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ADI METHOD FOR MHD PIPE FLOW WITH CONDUCTING WALLS PARALLEL TO EXTERNAL MAGNETIC FIELD

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The ADI method has been used to obtain the numerical solution of MHD flow in a rectangular pipe with conducting walls parallel to the external magnetic field, the other two walls being nonconducting. No exact solution of the problem is available. Comparison has been made with the results in authors' other paper where the problem is reduced to the numerical solution of a singular integral equation and Shercliff's solution with nonconducting walls.

Key Words : ADI Method; MHD Pipe Flow; Conducting Walls; External Magnetic Field; Shercliff's Solution; Hartmann Number MHD Flowmetry; MHD Power Generation

INTRODUCTION

IN a recent paper,[1] the authors have discussed the numerical solution of an MHD pipe flow problem in a rectangular pipe with walls parallel to the external applied magnetic field as perfectly conducting and those perpendicular to it as nonconducting. The study of this problem is important due to engineering applications in MHD flowmetry and MHD power generation. As pointed out by Hunt,[2] no satisfactory approximate or exact solution exists for this important practical situation. In our earlier paper we had reduced the problem to the numerical solution of a singular integral equation using Green's function. The object of the present paper is to provide an alternative numerical procedure using the ADI method. Comparison has been made between the results by the two methods. It is found that as the mesh size is reduced the results show a tendency to converge. As the Hartmann number is increased the convergence is seen to be faster.

The basic equations of MHD pipe flow are well known. The final equations for the present problem in the nondimensional form are

$$V_{xx} + V_{yy} + MB_x = -1, \text{ in } R \qquad ...(1)$$

and

$$B_{xx} + B_{yy} + MV_x = 0, \text{ in } R, \qquad ...(2)$$

where R is the section of the pipe bounded by the lines $x = \pm 1$, $y = \pm m$. $V(x, y)$, $B(x, y)$ are the velocity and the induced magnetic field and M is the

Hartmann number. The applied magnetic field is taken parallel to x-axis. The boundary conditions are

$$V = 0, \quad B = 0, \quad \text{on } x = \pm 1, \qquad \ldots(3)$$

$$V = 0, \quad B_y = 0, \quad \text{on } y = \pm m. \qquad \ldots(4)$$

The ADI Method

As shown in Fig. 1 the finite difference mesh in R is defined by

$$\left.\begin{aligned} &\Delta x = 2/(I+1) = h, \quad I \text{ prescribed}, \\ &\Delta y = 2m/(J-1) = k, \quad J \text{ prescribed}, \\ &x_i = -1 + ih, \quad i = O(1)\, I+1 \\ &y_j = -m + (j-1)\,k, \quad j = 1(1)\, J. \end{aligned}\right\} \qquad \ldots(5)$$

and

Also let $V(x_i, y_j) = V_{ij}, B(x_i, y_j) = B_{ij}$.

Further, to accommodate the boundary conditions at $y = \pm m$, we introduce two fictitious lines $j = 0$ and $j = J + 1$. Then the boundary conditions lead to the following results :

$$\left.\begin{aligned} &V_{0j} = V_{I+1,j} = B_{0j} = B_{I+1,j} = 0, \quad j = 0(1)\, J+1, \\ &V_{i1} = V_{iJ} = 0, \quad i = 0(1)\, I+1 \end{aligned}\right\} \qquad \ldots(6)$$

and

$$B_{i0} = B_{i,2}, \quad B_{i,J+1} = B_{i,J-1}, \quad i = 1(1)\, I.$$

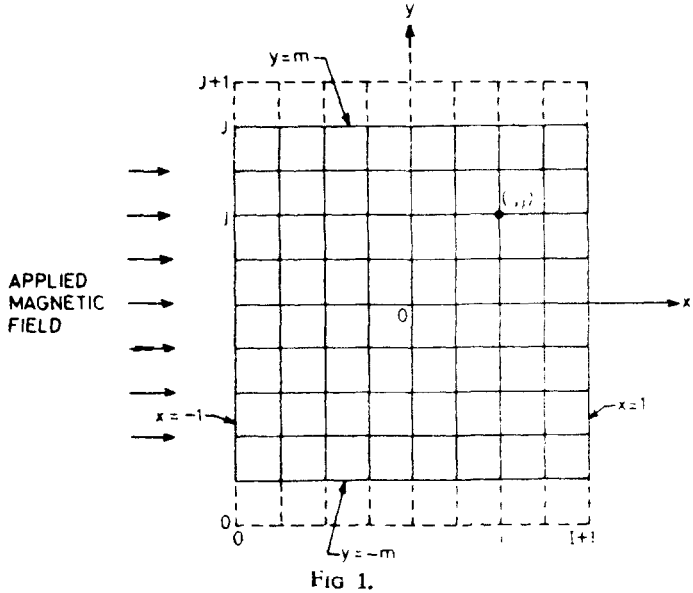

Fig 1.

Introducing superscript for iteration level, the ADI method leads to the following equations :

$$(w_1 - \delta_x^2)\, V_{ij}^* = (w_1 + r\delta_y^2)\, V_{ij}^n + Mh\mu_x\delta_x B_{ij}^n + h^2,$$

$$i = 1(1)\, I, \text{ for fixed } j, \qquad \ldots(7)$$

$$(w_2 - \delta_x^2)\, B_{ij}^* = (w_2 + r\delta_y^2)\, B_{ij}^n + Mh\mu_x\delta_x V_{ij}^*,$$

$$i = 1(1)\, I, \text{ for fixed } j, \qquad \ldots(8)$$

$$(w_3 - \delta_y^2)\, V_{ij}^{n+1} = \left(w_3 + \frac{1}{r}\,\delta_x^2\right) V_{ij}^* + \frac{Mh}{r}\,\mu_x\delta_x B_{ij}^* + k^2,$$

$$j = 2(1)\, J - 1, \text{ for fixed } i, \qquad \ldots(9)$$

and

$$(w_4 - \delta_y^2)\, B_{ij}^{n+1} = \left(w_4 + \frac{1}{r}\,\delta_x^2\right) B_{ij}^* + \frac{Mh}{r}\,\mu_x\delta_x V_{ij}^{n+1},$$

$$j = 1(1)\, J, \text{ for fixed } i, \qquad \ldots(10)$$

where $r = h^2/k^2$ and w_1, w_2, w_3, w_4 are acceleration parameters. The symbols μ_x, δ_x etc. have usual meaning. Making use of the boundary conditions, the following tridiagonal sets of equations are obtained :

$$\begin{bmatrix} 2+w_1 & -1 & & & 0 \\ -1 & 2+w_1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 2+w_1 & -1 \\ & & & -1 & 2+w_1 \end{bmatrix} \begin{bmatrix} V_{1j}^* \\ V_{2j}^* \\ \vdots \\ V_{Ij}^* \end{bmatrix} = \begin{bmatrix} p_{1j}^n \\ p_{2j}^n \\ \vdots \\ p_{Ij}^n \end{bmatrix}, \qquad \ldots(11)$$

$$\begin{bmatrix} 2+w_2 & -1 & & & \\ -1 & 2+w_2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+w_2 & -1 \\ & & & -1 & 2+w_2 \end{bmatrix} \begin{bmatrix} B_{1j}^* \\ B_{2j}^* \\ \vdots \\ B_{Ij}^* \end{bmatrix} = \begin{bmatrix} q_{1j}^n \\ q_{2j}^n \\ \vdots \\ q_{Ij}^n \end{bmatrix}, \qquad \ldots(12)$$

$$\begin{bmatrix} 2+w_3 & -1 & & & \\ -1 & 2+w_3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+w_3 & -1 \\ & & & -1 & 2+w_3 \end{bmatrix} \begin{bmatrix} V_{i2}^{n+1} \\ V_{i3}^{n+1} \\ \vdots \\ V_{i,J-1}^{n+1} \end{bmatrix} = \begin{bmatrix} r_{i2}^* \\ r_{i3}^* \\ \vdots \\ r_{i,J-1}^* \end{bmatrix}, \qquad \ldots(13)$$

and

$$\begin{bmatrix} 2+w_4 & -2 & & & \\ -1 & 2+w_4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 2+w_4 & -1 \end{bmatrix} \begin{bmatrix} B_{i1}^{n+1} \\ B_{i2}^{n+1} \\ \vdots \\ B_{iJ}^{n+1} \end{bmatrix} = \begin{bmatrix} s_{i1}^* \\ s_{i2}^* \\ \vdots \\ s_{iJ}^* \end{bmatrix}, \quad \ldots(14)$$

where

$$p_{ij}^n = r(V_{i,j-1}^n + V_{i,j+1}^n) - (2r - w_1) V_{ij}^n + \frac{Mh}{2}(B_{i+1,j}^n - B_{i-1,j}^n) + h^2,$$

$$i = 1(1)\ I,\ j = 2(1)\ J-1, \quad \ldots(15)$$

$$q_{ij}^n = r(B_{i,j-1}^n + B_{i,j+1}^n) - (2r - w_2) B_{ij}^n + \frac{Mh}{2}(V_{i+1,j}^* - V_{i-1,j}^*),$$

$$i = 1(1)\ I,\ j = 1(1)\ J, \quad \ldots(16)$$

$$r_{ij}^* = \frac{1}{r}(V_{i-1,j}^* + V_{i+1,j}^*) - \left(\frac{2}{r} - w_3\right) V_{ij}^* + \frac{Mh}{2r}(B_{i+1,j}^* - B_{i-1,j}^*)$$

$$+ k^2,\ i = 1(1)\ I,\ j = 2(1)\ J-1, \quad \ldots(17)$$

$$s_{ij}^* = \frac{1}{r}(B_{i-1,j}^* + B_{i+1,j}^*) - \left(\frac{2}{r} - w_4\right) B_{ij}^* + \frac{Mh}{2r}(V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1})$$

$$i = 1(1)\ I,\ j = 1(1)\ J. \quad \ldots(18)$$

In each case, we have to solve a system of tridiagonal equations. This can be done either by LU-decomposition or using Thomas algorithm. Note that in (11) j varies from 2 to $J-1$. Thus we have in all $J-2$ sets of equations each having I unknowns. After solving these we get all the unknowns V_{ij}^*, $i = 1(1)\ I$, $j = 2(1)\ J-1$. Similarly, (12) gives induced magnetic field B_{ij}^*, $i = 1(1)\ I$, $j = 1(1)\ J$. Note that although $V = 0$ on $y = \pm m$, the induced magnetic field B is not known on these boundaries. Equations (13) and (14) give velocity and induced magnetic field at $n+1$ iteration level. Here we solve along lines parallel to y-axis.

The above iterations are continued till the values get stablized to the desired accuracy. To start iterations, the initial values are taken zero over the entire section. The parameters w_1, w_2, w_3 and w_4 have been introduced to accelerate convergence. Since no analytical procedure is available to estimate these for the present problem, to have fastest convergence rate, several numerical experiments were performed. Since these depend upon $\alpha = M/2$, h and k, it will be quite a time consuming task to vary all the four independently. So we chose to have $w_1 = w_2 = w_3 = w_4 = w$, say. This gives only one parameter w. We have carried out calculations for a square pipe ($m = 1$) with $\alpha = 1(1)\ 20$ and

$h = k = 0.2, 0.1, 0.05$. It is found that w increases with increasing α and/or h. Table I gives the velocity in the first quadrant for $\alpha = 2, 5, 10$ for $h = k = .05$ after 50 iterations. The corresponding values of induced magnetic field are given in Table II. Table III gives the velocity at the centre for various values of α and h. The convergence of the results with decreasing h is immediately clear. For comparison the results obtained by Grinberg method of Green's function as discussed by the author[1] are also given for $h = .0625$. From the numerical results, we make the following observations :

1. As α is increased, the velocity profile shows a flattening tendency. The velocity at the centre and the flux through a section decrease. This effect is attributed to the presence of magnetic field whose lines of force act like stretched rubber bands and try to retard the flow rate. This is a very well-known effect in MHD.

2. If we compare the present case with the one with all walls nonconducting which has been studied by Shercliff[3] and computed in our earlier paper, we find that the velocity and flux are smaller in the present case i.e., wall conductance has a retarding effect on flow. In addition to this the induced magnetic

TABLE I

Velocity in 1st quadrant

α	y \ x	0.0	0.2	0.4	0.6	0.8	1.0
0	1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
	0.8	0.1162	0.1129	0.1025	0.0835	0.0523	0.0
	0.6	0.1988	0.1926	0.1734	0.1386	0.0835	0.0
	0.4	0.2535	0.2452	0.2194	0.1734	0.1025	0.0
	0.2	0.2846	0.2750	0.2452	0.1926	0.1129	0.0
	0.0	0.2947	0.2846	0.2535	0.1988	0.1162	0.0
2	1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
	0.8	0.0792	0.0788	0.0768	0.0705	0.0522	0.0
	0.6	0.1339	0.1326	0.1273	0.1138	0.0805	0.0
	0.4	0.1684	0.1663	0.1584	0.1396	0.0970	0.0
	0.2	0.1871	0.1844	0.1750	0.1533	0.1058	0.0
	0.0	0.1930	0.1901	0.1802	0.1576	0.1086	0.0
5	1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
	0.8	0.0454	0.0457	0.0465	0.0472	0.0436	0.0
	0.6	0.0732	0.0732	0.0732	0.0720	0.0634	0.0
	0.4	0.0878	0.0876	0.0868	0.0846	0.0738	0.0
	0.2	0.0943	0.0940	0.0931	0.0906	0.0791	0.0
	0.0	0.0960	0.0958	0.0948	0.0923	0.0807	0.0
10	1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
	0.8	0.0288	0.0289	0.0293	0.0300	0.0302	0.0
	0.6	0.0429	0.0429	0.0427	0.0424	0.0412	0.0
	0.4	0.0481	0.0480	0.0477	0.0473	0.0461	0.0
	0.2	0.0495	0.0495	0.0493	0.0491	0.0481	0.0
	0.0	0.0498	0.0498	0.0497	0.0495	0.0487	0.0

TABLE II

Induced magnetic field in 1st quadrant

α	y \ x	0.0	0.2	0.4	0.6	0.8	1.0
2	1.0	0.0	−.0163	−.0294	−.0352	−.0277	0.0
	0.8	0.0	−.0177	−.0326	−.0406	−.0348	0.0
	0.6	0.0	−.0212	−.0397	−.0512	−.0461	0.0
	0.4	0.0	−.0248	−.0469	−.0611	−.0554	0.0
	0.2	0.0	−.0274	−.0518	−.0676	−.0612	0.0
	0.0	0.0	−.0283	−.0536	−.0698	−.0631	0.0
5	1.0	0.0	−.0105	−.0200	−.0268	−.0256	0.0
	0.8	0.0	−.0112	−.0218	−.0307	−.0338	0.0
	0.6	0.0	−.0134	−.0267	−.0393	−.0458	0.0
	0.4	0.0	−.0158	−.0317	−.0468	−.0545	0.0
	0.2	0.0	−.0175	−.0349	−.0513	−.0593	0.0
	0.0	0.0	−.0180	−.0359	−.0527	−.0608	0.0
10	1.0	0.0	−.0058	−.0112	−.0156	−.0172	0.0
	0.8	0.0	−.0063	−.0124	−.0182	−.0233	0.0
	0.6	0.0	−.0078	−.0157	−.0237	−.0315	0.0
	0.4	0.0	−.0091	−.0182	−.0274	−.0361	0.0
	0.2	0.0	−.0097	−.0194	−.0291	−.0381	0.0
	0.0	0.0	−.0099	−.0197	−.0295	−.0387	0.0

TABLE III

Convergence of velocity at the centre

	ADI			GRIN
α \ h	.20	.10	.05	.0625
1	.25495	.25572	.25592	.25669
2	.19230	.19283	.19297	.19334
3	.14590	.14667	.14686	.14707
4	.11567	.11642	.11661	.11674
5	.09529	.09589	.09605	.09613
6	.08078	.08123	.08135	.08141
7	.07000	.07030	.07039	.07043
8	.06169	.06188	.06194	.06196
9	.05520	.05520	.05524	.05526
10	.05003	.04979	.04982	.04983
20	.02912	.02500	.02499	.02500

field is not zero on $y = \pm m$ as compared with zero value on the entire boundary when all walls were nonconducting.

Finally, we would conclude that, if one is interested in the solution at a point, the Green's function approach as discussed by us in an earlier paper is preferable.

However, if solution is desired in the entire cross-section, the present approach is recommended due to better computational efficiency. The computer time and storage involved is much less and the accuracy is roughly the same.

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