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APPLICATION OF TAPERED FINITE-ELEMENT IN THE VIBRATION ANALYSIS OF SPINNING PLATES

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A generalised algorithm is presented in this paper which may be conveniently utilised for the efficient analysis of certain broad class of spinning plate problems. Extensive applications of this methodology are envisaged in the design and analysis of the rotating turbomachinery blades.

Key Words : Finite-Element-Tapered; Vibration; Spinning Plate; Turbomachinery; Blade; Southwell Coefficient

Nomenclature

u, v	—	Inplane displacements.
w	—	Out of plane displacements.
NTU	—	Number of terms in the polynomial for u.
NN	—	Number of nodes in the element.
NT	—	Total number of constants.
ξ, η	—	Local coordinates.
L, W, t	—	Length, width and thickness of the plate respectively.
$\bar{w}$	—	Natural frequency.
N	—	Speed of rotation.
θ	—	Setting angle.
ρ	—	Density of the material.
μ, E	—	Poisson's ratio and Young's modulus.
w_0	—	Natural frequency of the static plate.
R	—	Radius of the disk.
δ	—	Nodal displacements.
D_{ij}	—	Elements of the stress-strain matrix.
S	—	Southwell coefficient.

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INTRODUCTION

THE static and dynamic analysis of a blade has always been one of the focal points of analysis of machine designers.

When a plate rotates about an axis perpendicular to its own axis, the resulting centrifugal forces have the effect of stiffening the plate to a certain degree, thereby increasing the natural frequency of bending vibration of the plate. The relation between the frequency of a rotating blade ($\bar{w}$) and the corresponding frequency of non-rotating blade (w_0) is generally expressed in the form

$$\bar{w}^2 = w_0^2 + SN^2, \qquad \ldots(1)$$

where N is the speed of rotation and S is commonly known as the Southwell coefficient which depends on the setting angle at which the blade is mounted on the disk, the disk radius and even on the speed of rotation. This coefficient is very much useful for the rapid estimation of the natural frequency.

The bending vibration of a rotating blade vibration in the plane of rotation has been analysed successfully by a practical method developed by Myklestad.[1] This problem has also been checked experimentally[2] and found to be accurate. The vibration analysis rotating blade, vibrating in the plane of rotation, was made by Sutherland[3] using the general method developed by Myklestad. The first three bending frequencies of a rotating beam have also been obtained by Yntema[4] by using the Rayleigh energy method.

In the present investigation, vibration of the spinning plate has been analysed by using tapered Finite elements. This element has wide range of flexibility in the application in various static and dynamic problems in Engineering. Although such formulation is straight forward theoretically, emphasis has been given on the programming where the amount of algebra involved in the matrix manipulation is avoided.

A generalised program BLDVIB has been developed for calculating the natural frequencies and the mode shapes of a tapered plate. Computations are carried out for various values of the disk radius, setting angle and the speed of rotation.

GENERAL THEORY AND TAPERED FINITE ELEMENT FORMULATION

Assuming the cantilever plate to be mounted on the periphery of a rigid rotating disk of radius 'R' (Fig. 1), cartesian global coordinate axes x, y, z are taken at one corner of the fixed edge of the plate. If 'w' is the deflection of an arbitrary point on the middle surface of the plate, the components of the centrifugal force per unit volume of the plate along the x, y and z axes are

$$F_x = \rho N^2(x + R); \; F_y = \rho N^2(y \cos^2\theta - w \sin\theta \cos\theta)$$

$$F_z = \rho N^2(-y \sin\theta \cos\theta + w \sin^2\theta) - \rho\ddot{w} \qquad \ldots(2)$$

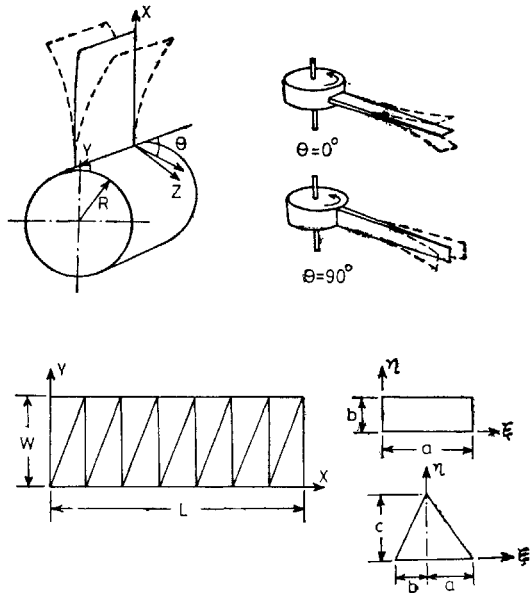

FIG 1

In case of a spinning plate the stresses produced in the middle surface of the plate due to in-plane centrifugal forces increases the bending stiffness. The total stiffness $[\mathbf{K_T}]$ for the plate element will be the summation of bending stiffness $[\mathbf{K_B}]$ and centrifugal stiffness $[\mathbf{K_G}]$. These stiffness matrices can be obtained by calculating the strain energy from the relation

$$\mathbf{U} = \int_v \epsilon^T D\epsilon \, dv, \qquad \ldots(3)$$

where ϵ is the strain vector and D is the stress-strain matrix.

The inplane displacements u, v, the out of plane displacement w and the variation of thickness t can be expressed in the local ξ, η axes as

$$\left.\begin{aligned} u &= \sum_{i=1}^{\mathrm{NTU}} \alpha_i \xi^{m_i} \eta^{n_i} \; ; \; V = \sum_{i=\mathrm{NTU}}^{\mathrm{NT}} \alpha_i \xi^{r_i} \eta^{s_i} \\ \text{and} \qquad & \\ w &= \sum_{i=1}^{\mathrm{NT}} \alpha_i \xi^{m_i} \eta^{n_i} \; ; \; t = \sum_{i=1}^{\mathrm{NN}} \beta_i \xi^{p_i} \eta^{q_i} \end{aligned}\right\} \qquad \ldots(4)$$

The values of m, n, r and s will depend on the element shape and the order of polynomial.

Bending Stiffness Formulation

The strain vector from classical plate bending theory can be written as,

$$\epsilon = -z \{w_{\xi\xi} \; w_{\eta\eta} \; 2w_{\xi\eta}\}, \qquad \ldots(5)$$

where, z is the distance from the neutral surface.

Expanding (3) and substituting w from (4), the strain energy can be expressed as

$$\mathbf{U} = \tfrac{1}{2}\,\mathbf{A}^{\mathbf{T}}\,\mathbf{K}\mathbf{A}, \qquad \ldots(6)$$

where $$\mathbf{A} = [\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots, \alpha_{NT}]. \qquad \ldots(7)$$

The elements of k can be evaluated automatically once the values of m_i and n_i are furnished and the final form of k can be expressed as

$$K_{ij} = \sum_{k_1=1}^{NN} \sum_{k_2=1}^{NN} \sum_{k_3=1}^{NN} \beta_{k_1}\beta_{k_2}\beta_{k_3} \{\mathbf{D}_{11}\, m_i\, m_j\, (m_i - 1)\, (m_j - 1)\, F(\mathrm{II} - 4, JJ) + \mathbf{D}_{22} n_i n_j (n_i - 1)(n_j - 1)\, F(\mathrm{II}, JJ - 4) + [\mathbf{D}_{12} m_i n_j (m_i - 1)\, (n_j - 1) + \mathbf{D}_{12} m_j n_i\, (m_j - 1)\, (n_i - 1) + 4\mathbf{D}_{33} m_i m_j n_i n_j]\, F(\mathrm{II} - 2, JJ - 2) + 2\mathbf{D}_{13}\, [m_i n_i m_j\, (m_j - 1) + m_j n_j m_i (m_i - 1)]\, F(\mathrm{II} - 3, JJ - 1) + 2\mathbf{D}_{23}\, [m_i n_i n_j (n_j - 1) + m_j n_j n_i (n_i - 1)]\, F(\mathrm{II} - 1, JJ - 3)\}, \qquad \ldots(8)$$

where $$\mathrm{II} = m_i + m_j + p_{k_1} + p_{k_2} + p_{k_3}$$

$$JJ = n_i + n_j + q_{k_1} + q_{k_2} + q_{k_3}$$

For triangular elements

$$F(m, n) = C^{n+1}\, [a^{m+1} - (-b)^{m+1}]\, \frac{m!\, n!}{(m + n + 2)!}. \qquad \ldots(9)$$

For rectangular elements

$$F(m, n) = \frac{a^{m+1}}{(m + 1)}\, \frac{b^{n+1}}{(n + 1)}, \qquad \ldots(10)$$

where a, b and c define the geometry of element as shown in Fig. 1.

From eq (4) the constants α_i can be expressed in terms of generalised displacement as

$$\{\alpha\}^i = [C^{-1}]\, \{\delta\} \qquad \ldots(11)$$

From eqs. (6) and (11) the bending stiffness matrix can be expressed as

$$[\mathbf{K_B}] = [\mathbf{C}^{-1}]^T\, [\mathbf{k}]\, [\mathbf{C}^{-1}] \qquad \ldots(12)$$

Elasttc Stiffness Formulation

The elastic stiffness matrix can be derived as before considering the strain vector as

$$\boldsymbol{\epsilon} = \{\mathbf{u}_\xi \mathbf{v}_\eta (\mathbf{u}_\eta + \mathbf{v}_\xi)\} \qquad \ldots(13)$$

The elastic stiffness matrix can be expressed as

$$[\mathbf{K_E}] = [\mathbf{C^{-1}}]^T \ [\bar{\mathbf{K}}] \ [\mathbf{C^{-1}}], \qquad ...(14)$$

where

$$\bar{K}_{ij} = \sum_{k=1}^{NN} \beta_k \ \{D_{11} m_i m_j F(\mathrm{II} - 2), \ JJ) + D_{22} s_i s_j (RR, \ SS - 2) + D_{33}$$

$$[n_i n_j F(\mathrm{II}, \ JJ - 2) + r_i r_j F(RR - 2, SS)] + [D_{12} m_i s_j + D_{33} n_i r_j]$$

$$F(MRI - 1, \ NSI - 1) + [D_{12} m_j s_i + D_{33} n_j r_i] \ F(MRJ - 1,$$

$$NSJ - 1) + D_{13} [m_i n_j + m_j n_i] \ F(\mathrm{II} - 1, \ JJ - 1) + D_{23} \ [r_i s_j +$$

$$r_j s_k] \ F(RR - 1, \ SS - 1) + D_{13} \ [m_i r_j \ F(MRI - 2, NSI) + m_j r_i$$

$$F(MRJ - 2, NSJ)] + D_{23} \ [n_i s_j F(MRI, \ NSI - 2) + n_j s_i F(MRJ,$$

$$NSJ - 2)]\} \qquad ...(15)$$

$$II = m_i + m_j + p_k; \ JJ = n_i + n_j + q_k; \ RR = r_i + r_j + p_k$$

$$SS = s_i + s_j + q_k; \ MRJ = m_j + r_i + q_k; \ NSJ = n_j + s_i +$$

$$q_k; \ MRI = m_i + r_j + p_k; NSI = n_i + s_j + q_k$$

Geometric Stiffness Formulation

The potential energy due to inplane stresses σ_η, σ_η and $\tau_{\xi\eta}$ and σ_η can be expressed as,

$$V = \tfrac{1}{2} \int_v (\sigma_\xi \omega_\xi^2 + \sigma_\eta \omega_\eta^2 + \tau_{\xi\eta} \omega_\xi \omega_\eta + \tau_{\xi\eta} \omega_\eta \omega_\xi) \, dv \qquad ...(16)$$

This additional energy results in an increase in the stiffness of the element. Proceeding as before the geometric stiffness matrix can be expressed as

$$[\mathbf{K}]_\mathbf{G} = [\mathbf{C^{-1}}]^T \ [\bar{\bar{\mathbf{K}}}] \ [\mathbf{C^{-1}}], \qquad ...(17)$$

where,

$$\bar{\bar{K}}_{ij} = \sum_{k=1}^{NN} \beta_k \{\sigma_\xi m_i m_j F(m_i + m_j + p_k - 2; \ n_i + n_j + q_k) + \sigma_\eta n_i n_j.$$

$$F(m_i + m_j + p_k, \ n_i + n_j + q_k - 2) + \tau_{\xi\eta} (m_i n_j + m_j n_i) \ F(m_i + m_j +$$

$$p_k - 1, \ n_i + n_j + q_k - 1)\} \qquad ...(18)$$

Consistent Mass Matrix Formulation

The consistent mass matrix is obtained from the kinetic energy of the element and the expression of the mass matrix

$$[\mathbf{M}] = [\mathbf{C^{-1}}]^T \ [\mathbf{M}] \ [\mathbf{C^{-1}}], \qquad ...(19)$$

where

$$M_{ij} = \sum_{k=1}^{NN} \beta_k \rho F(m_i + m_j + p_k, n_i + n_j + q_k).$$

Natural Frequencies and Mode Shapes

The total stiffness matrix $[\mathbf{K_T}]^e$ and the mass matrices $[\mathbf{M}]^e$ for all elements are grouped together according to the sequence of the node numbering on the plate. The rows and columns corresponding to the nodes on the fixed edge are deleted to obtain the complete stiffness matrix [K] and the complete mass matrix [M] for the rotating cantilever plate. The equation of motion for the plate can be written as,

$$\left.\begin{aligned} &[\mathbf{K}]\{\delta\} = N^2 \sin^2\theta\,[\mathbf{M}]\{\delta\} - [\mathbf{M}]\{\ddot{\delta}\} \\ \text{or} \\ &\{[\mathbf{K}] - (\bar{w}^2 + N^2 \sin^2\theta)\,[\mathbf{M}]\}\{\delta\} = 0 \end{aligned}\right\} \quad \ldots(20)$$

The eigenvalues and the eigenvectors of matrix $[\mathbf{K}^{-1}]$, [M], therefore, give the values of $1/(\bar{w}^2 + N^2 \sin^2\theta)$ and the mode shapes.

Numerical Example and Discussions

To show the application and to verify the accuracy of the tapered finite elements, vibration of a spinning plate has been studied for various speeds of rotation, setting angles and disk radii. The following numerical data were assumed in this investigation :

$L = 29$ cm., $W = 6.6$cm, $R = 29$cm., $t = 0.2$cm.

$E = 0.7242 \times 10^6$ kg/cm^2, $\mu = 0.334$,

$\rho = 28 \times 10^{-7}$kg. sec^2/cm^4.

The maximum and minimum thickness of the plate are taken as 0.2cm and 0.05cm respectively and the variation is assumed to be linear.

To examine the convergence and extent of accuracy of this methodology, computations of the natural frequencies and mode shapes of a static plate are carried out and the results have been compared with SAP-6 (Structural Analysis program, University of California, Berkeley, USA). The percentage variation of the fundamental frequencies are 0.8 per cent and 18.37 per cent for computation with rectangular and triangular elements respectively. The mode shapes calculated by rectangular elements, also observed to be very close to SAP-6 than triangular elements.

With the above information about the convergence and accuracy of this formulation the nondimensional natural frequencies expressed in the form

$$\bar{\beta} = \bar{w}L^2 \sqrt{\frac{12\rho(1-\mu^2)}{Et^2_{max}}} \qquad \ldots(21)$$

and mode shapes of the same plate were computed for various speed of rotation (1000 to 3000rpm), the various setting angles (0–90°) and various disk radius (29cm to 116cm).

The variation of Natural frequencies with speed were plotted in Fig. 2 for 0° setting angle. The influence of the thickness variation has also been incorporated in the same figure. It is clear that all the natural frequencies are increasing with speed of rotation since the resulting centrifugal forces have the effect of stiffening the plate to a certain degree. Again the natural frequencies are decreasing (Fig. 3) with the increase in setting angle. This gives the idea that the natural frequencies of vibration in the plane of rotation ($\theta = 90°$) is lower than the natural frequency of vibration perpendicular to the plane of rotation ($\theta = 0°$).

To show the effect of hub radius on the natural frequency the same analysis were performed for four different hub radii as 29cm, 58cm, 87cm and 116cm and the results were plotted in Fig. 4 and Fig. 5 for two different cases (*a*) vibration in the plane of rotation ($\theta = 90°$) and (*b*) vibration in a plane perpendicular to the plane of rotation ($\theta = 0°$). It is clear that all the natural frequencies are increasing with the disk radius.

The Southwell coefficient is very much useful for the rapid estimation of the natural frequencies of plate. Fig. 6 gives the values of Southwell coefficients for

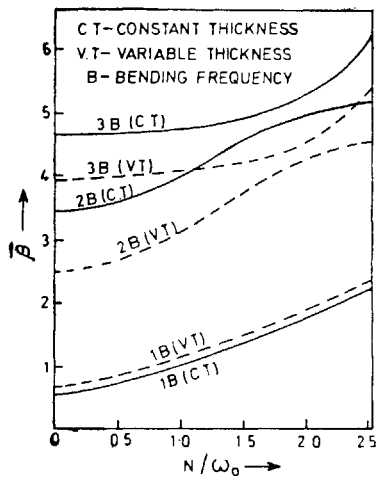

FIG 2

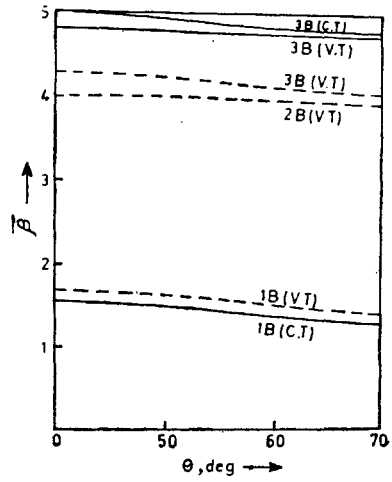

FIG 3

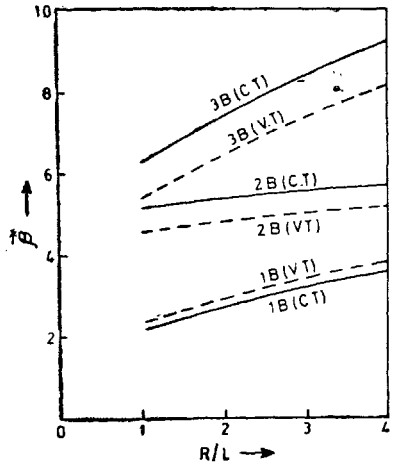

FIG 4

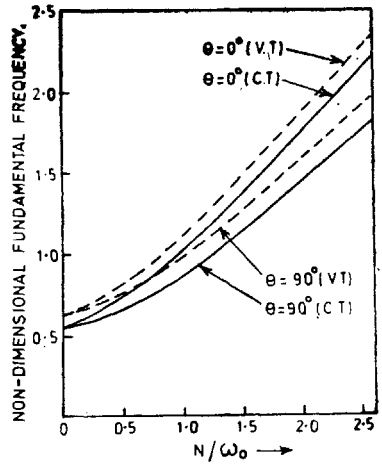

FIG 5

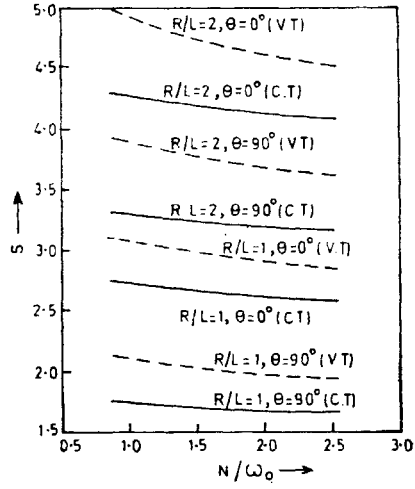

FIG 6

various parameters and this figure should be more useful, from practical consideration, than a long table.

CONCLUSION

The program BLDVIB developed for the present study is useful for static dynamic and stability problems.

Although the results given here are for constant and uniform tapered plates, this methodology can also be applied for twisted plates having aerofoil sections.

The Southwell coefficients given in Fig. 6 for various parameters are useful for the rapid estimation of natural frequencies for various conditions.

Finally, it should be mentioned that the methodology presented in this investigation permits a more orderly process and the designers can use this program to estimate the natural frequencies of the blades of different materials and having various cross-sections where no mathematically complete or close form solution exists.

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