

## CRACK IN A TRANSVERSELY ISOTROPIC SOLID UNDER TORSION

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The distribution of stress in the vicinity of a penny-shaped crack in a transversely isotropic solid under torsion has been determined and the stress intensity factor has been obtained. Numerical results of stress distribution within the interior of the medium are also calculated.

**Key Words :** Crack; Solid—Transversely Isotropic; Torsion; Stress

### INTRODUCTION

Sack<sup>1</sup> has shown that the presence of a penny-shaped crack of unit radius in an isotropic solid under a uniform torsion alters the free energy of the solid.

In this paper we consider the problem of determining the distribution of stress in a semi-infinite transversely isotropic solid  $z \geq 0$  under torsion with a penny-shaped crack of unit radius. It will be assumed that the axis of  $z$  is the axis of elastic symmetry and the position of a typical point of the solid may then be expressed in terms of cylindrical coordinates  $(r, \theta, z)$ . For symmetrical deformation of the solid the non-vanishing components of stress tensor are  $\sigma_{\theta z}$  and  $\partial_{r\theta}$  and the only non-vanishing components of displacement vector is  $u_\theta$  in this coordinate system when the solid  $z \geq 0$  is under torsion. Consequently, the boundary conditions are

$$\begin{aligned}\sigma_{\theta z}(r, 0) &= 0, & 0 \leq r \leq 1 \\ u_\theta(r, 0) &= 0, & r > 1\end{aligned}$$

### METHOD OF SOLUTION

We consider the problem of calculating the stress field in the vicinity of a penny-shaped crack defined by  $0 \leq r \leq 1, z = 0$  in a solid under torsion so that

$$\sigma_{\theta z}(r, \theta, z) \sim S \quad \dots(1)$$

where  $S$  is the "all round" torsion as  $\sqrt{r^2 + z^2} \rightarrow \infty$ . In such a stress-distribution, the displacement components  $u_r, u_z$  will be identically zero and the equation of equilibrium that is not automatically satisfied is

$$\frac{\partial}{\partial r}(\sigma_{r\theta}) + \frac{\partial}{\partial z}(\sigma_{\theta z}) + \frac{2}{r}\sigma_{r\theta} = 0. \quad \dots(2)$$

We have for transversely isotropic material, *vide* Love<sup>2</sup>

$$\sigma_{\theta z} = c_{44} \frac{\partial u_{\theta}}{\partial z} \quad \dots(3)$$

$$\sigma_{r\theta} = c_{66} \left( \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right) \quad \dots(4)$$

where  $C_{44}$  and  $C_{66}$  being the elastic constants.

The equation (2), therefore, reduces to

$$\frac{\partial^2 u_{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r^2} + k^2 \frac{\partial^2 u_{\theta}}{\partial z^2} = 0, \quad \dots(5)$$

where

$$k^2 = \frac{C_{44}}{C_{66}}. \quad \dots(6)$$

We take the solution of the equation (5) in the form

$$u_{\theta}(r, z) = \frac{S}{C_{44}} z + \frac{kS}{C_{44}} \int_0^{\infty} A(\alpha) (\exp(-\alpha z)/k) J_1(\alpha r) d\alpha, \quad \dots(7)$$

so that from the relation (3), we have

$$\sigma_{\theta z}(r, z) = S - S \int_0^{\infty} \alpha A (\exp(-\alpha z)/k) J_1(\alpha r) d\alpha \quad \dots(8)$$

The expression (8) ensures that the condition (1) will be satisfied if the Hankel transform exist. Moreover, the boundary conditions

$$\sigma_{\theta z}(r, 0) = 0, \quad 0 \leq r \leq 1$$

$$u_{\theta}(r, 0) = 0, \quad r > 1$$

will be satisfied if  $A(\alpha)$  satisfies the pair of dual integral equations

$$\int_0^{\infty} \alpha A(\alpha) J_1(\alpha r) d\alpha = 1, \quad 0 \leq r \leq 1 \quad \dots(9)$$

$$\int_0^{\infty} A(\alpha) J_1(\alpha r) d\alpha = 0, \quad r > 1$$

The solution of this pair of equations is known to be

$$A(\alpha) = \frac{1 - \cos \alpha}{\alpha^2} - \frac{\sin \alpha}{2\alpha}. \quad \dots(10)$$

Using the following standard results of definite integrals viz

$$\int_0^{\infty} \left( \frac{1 - \cos \alpha}{\alpha^2} - \frac{1}{2} \sin \alpha \right) J_1(\alpha r) d\alpha$$

$$= \begin{cases} 1 & 0 \leq r \leq 1 \\ \frac{1}{r} (r - \sqrt{r^2 - 1}) - \frac{1}{2r\sqrt{r^2 - 1}}, & r > 1 \end{cases}$$

$$\int_0^{\infty} \left( \frac{1 - \cos \alpha}{\alpha^2} - \frac{\sin \alpha}{2\alpha} \right) J_1(\alpha r) d\alpha$$

$$= \begin{cases} \frac{1}{2} r \log \frac{1 + \sqrt{1 - r^2}}{r}, & 0 \leq r \leq 1 \\ 0 & , r > 1 \end{cases}$$

we deduce from the relations (7) and (8)

$$u_{\theta}(r, 0) = \frac{kS}{2C_{44}} r \log \frac{1 + \sqrt{1 - r^2}}{r}, \quad 0 \leq r \leq 1 \quad \dots(11)$$

$$\sigma_{\theta z}(r, 0) = S \left[ \frac{\sqrt{r^2 - 1}}{r} + \frac{1}{2r\sqrt{r^2 - 1}} \right], \quad r > 1 \quad \dots(12)$$

So the stress intensity factor for torsion defined by

$$K_t = \lim_{r \rightarrow 1+} \sqrt{2(r - 1)} \cdot \sigma_{\theta z}(r, 0) \quad \dots(13)$$

has the value  $\frac{1}{2} S$  as we obtained in the isotropic case.

The energy of the crack is given by

$$W_t = 2\pi S \int_0^1 r u_{\theta}(r, 0) dr$$

so that from the relation (11) we obtain

$$W_t = \frac{\pi k S^2}{C_{44}} \int_0^1 r^2 \log \frac{1 + \sqrt{1 - r^2}}{r} dr$$

Evaluating the integral by the rule for integrating by parts we obtain

$$W_t = \frac{\pi^2 k S^2}{12 C_{44}} \quad \dots(14)$$

The result in case of isotropic solid is

$$W_t = \frac{\pi^2 S^2}{12 \mu} \quad \dots(15)$$

where  $\mu$  is the rigidity modulus.

Using the relation

$$\sigma_{r\theta} = C_{66} r \frac{\partial}{\partial r} \left( \frac{u_{\theta}}{r} \right)$$

and the formula

$$\frac{\partial}{\partial r} \left[ \frac{1}{r} J_1(\alpha r) \right] = - \frac{\alpha}{r} J_2(\alpha r)$$

we obtain from the relation (7) the remaining stress component to be

$$\sigma_{r\theta} = - \frac{S}{k} \int_0^{\infty} \alpha A(\alpha) (\exp(-\alpha z/k)) J_2(\alpha r) d\alpha. \quad \dots(16)$$

#### NUMERICAL RESULTS

We take a specimen of zinc whose elastic constants are

$$C_{44} = 0.4, \quad C_{66} = 0.634$$

Here the constants are expressed in terms of unit stress of  $10^{12}$  dynes/cm<sup>2</sup>. From the expression (14) the energy of crack in torsion is

$$W_t = 0.165 \pi^2 S^2. \quad \dots(17)$$

In case of an isotropic solid like copper, the energy of the crack is given by

$$(W_t)_{\text{copper}} = 0.186 \pi^2 S^2 \text{ and } (W_t)_{\text{steel}} = 0.102 \pi^2 S^2, \quad \dots(18)$$

where

$$(\mu)_{\text{copper}} = 0.447 \text{ and } (\mu)_{\text{steel}} = 0.819.$$

So we conclude that the energy of the crack in torsion in a transversely isotropic material is less than that in an isotropic solid when the rigidity modulus  $\mu < 0.5$ . If  $\mu > 0.5$ , the energy of the crack in torsion is less than that in a transversely isotropic material.

We now proceed to determine the distribution of stress at a general point of the solid. From the relations (8) and (10), we obtain

$$\sigma_{\theta z}(r, z)/S = 1 - \int_0^{\infty} \left[ \frac{1 - \cos \alpha}{\alpha} - \frac{\sin \alpha}{2} \right] \exp(-\alpha z_1) J_1(\alpha r) d\alpha \quad \dots(19)$$

where

$$z_1 = z/k.$$

Adopting the notations of George<sup>3</sup> namely,

$$S(m, n) = \int_0^{\infty} \alpha^n \sin \alpha J_m(\alpha r) (\exp(-\alpha z_1)) d\alpha \quad \dots(20)$$

$$C(m, n) = \int_0^{\infty} \alpha^{n-1} (1 - \cos \alpha) J_m(\alpha r) (\exp(\alpha z_1) d\alpha \quad \dots(21)$$

the value of the stress in (19) reduces to

$$\sigma_{\theta z}(r, z)/S = 1 - \left[ C(1, 0) - \frac{1}{2} S(1, 0) \right] \quad \dots(22)$$

Similarly from the relation (7) the component of displacement can be expressed as

$$u_{\theta}(r, z)/S = \frac{1}{C_{44}} \left[ z + k \int_0^{\infty} \left\{ \frac{1 - \cos \alpha}{\alpha^2} - \frac{\sin \alpha}{2\alpha} \right\} \times (\exp(-\alpha z_1/k) J_1(\alpha r) dx) \right]$$

So 
$$u_{\theta}(r, 0)/S = \frac{k}{2C_{44}} r \log \frac{1 + \sqrt{1 - r^2}}{r}, \quad 0 < r < 1 \quad \dots(23)$$

The deformation of the crack is obtained from the relation (23). The expression for  $u_{\theta}(r, 0)/S$  has been calculated for different values of  $r$  and the results are shown in Fig. 1.

To obtain the distribution of stress  $\sigma_{\theta z}(r, z)/S$  in the interior of the medium, we calculate the expression given in relation (22) by using George's  $s^3$  integrated values for different values of  $r$  and  $z$ . The variation of the stress distribution in the

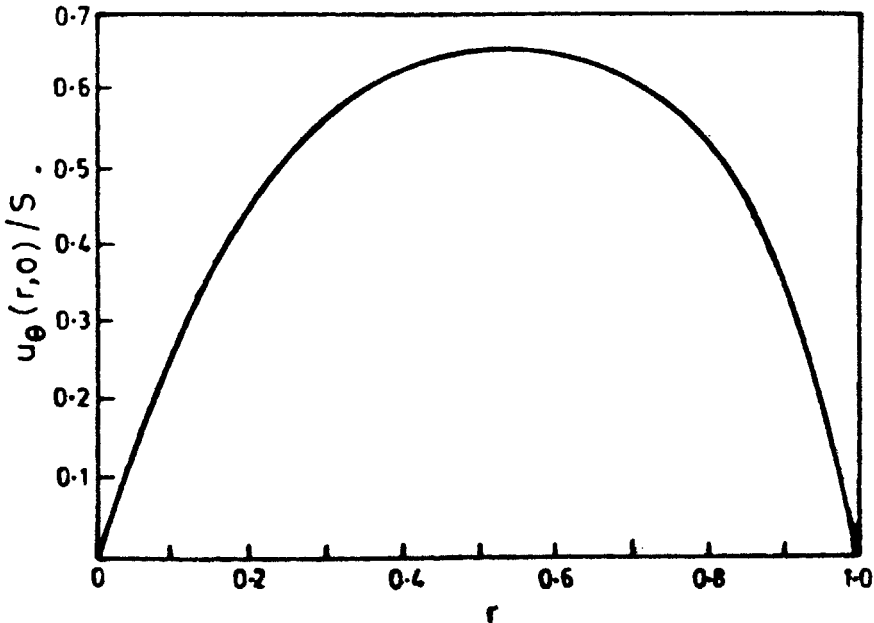


FIG 1 Variation of  $u_{\theta}(r, 0)/S$  for different values of  $r$ .

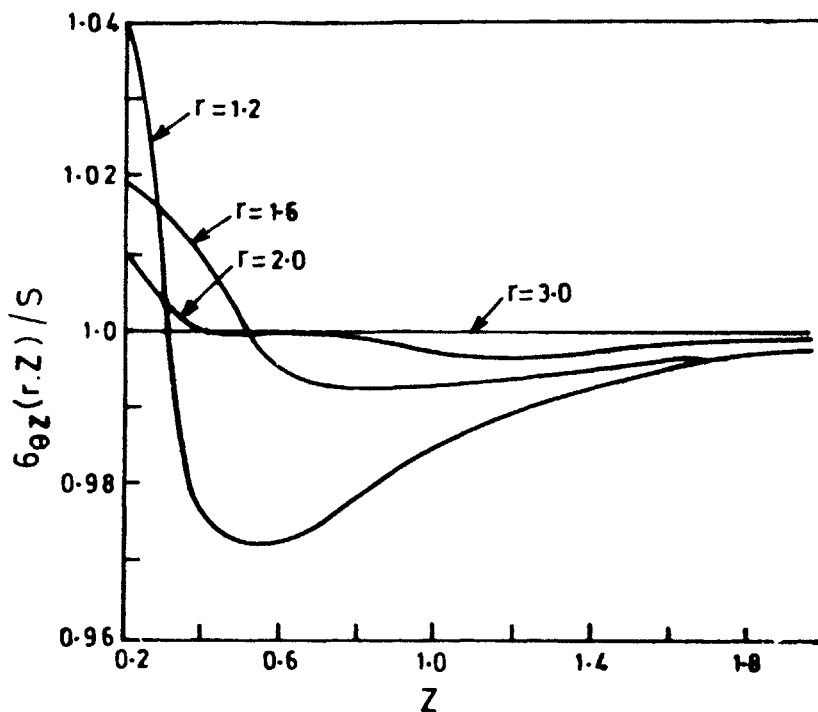


Fig 2 Variation of  $\sigma_{\theta z}(r, z)/S$  for different values of  $r$  and  $z$ .

planes  $r = 1.2$ ,  $r = 1.6$ ,  $r = 2.0$  and  $r = 3.0$  are shown in Fig. 2. The results have been calculated by using a computer.

So from the Fig. 2 it is obvious that  $\sigma_{\theta z} \sim S$  as  $\sqrt{r^2 + z^2} \rightarrow \infty$  as proposed in relation (1).

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