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SCATTERING OF WATER WAVES BY VERTICAL BARRIERS AND ASSOCIATED MATHEMATICAL METHODS

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There exists a number of mathematical methods to solve the water wave scattering problems involving fixed vertical barriers. Here we demonstrate these methods by considering a particular scattering problem.

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INTRODUCTION

WITHIN the framework of linearised theory of water waves, the problems involving scattering of normally incident surface water wave trains by fixed vertical plane barriers only, admit of exact solutions. These barrier problems can be attacked by a number of mathematical methods. Dean[1] first considered the scattering of a normally incident wave train by a submerged plane vertical barrier and solved the problem by employing a technique which we call as the 'complex variable method.' However, he gave only the expressions for the reflection and transmission coefficients. Ursell[2] considered the complementary problem of a partially immersed vertical barrier and used Havelock's[3] expansion theorem to solve it. We call this 'expansion method.' He obtained the velocity potential from which the reflection and transmission coefficients are readily available. The completely submerged barrier problem was also considered by Ursell[2] and he gave only the results for the velocity potential and the reflection and transmission coefficients. By a simple application of the Green's integral theorem in the fluid region Levine and Rodemich[4] reduced the immersed barrier problem to a singular integral equation whose solution is known. We call this as the 'integral equation method.' Finally, Williams[5] gave a simplified method in which he reduced the boundary value problem (BVP) associated with the immersed barrier problem to another BVP whose solution is obtained rather quickly. We call this 'reduction method.'

Although various scattering problems involving a single barrier or a number of barriers in the same vertical plane were solved by various authors from time to time by employing one of these methods,[6–9] it appears to be instructive to solve a particular problem by using all these methods. This has motivated us to present all these methods in some detail to solve the scattering problem involving a submerged barrier considered first by Dean.[1]

FORMULATION OF THE PROBLEM

A rectangular cartesian co-ordinate system is chosen in which the y-axis is taken vertically downwards and the xz-plane as the mean free suface (FS). A completely submerged plane vertical barrier is present in the fluid and occupies the position $x = 0, a \leqslant y < \infty$. The fluid is assumed to be inviscid, incompressible, the motion is irrotational and simple harmonic in time with angular frequency σ so that it can be described by a velocity potential which can be taken as $Re\ \Phi \exp(-i\sigma t)$. Then Φ satisfies

$$\nabla^2\Phi = 0 \text{ in the fluid region,} \qquad \ldots(1)$$

and

$$K\Phi + \frac{\partial\Phi}{\partial y} = 0 \text{ on } y = 0, \qquad \ldots(2)$$

where $K = \sigma^2/g$, g being the gravity. (1) is the equation of continuity and (2) is the linearised FS condition.

A train of surface waves represented by $\varphi_0 = \exp(-Ky + iKx)$ is assumed to be incident normally on the barrier from negative infinity. The wave train will be partially reflected by and transmitted over the barrier. Φ now depends on x and y only and satisfies the barrier condition

$$\frac{\partial\Phi}{\partial x} = 0 \text{ on } x = 0,\ a < y < \infty. \qquad \ldots(3)$$

Φ and its partial derivatives are continuous everywhere except possibly across the barrier, bounded everywhere away from the line $x = 0, y = a$, and near this line $\{x^2 + (y - a)^2\}^{1/2}\,\nabla\Phi$ is bounded. This is the edge condition. Also as $|x| \to \infty$, $\Phi(x, y)$ has the asymptotic forms

$$\left.\begin{aligned} &\Phi(x, y) \sim T\exp(iKx - Ky) \text{ as } x \to +\infty, \\ \text{and}\quad &\Phi(x, y) \sim \exp(iKx - Ky) + R\exp(-iKx - Ky) \text{ as } x \to -\infty, \end{aligned}\right\} \ldots(4)$$

where T and R are the (complex) transmission and reflection coefficients respectively. Our problem is to determine T, R and $\Phi(x, y)$. We solve the BVP described by (1) to (4) by employing the different mathematical techniques mentioned above.

EXPANSION METHOD

This method was employed by Ursell[2] in connection with the problem of a partially immersed plane vertical barrier. He gave only the results for the submerged barrier problem. We demonstrate below in some detail the expansion method to solve the BVP described by (1) to (4). As the dependence of time on Φ is omitted by suppressing the factor $\exp(i\omega t)$, it may be noted that the method presented here is somewhat a modified version of Ursell's method.

In view of (4), $\Phi(x, y)$ can be represented by

$$\left.\begin{aligned} \Phi(x, y) &= T \exp(-Ky + iKx) \\ &\quad + \int_0^\infty D(k) \exp(-kx)(k \cos ky - K \sin ky)\, dk, \\ &\qquad x > 0, \\ \text{and} \\ \Phi(x, y) &= \exp(-Ky + iKx) + R \exp(-Ky - iKx) \\ &\quad + \int_0^\infty E(k) \exp(kx)(k \cos ky - K \sin ky)\, dk, \\ &\qquad x < 0, \end{aligned}\right\} \quad \ldots(5)$$

where $D(k)$ and $E(k)$ are unknown functions.

$$\text{Let } \frac{\partial \Phi}{\partial x} = f(y) \text{ on } x = \pm 0, \quad \ldots(6)$$

then in view of (3)

$$f(y) = 0, \; a < y < \infty, \quad \ldots(7)$$

and by the edge condition,

$$f(y) \sim (a^2 - y^2)^{-1/2} \text{ as } y \to a. \quad \ldots(8)$$

For $y > 0, f(y)$ can be expanded as[3]

$$\left.\begin{aligned} f(y) &= A_0 \exp(-Ky) + \int_0^\infty A(k)(k \cos ky - K \sin ky)\, dk, \\ \text{where} \\ A_0 &= 2K \int_0^\infty f(y) \exp(-Ky)\, dy \\ \text{and} \quad A(k) &= \frac{2}{\pi} \frac{1}{k^2 + K^2} \int_0^\infty f(y)(k \cos ky - K \sin ky)\, dy. \end{aligned}\right\} \quad \ldots(9)$$

Using (6) in (5), we obtain

$$iKT \exp(-Ky) - \int_0^\infty kD(k)(k \cos ky - K \sin ky)\, dk$$

$$= iK(1 - R) \exp(-Ky) + \int_0^\infty kE(k)(k \cos ky - K \sin ky)\, dk$$

$$= f(y).$$

Using (9) and noting (7), this gives

$$\left.\begin{aligned}\frac{i}{2}T=\frac{i}{2}(1-R)&=\int_0^a f(y)\exp(-Ky)\,dy,\\ \text{and}\qquad D(k)=-E(k)&=-\frac{2}{\pi}\frac{1}{k(k^2+K^2)}\int_0^a f(y)(k\cos ky\\ &\qquad -K\sin ky)\,dk,\end{aligned}\right\}\quad\ldots(10)$$

so that $\quad T+R=1. \qquad\ldots(11)$

Now $\Phi(x, y)$ is continuous across the gap $x = 0$, $a > y > 0$, so that from (5)

$$T\exp(-Ky)+\int_0^\infty D(k)(k\cos ky-K\sin ky)\,dk$$
$$=(1+R)\exp(-Ky)+\int_0^\infty E(k)(k\cos ky-K\sin ky)\,dk,$$
$$0<y<a.$$

Using (10) and (11) this gives

$$2\int_0^\infty D(k)(k\cos ky-K\sin ky)\,dk=2R\exp(-Ky),\ 0<y<a.$$

This is reduced to the integral equation given by

$$\frac{2}{\pi}\int_0^a f(u)\left\{\int_0^\infty\frac{(k\cos ky-K\sin ky)(k\cos ku-K\sin ku)}{k(k^2+K^2)}\,dk\right\}du$$
$$=-R\exp(-Ky),\ 0<y<a. \qquad\ldots(12)$$

Applying the operator $\frac{\partial}{\partial y}+K$ to (12) we obtain after simplification

$$\int_0^a f(u)\left\{K\ln\left|\frac{y+u}{y-u}\right|-\frac{1}{y-u}-\frac{1}{y+u}\right\}du=0,\ 0<y<a, \qquad\ldots(13)$$

where the integral is in the sense of Cauchy principal value. Defining

$$\Psi(y)=f(y)+K\int_a^y f(u)\,du, \qquad\ldots(14)$$

(13) reduces to the singular IE

$$\int_0^a\frac{\Psi(u)}{y^2-u^2}\,du=0,\ 0<y<a, \qquad\ldots(15)$$

where $\Psi(y)$ has integrable singularity near the end $y = a$. Following Mikhlin[10] solution of (14) is

$$\Psi(y) = C(a^2 - y^2)^{-1/2}, \quad \ldots(16)$$

where C is an arbitrary constant to be determined. Using (14) we obtain

$$f(y) = C \frac{\mathrm{d}}{\mathrm{d}y} \left\{ \exp(-Ky) \int_a^y \frac{\exp(Kv)}{(a^2 - v^2)^{1/2}} \, \mathrm{d}v \right\}. \quad \ldots(17)$$

To find C we make $y \to 0$ in (12) and substitute for $f(u)$ from (17). This gives an equation in C. Various integrals can be evaluated in terms of known functions and the final result is

$$C = -R\,\{K_0(Ka)\}^{-1}, \quad \ldots(18)$$

where $K_0(z)$ is the modified Bessel function of second kind.

From (10) we obtain

$$\frac{i}{2} T = \frac{i}{2}(1 - R) = C \frac{\pi}{2} I_0(Ka),$$

where $I_0(z)$ is the modified Bessel function of first kind. Noting (18) we obtain

$$\left.\begin{aligned} & R = K_0(Ka)\,\Delta^{-1} \\ \text{and} \quad & T = i\pi\, I_0(Ka)\,\Delta^{-1} \\ \text{where} \quad & \Delta(Ka) = i\pi\, I_0(Ka) + K_0(Ka). \end{aligned}\right\} \quad \ldots(19)$$

Again, from (10)

$$D(k) = -E(k) = -C \frac{J_0(ka)}{k^2 + K^2},$$

where $J_0(z)$ is the Bessel function of first kind. Substituting in (5) we obtain

$$\Phi(x, y) = \Delta^{-1} \Big[i\pi\, I_0(Ka) \exp(-Ky + iKx) + \int_0^\infty \frac{J_0(ka)}{k^2 + K^2} \exp(-kx)(k \cos ky - K \sin ky)\, \mathrm{d}k \Big], \; x > 0, \quad \ldots(20)$$

and

$$\Phi(x, y) = \exp(-Ky + iKx) + \Delta^{-1} \Big[K_0(Ka) \exp(-Ky + iKx) - \int_0^\infty \frac{J_0(ka)}{k^2 + K^2} \exp(kx)(k \cos ky - K \sin ky)\, \mathrm{d}k \Big], \; x < 0. \quad \ldots(21)$$

INTEGRAL EQUATION METHOD

In this method by an appropriate use of the Green's integral theorem in the fluid region the velocity potential is expressed in terms of an unknown function defined on the plane of the barrier. By applying the condition on the barrier, this unknown function can be shown to satisfy an integral equation. This unknown function is either the unknown horizontal component of velocity in the gap above the barrier or the difference of velocity potential across the barrier. For application of the Green's integral theorem we require the Green's function $G(x, y; \xi, \eta)$ which satisfy

$$\nabla^2 G = 0, \; y > 0 \text{ except at } (\xi, \eta), \; \eta > 0,$$

$$KG + \frac{\partial G}{\partial y} = 0 \text{ on } y = 0,$$

$$G \sim -\ln r \quad \text{as } r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2} \to 0,$$

G, ∇G are bounded as $y \to +\infty$, and G represents an outgoing wave as $|x| \to \infty$. Following Thorne,[11] G is given by

$$G(x, y; \xi, \eta) = -\ln \frac{r}{r'} + 2 \int_0^\infty \frac{k \cos k(y+\eta) - K \sin k(y+\eta)}{k^2 + K^2} \exp(-k|x|)\, dk + 2\pi i \exp\{-K(y+\eta) + iK|x-\xi|\}, \quad \ldots(22)$$

where $r' = \{(x-\xi)^2 + (y+\eta)^2\}^{1/2}$.

We now give two approaches to solve this BVP by the IE method.

First Approach

This approach was used in Mandal and Goswami[12] in connection with the scattering of an obliquely incident wave train by a submerged plane vertical barrier. Let

$$\mathcal{G}(x, y; \xi, \eta) = G(x, y; \xi, \eta) + G(-x, y; \xi, \eta), \quad \ldots(23)$$

then $$\frac{\partial \mathcal{G}}{\partial x} = 0 \text{ on } x = 0,$$

$$\mathcal{G}(0, y; \xi, \eta) = 2G(0, y; \xi, \eta) \quad \ldots(24)$$

and $\mathcal{G}$ behaves as an outgoing wave as $|x| \to \infty$. Noting that Φ behaves as an outgoing wave as $x \to \infty$, by applying the Green's integral theorem to $\Phi(x, y)$ and $G(x, y; \xi, \eta)$ in the fluid region in the right of the y-axis we obtain

$$-\pi\Phi(\xi, \eta) = \int_0^a G(0, y; \xi, \eta) \frac{\partial \Phi}{\partial x}(0, y)\, dy, \; \xi > 0. \quad \ldots(25)$$

Again applying the Green's integral theorem to $\Phi(x, y) - \varphi_0(x, y)$ and $\mathcal{G}(x, y; \xi, \eta)$ in the fluid region in the left of the y-axis we obtain

$$\pi\Phi(\xi, \eta) = \pi\varphi_0(\xi, \eta) + \int_0^a G(0, y; \xi, \eta) \frac{\partial \Phi}{\partial x}(0, y)\, dy$$

$$- \int_0^\infty G(0, y; \xi, \eta) \frac{\partial \varphi_0}{\partial x}(0, y)\, dy, \ \xi < 0. \qquad ...(26)$$

Using the condition that $\Phi(\xi, \eta)$ is continuous across the gap $\xi = 0$, $0 < \eta < a$ and noting that

$$\int_0^\infty G(0, y; 0, \eta) \frac{\partial \varphi_0}{\partial x}(0, y)\, dy = -\pi \exp(-K\eta),$$

we obtain the IE

$$\int_0^a G(0, y; 0, \eta) f(y)\, dy = -\pi \exp(-K\eta), \ 0 < \eta < a, \qquad ...(27)$$

where $\quad f(y) \equiv \frac{\partial \Phi}{\partial x}(0, y), \ 0 < y < a.$

The kernel of this IE is slightly different from the kernel in the IE (12). The two kernels differ by the term $2\pi i \exp\{-K(y + \eta)\}$. Solution of (27) can be obtained as in (12). Applying the operator $\frac{\partial}{\partial \eta} + K$ to (27) we obtain

$$\int_0^a \left(K + \frac{\partial}{\partial \eta}\right) G(0, y; 0, \eta) f(y)\, dy = 0, \ 0 < \eta < a. \qquad ...(28)$$

But from (22)

$$\left(K + \frac{\partial}{\partial \eta}\right) G(0, y; 0, \eta) = -K \ln\left|\frac{y - \eta}{y + \eta}\right| + \frac{1}{y - \eta} - \frac{1}{y + \eta}$$

so that (28) reduces to the singular IE

$$\int_0^a \frac{\Psi(y)}{\eta^2 - y^2}\, dy = 0, \ 0 < \eta < a, \qquad ...(29)$$

where $\Psi(y)$ is the same as in (14). Thus

$$\Psi(y) = C_0(a^2 - y^2)^{-1/2},$$

where C_0 is an arbitrary constant. To find C_0 we make $\eta \to 0$ in (27). After evaluating the different integrals we obtain

$$C_0 = -\Delta^{-1} \qquad \ldots(30)$$

To obtain the transmission and reflection coefficients we make $\xi \to \infty$ in (25) and $\xi \to -\infty$ in (26). Noting (22) and the asymptotic form of $\Phi(\xi, \eta)$ as $\xi \to \pm\infty$ in (4) we obtain after evaluating the integrals

$$C = -C_0 \pi I_0(Ka)$$

and

$$R = 1 + C_0 \pi I_0(Ka).$$

Using (30) we obtain for T and R the results given in (19).

To evaluate $\Phi(\xi, n)$ we note that

$$G(0, y; \xi, \eta) = 2 \int_0^\infty \frac{(k \cos ky - K \sin ky)\ (k \cos k\eta - K \sin k\eta)}{k(k^2 + K^2)} \exp(-k\,|\,\xi\,|)\,\mathrm{d}k + 2\pi i \exp\{-K(y+\eta) + iK|\,\xi\,|\}.$$

Thus for $\xi > 0$, (25) gives

$$-\Phi(\xi, \eta) = 2i \exp(-K\eta + iK\xi) \int_0^a \exp(-Ky)\, f(y)\,\mathrm{d}y + \frac{2}{\pi} \int_0^\infty \frac{k \cos k\eta - K \sin k\eta}{k(k^2 + K^2)} \exp(-k\xi) \left\{ \int_0^a (k \cos ky - K \sin ky)\, f(y)\,\mathrm{d}y \right\} \mathrm{d}k.$$

This gives finally the result (19) with (x, y) replaced by (ξ, η). Again noting that

$$\frac{1}{\pi} \int_0^\infty G(0, y; \xi, \eta) \frac{\partial \varphi_0}{\partial x}(0, y)\,\mathrm{d}y = -\exp(-K\eta - iK\xi), \text{ for } \xi < 0,$$

we obtain from (26) the result (20) with (x, y) replaced by (ξ, η).

Second Approach

This approach is given in Goswami.[13]

Let $\qquad \varphi(x, y) = \Phi(x, y) - \varphi_0(x, y).$

Then from (4), $\varphi(x, y)$ behaves as an outgoing wave as $|\,x\,| \to \infty$. Applying the Green's integral theorem to $\varphi(x, y)$ and $G(x, y; \xi, \eta)$ in the fluid region we obtain

$$2\pi\varphi(\xi, \eta) = \int_a^\infty g(y) \frac{\partial G}{\partial x}(0, y; \xi, \eta)\, dy, \qquad \ldots(31)$$

where $g(y) = \varphi(+0, y) - \varphi(-0, y),\ a < y < \infty,$ ...(32)

so that $g(y)$ denotes the difference of the scattered potential across the barrier and hence

$$g(y) \to 0 \text{ as } y \to a.$$

Now using the condition (3), i.e.

$$\frac{\partial\varphi}{\partial\xi}(0, \eta) = -iK \exp(-K\eta),\ a < \eta < \infty,$$

we obtain an IE for $g(y)$ as

$$\int_a^\infty g(y) \frac{\partial^2 G}{\partial\xi\partial x}(0, y; 0, \eta) = -2\pi iK \exp(-K\eta),\ a < \eta < \infty. \qquad \ldots(33)$$

But $\frac{\partial^2 G}{\partial\xi\partial x}(0; y; 0, \eta) = \frac{\partial^2}{\partial\eta^2} G(0, y; 0, \eta)$

so that (33) can be written as

$$\frac{d^2}{d\eta^2}\int_0^\infty g(y)\, G(0, y; 0, \eta)\, dy = -2\pi iK \exp(-K\eta),\ a < \eta < \infty. \qquad \ldots(34)$$

This can be reduced to the singular IE

$$\int_a^\infty \{Kg(y) + g'(y)\} \frac{y}{y^2 - \eta^2}\, dy = 0. \qquad \ldots(35)$$

Although $g(y) \to 0$ as $y \to a$, by the edge condition, as $y \to a$, $g'(y) \sim (y^2 - a^2)^{-1/2}$. Following Ursell[2] the solution of this IE is

$$Kg(y) + g'(y) = C_1(y^2 - a^2)^{-1/2},$$

where C_1 is an arbitrary constant. Thus

$$g(y) = C_1 \exp(-Ky) \int_a^y \frac{\exp(Kv)}{(v^2 - a^2)^{1/2}}\, dv. \qquad \ldots(36)$$

To determine this constant C_1 we substitute $g(y)$ from (36) into (34) and after evaluating the integrals we obtain

$$C_1 = -2\,\Delta^{-1} \qquad \ldots(37)$$

To find the transmission and reflection coefficients we make $\xi \to \pm\infty$ in (27). This gives

$$T = 1 + K \int_a^\infty g(y)\,\mathrm{d}y, \quad R = -K \int_a^\infty g(y)\,\mathrm{d}y.$$

But
$$K \int_a^\infty g(y)\,\mathrm{d}y = \frac{C_1}{2} K_0(Ka)$$

so that using (37), we arrive at the same results for T and R as given in (19).

$\Phi(\xi, \eta)$ can be obtained from (31) by noting that $\frac{\partial G}{\partial x}(0, y; \xi, \eta)$ needs to be calculated separately for $\xi > 0$ and $\xi < 0$ from the result

$$G(x, y; \xi, \eta) = 2 \int_0^\infty \frac{(k \cos ky - K \sin ky)(k \cos k\eta - K \sin k\eta)}{k(k^2 + K^2)}$$

$$\exp(-k \mid x - \xi \mid)\,\mathrm{d}k + 2\pi i \exp\{-K(y + \eta) + iK \mid x - \xi \mid\}.$$

After carrying out the details of the calculations the same expressions for $\Phi(\xi, \eta)$ as given earlier are obtained.

Reduction Method

This method was introduced by Williams[5] in connection with the partially immersed barrier problem. Let

$$\chi(x, y) = -\exp(Ky) \int_y^\infty \Phi(x, \eta) \exp(-K\eta)\,\mathrm{d}\eta, \qquad \ldots(38)$$

then
$$\Phi = \frac{\partial \chi}{\partial y} - K\chi \qquad \ldots(39)$$

and χ satisfies

$$\nabla^2 \chi = 0$$

in the fluid region. The FS condition (2) becomes

$$\frac{\partial^2 \chi}{\partial x^2} + K^2 \chi = 0 \text{ on } y = 0. \qquad \ldots(40)$$

Using (4) in (38) and noting (40), it is obvious that

$$\chi(x, 0) = \begin{cases} -\dfrac{1}{2K}[\exp(iKx) + R\exp(-iKx)], & x > 0 \\ \\ \dfrac{T}{2K}\exp(iKx), & x < 0, \end{cases} \qquad \ldots(41)$$

Now since $\frac{\partial \Phi}{\partial x}$ is continuous across the plane $x = 0$ (on the gap this is obvious while on the barrier $\frac{\partial \Phi}{\partial x}$ is zero on both sides and so is continuous), $\frac{\partial \chi}{\partial x}$ is also so and hence, (41) gives

$$R + T = 1. \qquad \text{...(42)}$$

We now define a function $\Psi(x, y)$ by

$$\Psi(x, y) = \begin{cases} \chi(x, y) + \dfrac{T}{2K} \exp(iKx - Ky), \; x > 0, \\ \chi(x, y) + \dfrac{1}{2K} \exp(-Ky) \, [\exp(iKx) + R \exp(-iKx)], \\ \qquad x < 0 \end{cases} \qquad \text{...(43)}$$

The original BVP described by (1) to (4) is then reduced to the problem described by

$$\nabla^2 \Psi = 0, \; y > 0, \qquad \text{...(44)}$$

$$\Psi(x, 0) = 0, \; |x| < \infty, \qquad \text{...(45)}$$

$$\Psi(+0, y) - \Psi(-0, y) = \frac{2R}{K} \sinh Ky, \; 0 < y < a, \qquad \text{...(46)}$$

$$\frac{\partial \Psi}{\partial x} \text{ is continuous across } x = 0, \; y > 0, \qquad \text{...(47)}$$

$$\frac{\partial \Psi}{\partial x}(\pm 0, y) = \frac{i(1 - R)}{2} \exp(-Ky), \; y > a \qquad \text{...(48)}$$

and grad Ψ is bounded near the edge of the barrier. Using Green's integral theorem to $\Psi(x, y)$ and $G(x, y; \xi, \eta)$ in the regions $y > 0$, $x > 0$ and $y > 0$, $x < 0$, we obtain

$$\Psi(\xi, \eta) = \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int_0^\infty \frac{\partial \Psi}{\partial x}(+0, y) \, G_0(0, y; \xi, \eta) \, dy, \; \xi > 0 \\ \dfrac{1}{2\pi} \displaystyle\int_0^\infty \frac{\partial \Psi}{\partial x}(-0, y) \, G(0, y; \xi, \eta) \, dy, \; \xi < 0, \end{cases} \qquad \text{...(49)}$$

where $\quad G_0(x, y; \xi, \eta) = -2 \ln \dfrac{r r_1}{r' r_1'}$

with $\quad r^2 = (x - \xi)^2 + (y - \eta)^2, \; r'^2 = (x - \xi)^2 + (y + \eta)^2$

$\quad r_1^2 = (x + \xi)^2 + (y - \eta)^2, \; r_1'^2 = (x + \xi)^2 + (y + \eta)^2.$

We note that $\frac{\partial \Psi}{\partial x}(+0, y) = \frac{\partial \Psi}{\partial x}(-0, y)$ and is unknown for $0 < y < a$. Let it be denoted by $h(y)$, Then using the condition (46) we obtain the IE for $h(y)$ as

$$\frac{1}{\pi}\int_0^a h(y) \ln \left|\frac{y-\eta}{y+\eta}\right| dy$$

$$= \frac{R}{K}\sinh K\eta - \frac{i(1-R)}{2\pi}\int_0^\infty \exp(-Ky) \ln \left|\frac{y-\eta}{y+\eta}\right| dy,\ 0 < \eta < a, \quad \ldots(50)$$

where $h(y)$ is bounded near $y = a$. Using the result

$$\int_0^{\min(y,\eta)} \frac{u\,du}{\{(y^2-u^2)(\eta^2-u^2)\}^{1/2}} = -\frac{1}{2}\ln\left|\frac{y-\eta}{y+\eta}\right|,$$

the IE (50) reduces to the Volterra IE

$$\int_0^\eta \frac{u\,S(u)\,du}{(\eta^2-u^2)^{1/2}} = -\frac{\pi R}{2K}\sinh K\eta$$

$$-\frac{i(1-R)}{4}\int_a^\infty \exp(-Ky) \ln\left|\frac{y-\eta}{y+\eta}\right| dy,\ 0 < \eta < a, \quad \ldots(51)$$

where $$S(u) = \int_u^a \frac{h(y)}{(y^2-u^2)^{1/2}}\,dy,\ 0 < u < a. \quad \ldots(52)$$

Following Copson[14]

$$h(y) = -\frac{2}{\pi}\frac{d}{dy}\int_y^a \frac{u\,S(u)}{(u^2-y^2)^{1/2}}\,du,\ 0 < y < a, \quad \ldots(53)$$

and $S(u)$ is obtained from (51) as

$$S(u) = -R\int_0^u \frac{\cosh Ky}{(u^2-y^2)^{1/2}}\,dy - \frac{i(1-R)}{2}\int_0^\infty \frac{\exp(-Ky)}{(y^2-u^2)^{1/2}}\,dy. \quad \ldots(54)$$

But $h(y)$ is required to be bounded near the end $y = a$ and this requires that $S(a) = 0$. Using this in (54) we obtain R and hence T, and the results coincide with (19)

To obtain $\Phi(\xi, \eta)$ we note that for $\xi > 0$,

$$\Phi(\xi, \eta) = T \exp\ (iK\xi - K\eta) + \frac{\partial \Psi}{\partial \eta} - K\Psi.$$

Now for $\xi > 0$, using (49),

$$\frac{\partial \Psi}{\partial \eta} - K\Psi = -\frac{1}{2\pi}\int_0^\infty \frac{\partial \Psi}{\partial x}(0,\ y)\left(\frac{\partial G_0}{\partial \eta} - KG_0\right)(0,\ y;\ \xi,\ \eta)\ \mathrm{d}y$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\exp\ (-\ k\xi)}{k}\ (k \cos k\eta - K \sin k\eta)$$

$$\left[\int_0^\infty \frac{\partial \Psi}{\partial x}(0,\ y) \sin ky\ \mathrm{d}y\right] \mathrm{d}k.$$

But
$$\int_0^\infty \frac{\partial \Psi}{\partial x}(0,\ y) \sin ky\ \mathrm{d}y$$

$$= \int_0^a h(y) \sin ky\ \mathrm{d}y + \frac{i(1 - R)}{2}\int_a^\infty \exp\ (-\ Ky) \sin ky\ \mathrm{d}y$$

$$= -\ Rk\int_0^a uJ_0(ku)\, I_0(Ku)\ \mathrm{d}u + \frac{i(1 - R)}{2\pi}\ k\int_a^\infty uJ_0(ku)\, K_0(Ku)\ \mathrm{d}u$$

$$= \frac{ki}{i\pi I_0(Ka) + K_0(Ka)}\ \frac{J_0(ka)}{K^2 + k^2}$$

so that for $\xi > 0$

$$\frac{\partial \Psi}{\partial \eta} - K\Psi = \frac{1}{i\pi I_0(Ka) + K_0(Ka)}\int_0^\infty \exp\ (-\ k\xi)\,\frac{(k \cos k\eta - K \sin k\eta)}{k^2 + K^2}\ J_0(ka)\ \mathrm{d}k.$$

Hence, for $\xi > 0$, $\Phi(\xi, \eta)$ coincides with (20). Similarly we can show that for $\xi < 0$, the result (21) for $\Phi(\xi, \eta)$ is readily obtained.

Complex Variable Method

This method was first used by Dean[3] for the submerged barrier problem. Later Evans[15] and Porter[8] used this method to solve the scattering problem involving a submerged vertical plate and a gap along a vertical plate respectively.

Let us introduce a second complex unit j, which does note interact with i, and is used to denote the complex variable $z = x + jy$.

Let $\quad W(z) = \Phi(x, y) + j\psi(x, y), \text{ Im } z \geqslant 0,$ ...(55)

and $\quad W(z) = -\frac{dw}{dz} + jKw(z), \text{ Im } z \geqslant 0,$ ...(56)

so that

$$W(z) = -\left(K\psi + \frac{\partial \Phi}{\partial x}\right), + j\left(K\,\Phi + \frac{\partial \Phi}{\partial y}\right) y \geqslant 0.$$

Hence, the FS condition (2) is equivalent to

$$\text{Im}_j\, W(z) = 0,\ y = 0. \qquad \text{...(57)}$$

The analytic continuation of $W(z)$ into $\text{Im } z < 0$ is permitted by the Schwartz's reflection principle. Thus we may write

$$F(z) = \begin{cases} W(z) & , \text{Im } z \geqslant 0, \\ \overline{W(\bar{z})} & , \text{Im } z < 0. \end{cases} \qquad \text{...(58)}$$

Now as $\frac{\partial \Phi}{\partial x}(\pm\, 0, y) = 0$ for $a < y < \infty$, we obtain by the Cauchy-Riemann conditions $\frac{\partial \psi}{\partial y}(\pm\, 0, y) = 0$ for $a < y < \infty$ which is equivalent to

$$\psi(\pm\, 0, y)\ 0 \text{ for } a < y < \infty$$

by choosing the constant of integration to be zero. Thus the condition on the barrier is equivalent to

$$\text{Re}_j\, W(z) = 0 \text{ on } x = \pm\, 0, c < y < \infty. \qquad \text{...(59)}$$

Condition (59) can be continued into $\text{Im } z < 0$ with the help of (58) so that we obtain

$$\text{Re}_j\, F(z) = 0 \text{ on } z = \pm\, 0, a < |\, y\,| < \infty,$$

which is equivalent to

$$F(z) + \overline{F(z)} = 0 \text{ on } x = \pm\, 0, a < |\, y\,| < \infty, \qquad \text{...(60)}$$

The condition (60) can be written as

$$F_+(iy) + F_-(iy) = 0,\ a < |\, y\,| < \infty \qquad \text{...(61)}$$

provided

$$F(z) = \bar{F}(-\, z) \qquad \text{...(62)}$$

where the notation $F_\pm(iy) = \lim_{x \to \pm 0} F(z)$ has been used.

Now (61) is a homogeneous Riemann Hilbert problem for the sectionally analytic function $F(z)$ in the complex z-plane cut along the real axis from ja to $j\infty$ and from $-j\infty$ to $-ja$. Also, by definition $W(z)$ and hence F(z) is bounded as $|z| \to \infty$ and $F(z) \sim (a^2+z^2)^{-1/2}$ as $z \to \pm ja$. Then the solution of (44) is

$$F(z) = \frac{Az+B}{(a^2+z^2)^{1/2}},$$

where A and B are in general complex constants with respect to j. Because of the conditions (62), A is purely imaginary and B is purely real with respect to j. Thus,

$$F(z) = \frac{jD_0 z + D}{(a^2+z^2)^{1/2}}.$$

But by the condition (57), D_0 must be zero so that $F(z) = D(a^2+z^2)^{-1/2}$ where D is purely real with respect to j. Hence,

$$W(z) = D(a^2+z^2)^{-1/2}$$

so that

$$w(z) = \exp(jKz)\left[D_1 - D\int_{ja}^{z} \frac{\exp(-jK\zeta)}{(\zeta^2+a^2)^{1/2}}\,d\zeta\right] \qquad \ldots(63)$$

where D_1 is constant and real with respect to j. This result was guessed by Dean.[1]

The constants D and D_1 can be determined by identifying with the asymptotic form of $\Phi(x, y)$ as $|x| \to +\infty$. Now as $x \to \infty$, the contour in the integral in (63) can be deformed into the part of the imaginary axis from ja to $-ja$, then just to the right of the line from $-ja$ to $-j\infty$ and finally the quarter circle of infinite radius in the fourth quadrant. There is no contribution from the quarter circle while the other contributions from ja to $-ja$ and from $-ja+0$ to $-j\infty+0$ can be calculated. Thus it is seen that as $z \to +\infty$

$$w(z) \sim \exp(jKz)\,[D_1 - D\{K_0(Ka) - j\pi I_0(Ka)\}].$$

Similarly as $z \to -\infty$, by deforming the contour in the integral in (63) in an appropriate manner it can be shown that

$$w(z) \sim \exp(jKz)\,[D_1 + D\{K_0(Ka) + j\pi I_0(Ka)\}].$$

Thus as $x \to \pm\infty$,

$$\Phi(x, y) \sim \exp(-Ky)\,[\{D_1 \pm DK_0(Ka)\}\cos kx - D\pi I_0(Ka)\sin kx]. \qquad \ldots(64)$$

But from (4)

$$\Phi(x, y) \sim \begin{cases} T\exp(-Ky)(\cos kx + i\sin Kx) \text{ as } x \to +\infty, \\ \exp(-Ky)\,[(1+R)\cos Kx + i(1-R)\sin Kx] \text{ as} \\ x \to -\infty. \end{cases}$$

Comparing this with (58) we obtain

$$D_1 - DK_0(Ka) = T,\ D_1 + DK_0(Ka) = 1 + R,$$
$$-D\pi I_0(Ka) = iT,\ -D\pi I_0(Ka) = i(1 - R).$$

These give

$$D_1 = 1,\ D = \{i\pi I_0(Ka) + K_0(Ka\}^{-1} \qquad \ldots(65)$$

so that $R + T = 1$ and we obtain the same expression for R and T given in (19).

To obtain $\Phi(x, y)$ for $x > 0$ we deform the contour in the integral in (63) into the part of the imaginary axis from ja to $-ja$, a line just the right of the cut from $-ja$ to $-j\infty$, the quarter of a circle of infinite radius in the fourth quadrant and finally a line from $+\infty$ to z. Thus

$$w(z) = \exp(jKz)\left[1 - D\{-j\pi I_0(Ka) + K_0(Ka)\} + D\int_z^\infty \frac{\exp(-jK\zeta)}{(\zeta^2 + a^2)^{1/2}}\,d\zeta\right].$$

But
$$\int_z^\infty \frac{\exp(-jK\zeta)}{(\zeta^2 + a^2)^{1/2}}\,d\zeta = \int_0^\infty \exp(-jK\zeta)\left\{\int_0^\infty \exp(-k\zeta)\,J_0(ka)\,dk\right\}d\zeta$$
$$= \int_0^\infty J_0(k\mathrm{a})\left[\int_z^\infty \exp\{-(k + jK)\zeta\}\,d\zeta\right]dk$$
$$= \int_0^\infty J_0(ka)\,\frac{\exp\{-(k + jK)\}}{k + jK}\,dk.$$

Thus for $x > 0$

$$w(z) = \exp(-Ky)(\cos Kx + j\sin Kx)\left[1 - D\{K_0(Ka) - j\pi I_0(Ka)\} + D\int_0^\infty \exp(-kx)\,J_0(ka)\,\frac{(\cos ky - j\sin ky)(k - jK)}{k^2 + K^2}\,dk\right].$$
$$\ldots(66)$$

Taking the real part of (66) and using (65), we see that it coincides with the result given in (20). Similarly for $x < 0$, it can be shown that

$$w(z) = \exp(-Ky)(\cos Kx + j\sin Kx)\left[1 + D\{K_0(Ka) + j\pi I_0(Ka)\} - D\int_0^\infty \exp(kx)\,J_0(ka)\,\frac{(\cos ky + j\sin ky)(k + jK)}{k^2 + K^2}\,dk\right]. \quad \ldots(67)$$

The real part of (67) coincides with the result given in (21).

Discussion

Four mathematical methods are demonstrated in some detail to solve a particular scattering problem involving a submerged fixed vertical plane barrier. If we consider scattering problems involving obstacles of any geometrical shape, it is then obvious that the expansion method is suitable only when the obstacles are in the form of plane vertical barriers. The same remark also applies for the reduction method. The complex variable method is applicable only for two-dimensional problems with Laplace's equation as the governing equation. However, the IE method is suitable for obstacles of any geometrical shape. By a suitable use of Green's integral theorem in the fluid region the problem can always be reduced to finding the solution of certain IE of second kind. This IE then can be solved at least numerically. For problems involving plane vertical barriers we can obtain, however, singular IE whose solution can sometimes be obtained in closed form.

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