (23) and (24) admit of the solutions

$$u = u_1 \exp\left(-\frac{M}{R}\eta\right) \qquad \ldots(26)$$

and the consequent differential eqn. for v is

$$\frac{d^2v}{dx^2} = \frac{M^2}{R^2}(v - 1) \qquad \ldots(27)$$

whose integral in turn is

$$v = 1 + C_1 \exp\left(\frac{M}{R}x\right) + C_2 \exp\left(-\frac{M}{R}x\right).$$

Making use of the boundary conditions that $u = u_1$, $v = 1$ at the origin situated on the curved wall, and $v = 0$ at $x = b$, there being no change in dimensionless parameters as in (13), one can evaluate the constants C_1 and C_2 to get

$$v = 1 - \frac{\sinh \frac{M}{R}x}{\sinh \frac{M}{R}b}. \qquad \ldots(28)$$

Hence, in this model of flow the suction across the flat wall does not remain constant as in the previous case but decreases from the velocity V_0 at the inlet to zero at the outlet.

The equation of the curved boundary wall (streamlined) to be employed for creating such a flow is

$$\frac{d\eta}{dx} = \frac{v}{u} \qquad \ldots(29)$$

which, by means of (26) along with the relevant boundary conditions, yields

$$\frac{Ru_1}{M}\left(1-\exp\left(-\frac{M}{R}\eta\right)\right)-x-\frac{R}{M}\frac{\left(\cosh\frac{R}{M}x-1\right)}{\operatorname{Sinh}\frac{R}{M}b}=0 \qquad \ldots(30)$$

and the ordinate (η_1) of the highest point of this curved boundary surface satisfies the transcendental relationship

$$\frac{Ru_1}{M}\left(1-\exp\left(-\frac{M}{R}\eta_1\right)\right)-b+\frac{R}{M}\frac{\cosh\frac{R}{M}l-1}{\operatorname{Sinh}\frac{R}{M}b}=0 \qquad \ldots(31)$$

To find the fluid pressure at any point of fluid inside the system, eqns (23) and (24) can be combined as

$$-\frac{dP}{R}=\left(v\frac{du}{d\eta}+\frac{M}{R}u\right)dx+\left(u\frac{dv}{dx}\right)d\eta$$

Now in consequence of the boundary conditions for this model, and eqns (27) and (28), the above differential eqn of pressure gives rise to

$$P(x,\eta)=P_0-\frac{Mu_1\left\{\left(\operatorname{Cosh}\frac{M}{R}x\right)\exp\left(-\frac{M}{R}\eta\right)-1\right\}}{\operatorname{Sinh}\frac{Mb}{R}}, \qquad \ldots(32)$$

where P_0 is the pressure at the origin.

The vorticity ξ and stream function can be computed here under by employing (26) and (28) :

$$\xi=-\frac{M}{R}\left(\frac{\cosh\frac{R}{M}x}{\operatorname{Sinh}\frac{R}{M}b}-u_1\exp\left(-\frac{R}{M}\eta\right)\right) \qquad \ldots(33)$$

and

$$\psi=\frac{Ru_1}{M}\exp\left(-\frac{M}{R}\eta\right)+x-\frac{R}{M}\frac{\operatorname{Cosh}\frac{M}{R}x}{\operatorname{Sinh}\frac{M}{R}b}. \qquad \ldots(34)$$

It can be noted that in this section the contrived boundary conditions have to be somewhat different from those taken in the previous section so as to agree with the differential equations of fluid motion and their solution pertinent to the nature of flow studied herein.

Case III

Finally the possibility of producing a MHD flow with $U = U(X)$ and $V = V(Y)$ has been explored here. Then eqns (6) and (13) convert eqns (1) to (3) into the form

$$Ru\frac{du}{dx} + Mu = -\frac{\partial P}{\partial x}, \qquad \ldots(35)$$

$$Rv\frac{dv}{d\eta} = -\frac{\partial P}{\partial \eta} \qquad \ldots(36)$$

and

$$-\frac{du}{dx} = \frac{dv}{d\eta} = A \text{ (constant)}. \qquad \ldots(37)$$

Eqns (3) and (37) imply

$$u = -Ax + B \text{ and } v = A\eta + C, \qquad \ldots(38)$$

where A, B, C are constants. Now some boundary conditions suited for the solutions (38) have to be introduced;

$$\text{At } x = 0,\ u = u_0; \quad \text{at } \eta = 0,\ v = v_0;$$
$$\text{at } \eta = d,\ v = 1, \qquad \ldots(39)$$

which also bring about a continuous suction through the flat wall, but the suction velocity herein remains constant and at the same as that in case I, although η-component velocity varies from the curved surface to the flat surface. As a result of the boundary conditions (39), eqns (38) can be rewritten as

$$u = u_0 - \frac{(1 - v_0)\,x}{d},\ v = v_0 + \frac{(1 - v_0)\,\eta}{d} \qquad \ldots(40)$$

which represent the velocity components in non-dimensional form. Now the velocity potential and the stream function can be obtained as

$$\phi = -u_0x - v_0\eta + \frac{(1 - v_0)\,(x^2 - \eta^2)}{2d} \qquad \ldots(41)$$

and

$$\psi = -u_0\eta + v_0x + \frac{(1 - v_0)\,x\eta}{d}. \qquad \ldots(42)$$

The equation of the curved wall (streamlined) essential to the irrotational contrived flow is

$$\left\{\frac{(1 - v_0)\,x\eta}{d}\right\} + d(v_0x - u_0\eta) = 0$$

(By the use of the condition that the surface passes through the origin)

or
$$\frac{v_0}{\eta} - \frac{u_0}{x} + \frac{1 - v_0}{d} = 0. \qquad \ldots(43)$$

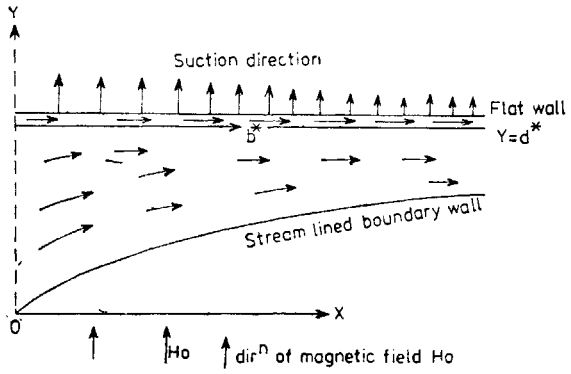

FIG 1

In closing, utilising eqns (35), (36) and (40), the pressure function $P(x, \eta)$ amenable to this typical MHD flow is given by

$$dp = \frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy;\quad P = P_0 - \frac{R}{d}(1 - v_0)\left\{\frac{(1 - v_0)(x^2 + \eta^2)}{d} - (u_0 x + v_0 \eta)\right\} - M\left(u_0 - \frac{1 - v_0}{2d}\,x\right)x. \qquad \ldots(44)$$

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