Proc. Indian natn. Sci. Acad, 54, A, No. 1, 1988, pp. 121-137.

NONLINEAR PARAMETRIC INTERACTIONS IN A PIEZOELECTRIC-SEMICONDUCTING MEDIUM WITH FIELD-DEPENDENT MOBILITY

M GANGULY *and* A K PAL

Department of Mathematics, Jadavpur University, Calcutta-700032, India

(*Received 23 March 1987; Accepted 8 July 1987*)

The present paper seeks to investigate the nonlinear parametric interaction of acoustic waves propagating through a piezoelectric semiconducting medium, characterized by a field-dependent mobility and subjected to a dc electric field. The effect of a random time-dependent elastic stiffness is also reckoned here. Assuming the drift velocity of the carriers to be exactly equal to the sound velocity and the interaction to be of degenerate type, several statistics of the gain/loss of the signal wave due to pump wave have been derived.

Key Words : Nonlinear Parametric Interactions; Piezoelectric-Semiconducting Medium; Field-dependent Mobility; Velocity-Drift and Sound; Pump Wave; Voight's Model

INTRODUCTION

RECENT years have witnessed a number of studies in nonlinear acoustics arising mainly due to space charge nonlinearity with propagation of waves in piezoelectic semiconductors, vide, Rudenko and Soluyan,[1] Carleton[2] etc.

However, one can accommodate such nonlinear effects by taking recourse to methods such as "parametric interaction." The pioneer in this direction is Shiren.[3] After his pioneering contribution a number of researchers have considered the problem of parametric interactions from various standpoints, vide, Pal and Gupta[4] Sinha *et al.*,[5] Spector[6] etc. But all these problems are of deterministic type.

There exists another class of problems called stochastic problems for which the initial/boundary conditions and/or external forces regulating the motion may not be definitely known but only their probabilities of occurrence are known, vide, Paria,[7] Ziegler[8] etc.

The object of the present paper is to study the case of a degenerate parametric interaction of waves in a piezoelectric-semiconducting medium which is subjected to a random time-dependent elastic stiffness as well as acted upon by a field-dependent mobility, vide, Wintle[9] and Aldert Van der Ziel.[10] As is customary, the whole analysis is based upon the assumption of equality of the drift velocity of electrons and the velocity of acoustic waves. In the present study attempts have been made to obtain some simple statistics such as mean square displacement field, mean square strain amplitude, mean energy component and mean fractional energy change of the signal wave on account of a pump wave.

Statement of the Problem and Basic Equations

Let us consider a weak signal wave and a strong pump wave propagating through a piezoelectric-semiconducting medium whose elastic stiffnesses are time-dependent as in Voight's viscoelastic model and are random in nature. The mobility constant of the electrons are assumed to be field-dependent and the medium is subjected to a dc electric field. We take the drift velocity of the carriers to be exactly equal to the sound velocity and the collision frequency to be large compared to the frequency of wave motion as is customary, vide, White[11] and Seeger.[12] Moreover, we assume the interaction to be of degenerate type so that the pump frequency is equal to twice the signal wave frequency and the direction of propagation of waves is taken along the x-axis.

The constitutive equations of the piezoelectric medium are given by

$$T_{ij} = c_{ijkl}\, s_{kl} - e_{mij}\, E_m \qquad \ldots(1)$$

and

$$D_m = e_{mkl}\, s_{kl} + \epsilon_{mn}\, E_n \qquad \ldots(2)$$

where T_{ij}, S_{kl}, D_m and E_m are the components of stress tensor, strain tensor, electric displacement and electric field, respectively. c_{ijkl}, e_{mij}, ϵ_{mn} are the components of the elastic stiffness tensor, piezoelectric and dielectric constant tensors.

In the present problem, since we are interested in the propagation of a transverse wave only along the x-direction, the constitutive equations can be taken in the following simplified form :

$$T = cs - eE \qquad \ldots(3)$$

and

$$D = es + \epsilon E. \qquad \ldots(4)$$

The symbols used in the above two equations represent the quantities mentioned before. The other relevent equations in one dimensional form are the following :

The Poission equation is given by

$$\frac{\partial D}{\partial x} = - q_{n_s} \qquad \ldots(5)$$

The continuity equation of charge flow is the following

$$\frac{\partial J}{\partial x} = q\frac{\partial n_s}{\partial t} \qquad \ldots(6)$$

The equation for the current density J is given by

$$J = \mu q_{n_c} E + qD_n \frac{\partial n_c}{\partial x} \qquad \ldots(7)$$

vide, Seeger[11] and

$$n_e = n_0 + n_s \qquad \ldots(8)$$

where n_e is the number of electron in the conduction band, n_0 the equilibrium number of electrons producing electrical neutrality with no ultrasonic wave present and q the electron charge, μ and D_n are respectively the mobility constant and electron diffusion constant.

We take the electric field E in the following form

$$E = E_0 + \tilde{E}, \qquad \ldots(9)$$

where E_0 is the applied electric field and is such that the drift velocity $v_d(= -\mu E_0)$ is equal to sound velocity and $\tilde{E}$ is the perturbed electric field.

The equation of mechanical motion is given by

$$\frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots(10)$$

where ρ is the material density and μ the mechanical displacement.

In semi-conductors, the mobility of a carrier (free electron) depends on its kinetic energy in a manner which depends on the scattering mechanism. In a low mobility material i.e., insulators, the mobility is governed by mechanisms other than scattering, but it is possible to envisage that as the carrier concentration increases, some trapping sites may become permanently occupied and no longer contribute any delay to the moving carriers. As a result, the average mobility could conceivably vary with the carrier concentration.

Hence we can assume the mobility constant μ in the following form

$$\mu = \mu_0 \frac{\partial E}{\partial x}, \qquad \ldots(11)$$

where μ_0 is the constant of proporsionality, vide, Wintle.[9]

The diffusion constant D_n given by Einstein's relation is the following

$$D_n = \frac{\mu K_B T}{q} \qquad \ldots(12)$$

Moreover, we assume the elastic stiffness constant c to be time-dependent as in Voight's model of viscoelasticity.

$$c = c_0 + c_1(\bar{\alpha}) \frac{\partial}{\partial t} \qquad \ldots(13)$$

where the term c_1 indicating the viscous effect is taken to be a function of the random parameter $\bar{\alpha}$ defined over a space S over which a probability density $p(\bar{\alpha})$ is defined.

Using (3), (9), (10) and (13) we find

$$c_0 \frac{\partial^2 u}{\partial x^2} + c_1(\bar{\alpha}) \frac{\partial^3 u}{\partial x^2 \partial t} - e \frac{\partial \widetilde{E}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad ...(14)$$

Again using (4), (5), (6), (7), (8), (9), (11) and (12) we find

$$\epsilon \frac{\partial^2 \widetilde{E}}{\partial x \partial t} + e \frac{\partial^3 u}{\partial x^2 \partial t} = e\mu_0 \left(\frac{\partial \widetilde{E}}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} + e\mu_0$$

$$\times (E_0 + \widetilde{E}) \left\{\frac{\partial^2 \widetilde{E}}{\partial x^2} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial \widetilde{E}}{\partial x} \cdot \frac{\partial^3 u}{\partial x^3}\right\}$$

$$- \sigma \left(\frac{\partial \widetilde{E}}{\partial x}\right)^2 - \sigma(E_0 + \widetilde{E}) \frac{\partial^2 \widetilde{E}}{\partial x^2} + \epsilon\mu_0 \left(\frac{\partial \widetilde{E}}{\partial x}\right)^3$$

$$+ 2\epsilon\mu_0(E + \widetilde{E}) \cdot \frac{\partial \widetilde{E}}{\partial x} \frac{\partial^2 \widetilde{E}}{\partial x^2} + \frac{e k_B T}{q} \frac{\partial^4 u}{\partial x^4}$$

$$+ \frac{\epsilon k_B T}{q} \frac{\partial^3 \widetilde{E}}{\partial x^3}, \qquad ...(15)$$

where

$$\sigma = \mu_0 q n_0.$$

Equations (14) and (15) constitute the basic equations of the problem.

Solution of the Problem

As in Pal and Gupta,[4] Sinha *et al.*[5] etc. we treat the interaction of waves by taking the displacement u and the perturbed electric field $\widetilde{E}$ in the following form.

$$u = u_1(x) \exp (i(\omega_1 t - k_1 x)) + u_3(x) \exp (i(\omega_3 t - k_3 x)) + c \cdot c \qquad ...(16)$$

and

$$\widetilde{E} = E_1(x) \exp (i(\omega_1 t - k_1 x)) + E_3(x) \exp (i(\omega_3 t - k_3 x)) + c \cdot c \qquad ...(17)$$

where $c \cdot c$ is the complex conjugate.

The frequencies and wave numbers ω_3, k_3 of the pump wave and ω_1, k_1 of the signal wave are connected by the following relation

$$\omega_3 = 2\omega_1 \qquad \text{and} \qquad k_3 = 2k_1 \qquad ...(18)$$

Since the amplitudes vary slowly with x we assume as in Shiren[3]

$$\frac{\partial^2 u}{\partial x^2} \ll k \frac{\partial u}{\partial x} \ll k^2 u \qquad ...(19)$$

Substituting (16) and (17) in the basic equations (14) and (15) and then equating terms of like frequencies subject to the above approximation (18) and (19) we find the following four equations

$$\left(1 + i\omega_1 \frac{c_1(\bar{\alpha})}{c_0}\right) \frac{\partial u_1}{\partial x} + \left(\frac{k_1}{2}\, \omega_1 \frac{c_1(\bar{\alpha})}{c_0)}\right) u_1^* = \left(\frac{e}{2c_0}\right) E_1 + \left(\frac{ie}{2c_0 k_1}\right) \frac{\partial E_1}{\partial x} \quad \text{...(20)}$$

$$\left(1 + i\omega_3 \frac{c_1(\bar{\alpha})}{c_0}\right) \frac{\partial u_3}{\partial x} + \left(\frac{k_3}{2}\, \omega_3 \frac{c_1(\bar{\alpha})}{c_0}\right) u_3 = \left(\frac{e}{2c_0}\right) E_3 + \left(\frac{ie}{2c_0 k_3}\right) \frac{\partial E_3}{\partial x} \quad \text{...(21)}$$

$$L_1 \frac{\partial u_1}{\partial x} + M_1 u_1 + N_1 \frac{\partial E_1}{\partial x} + P_1 E_1 = Q_1 u_1^* E_3 + R_1 u_3 E_1^* + S_1 E_1^* E_3 \quad \text{...(22)}$$

and

$$L_3 \frac{\partial u_3}{\partial x} + M_3 u_3 + N_3 \frac{\partial E_3}{\partial x} + P_3 E_3 = 0, \quad \text{...(23)}$$

where

$$L_1 = \left(2e\omega_1 k_1 - \frac{4iek_1 k_B T}{q}\right)$$

$$M_1 = \left(- i\omega_1 k_1^2 e - \frac{ek_B T}{q} k_1^4\right)$$

$$N_1 = \left(i\epsilon\omega_1 + \frac{3k_1^2 \epsilon k_B T}{q} - 2ik_1 \sigma E_0\right)$$

$$P_1 = \left(\epsilon\omega_1 k_1 - \frac{ik_1^3 \epsilon k_B T}{q} - \sigma E_0 k_1^2\right)$$

$$Q_1 = (2ek_1^4 \mu_0 E_0)$$

$$R_1 = (- 4ek_1^4 \mu_0 E_0)$$

$$S_1 = 3\sigma k_1^2 - 4ik_1^3 \epsilon\mu_0 E_0)$$

$$L_3 = \left(2e\omega_3 k_3 - \frac{4ek_B T}{q} k_3\right)$$

$$M_3 = \left(- ei\omega_3 k_3^2 - \frac{ek_B T}{q} k_3^4\right)$$

$$N_3 = \left(\frac{2\epsilon k_B T}{q} k_3^2 + i\epsilon k_3 - 2i\sigma E_0 k_3\right)$$

and

$$P_3 = \left(- \frac{ek_B T}{q} k_3^4 - ei\omega_3 k_3^3\right) \quad \text{...(24)}$$

Since the drift velocity is exactly equal to the sound velocity and since there is no linear gain/loss when the drift velocity is equal to the sound velocity, vide, White[11] we can take the amplitude of the pump wave $u_3(x)$ as

$$u_3(x) = u_3(0)\, e^{i\gamma x} \qquad \ldots(25)$$

then γ is given by

$$\gamma = \frac{k_3}{2} k^2 \frac{\dfrac{\omega_3}{\omega_D}}{\left(\dfrac{\omega_c}{\omega_3} + \dfrac{\omega_3}{\omega_D}\right)} \qquad \ldots(26)$$

where

$$\omega_c = \frac{\sigma}{\epsilon}, \; \omega_D = \frac{us^2}{D_n} \text{ and } K^2 \simeq \frac{e^2}{c\epsilon}$$

Using (21) we can express E_3 in terms of $u_3(0)$ as follows

$$E_3 = \frac{2c_0k_3}{e(k_3-\gamma)}\left\{i\gamma + \frac{\omega_3}{2}\cdot\frac{c_1(\bar{\alpha})}{c_0}(k_3 - 2\gamma)\right\}\exp(i\gamma x)\cdot u_3(0) \qquad \ldots(27)$$

Using (25) and (27) in (22) and then substituting

$$u_1 = \bar{u}_1 \exp\left(\frac{i\gamma x}{2}\right)$$

and

$$E_1 = \bar{E}_1 \exp\left(\frac{i\gamma x}{2}\right)$$

we obtain

$$(L_1' + iL_1'')\frac{\partial \bar{u}_1}{\partial x} + (M_1' + iM_1'')\bar{u}_1 + (N_1' + iN_1'')\frac{\partial \bar{E}_1}{\partial x} + (P_1' + iP_1'')\bar{E}_1$$

$$= \Bigg[(Q_1' + iQ_1'')\frac{2c_0k_3}{e(k_3-\gamma)}\left\{i\gamma + \frac{\omega_3}{2}\cdot\frac{c_1(\bar{\alpha})}{c_0}(k_3 - 2)\right\}\bar{u}_1^* + (R_1' + iR_1'')\bar{E}_1^*$$

$$+ (s_1' + is_1'')\frac{2c_0k_3}{e(k_3-\gamma)}\left\{i\gamma + \frac{\omega_3}{2}\cdot\frac{c_1(\bar{\alpha})}{c_0}(k_3 - 2\gamma)\right\}\bar{E}_1^*\Bigg]u_3(0) \qquad \ldots(28)$$

where

$$L_1' = 2e\omega_1k_1 \qquad L_1'' = -\frac{4k_1ek_BT}{q}$$

$$M_1' = \frac{ek_BTk_1(2\gamma - k_1^2)}{q} \qquad M_1'' = e\omega_1k_1(\gamma - k_1)$$

$$N_1' = \frac{3k_1^2\epsilon k_BT}{q} \qquad N_1'' = (\epsilon\omega_1 - 2k_1\sigma E_0)$$

$$P_1' = \left[\epsilon\omega_1\left(k_1 - \frac{\gamma}{2}\right) - \sigma E_0 k_1(k_1 - \gamma)\right],$$

$$P_1'' = \left[\frac{k_1^2 k_B T}{q}\left(\frac{3\gamma}{2} - k_1\epsilon\right)\right]$$

$$Q_1' = 2e\mu_0 E_0 k_1^4 \qquad Q_1'' = 0$$

$$R_1' = -\,4k_1^4 e\mu_0 E_0 \qquad R_1'' = 0$$

$$S_1' = 3k_1^2\sigma \quad \text{and} \quad S_1'' = -\,4\epsilon\mu_0 E_0 k_1^3 \qquad \ldots(29)$$

Similarly substituting

$$u_1 = \bar{u}_1 \exp\left(\frac{i\gamma x}{2}\right)$$

and

$$E_1 = \bar{E}_1 \exp\left(\frac{i\gamma x}{2}\right)$$

in (20), we find

$$\left(1 + i\omega_1\frac{c_1(\bar{\alpha})}{c_0}\right)\frac{\partial\bar{u}_1}{\partial x} + \bar{u}_1\left\{\frac{i\gamma}{2} - \frac{\omega_1\gamma c_1(\bar{\alpha})}{2c_0} + \frac{k_1 c_1(\bar{\alpha})\,\omega_1}{2c_0}\right\}$$

$$= \left(\frac{e}{2c_0} - \frac{e\gamma}{4c_0 k_1}\right)\bar{E}_1 + \frac{ie}{2c_0 k_1}\frac{\partial\bar{E}_1}{\partial x} \qquad \ldots(30)$$

Now putting

$$\bar{u}_1 = \alpha + i\beta$$

and

$$\bar{E}_1 = \gamma + i\delta$$

in equations (28) and (30) and then equating the real and imaginary parts, we find the following four equations in α, β, γ and δ :

$$f_1(D)\,\alpha - f_2(D)\,\beta + f_3(D)\,\gamma - f_4(D)\,\delta = 0 \qquad \ldots(31)$$

$$\bar{f}_1(D)\,\alpha + \bar{f}_2(D)\,\beta + \bar{f}_3(D)\,\gamma + \bar{f}_4(D)\,\delta = 0 \qquad \ldots(32)$$

$$D\alpha + (k_1 - \gamma)\,\frac{c_1(\bar{\alpha})\,\omega_1}{2c_0}\,\alpha - \beta\,\frac{\gamma}{2} - \left(\omega_1\frac{c_1(\bar{\alpha})}{c_0}\right)D\beta$$

$$+\left(\frac{e\gamma}{4c_0 k_1} - \frac{e}{2c_0}\right)\gamma + \left(\frac{e}{2c_0 k_1}\right)D\delta = 0 \qquad \ldots(33)$$

and

$$\alpha\,\frac{\gamma}{2} + \frac{\omega_1 c_1(\bar{\alpha})}{c_0}\,D\alpha + D\beta + \beta\left\{\frac{\omega_1 c_1(\bar{\alpha})}{2c_0}(k_1 - \gamma)\right\}$$

$$-\frac{e}{2c_0 k_1}\,D\gamma + \left(\frac{e\gamma}{4c_0 k_1} - \frac{e}{2c_0}\right)\delta = 0, \qquad \ldots(34)$$

where

$$f_1(D) = L'_1 \frac{\partial}{\partial x} + M'_1 - \frac{2c_0k_3}{e(k_3-\gamma)} u_3(0) \frac{\omega_3}{2} (k_3 - 2\gamma) \; Q'_1 \frac{c_1(\bar{\alpha})}{c_0}$$

$$f_2(D) = L''_1 \frac{\partial}{\partial x} + M''_1 + \frac{2u_3(0)\; c_0k_3}{e(k_3-\gamma)} Q'_1 \gamma$$

$$f_3(D) = N'_1 \frac{\partial}{\partial x} + P'_1 - S'_1 \frac{\omega_3}{2} \frac{c_1(\bar{\alpha})}{c_0} (k_3 - 2\gamma)\, u_3(0)$$

$$\times \frac{2c_0k_3}{e(k_3-\gamma)} + S'_1 \gamma u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)} - R'_1 u_3(0)$$

$$f_4(D) = N''_1 \frac{\partial}{\partial x} + P''_1 + S''_1 \frac{\omega_3}{2} \cdot \frac{c_1(\bar{\alpha})}{c_0} (k_3 - 2\gamma)$$

$$\times\, u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)} + S'_1 \gamma u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)}$$

$$\bar{f}_1(D) = L''_1 \frac{\partial}{\partial x} + M''_1 - u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)} Q'_1 \gamma$$

$$\bar{f}_2(D) = L'_1 \frac{\partial}{\partial x} + M'_1 + u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)} Q'_1 \frac{\omega_3}{2} \frac{c_1(\bar{\alpha})}{c_0} (k_3 - 2\gamma)$$

$$\bar{f}_3(D) = N''_1 \frac{\partial}{\partial x} + P''_1 - S'_1 \gamma u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)} - S''_1 \frac{\omega_3}{2} \cdot \frac{c_1(\bar{\alpha})}{c_0}$$

$$\times\, (k_3 - 2\gamma)\, u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)}$$

$$\bar{f}_4(D) = N'_1 \frac{\partial}{\partial x} + P'_1 + S'_1 \frac{\omega_3}{2} \frac{c_1(\bar{\alpha})}{c_0} (k_3 - 2\gamma)\, u_3(0)$$

$$\times \frac{2c_0k_3}{e(k_3-\gamma)} - S''_1 \gamma u_3(0) \frac{2c_0k_3}{e(k_3-\gamma)} + R'_1 u_3(0) \qquad \ldots(35)$$

Eliminating γ and δ from equations (31) to (32) we find the following equation in α or β

$$[F(D)\, \bar{\varphi}(D) + \varphi(D)\, \bar{F}(D)]\, [\alpha, \beta] = 0, \qquad \ldots(36),$$

where

$$F(D) = \left\{ \frac{e}{2c_0k_1} [f_3(D)\, \bar{f}_4(D) + \bar{f}_3(D)\, f_4(D)] \right.$$

$$\left. + \frac{e^2}{4c_0^2 k_1^2} [\bar{f}_3(D)\, f_1(D) - f_3(D)\, \bar{f}_1(D)] \right\} D^2$$

$$- \frac{\omega_1 c_1(\bar{\alpha})}{2c_0} \cdot \frac{e}{c_0} D \{f_3(D)\, \bar{f}_4(D) + \bar{f}_3(D)\, f_4(D)\}$$

$$+\frac{\gamma}{2}\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)\{f_3(D)\bar{f}_4(D)+\bar{f}_3(D)f_4(D)\}$$

$$+\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)^2\{\bar{f}_3(D)f_1(D)-f_3(D)\bar{f}_1(D)\}$$

$$\varphi(D)=\left[\frac{\omega_1C_1(\bar{\alpha})}{c_0}\cdot\frac{e}{2c_0k_1}\{\bar{f}_3(D)f_4(D)+f_3(D)\bar{f}_4(D)\}\right.$$

$$\left.+\frac{e^2}{4c_0^2k_1^2}\{f_3(D)\bar{f}_2(D)+\bar{f}_3(D)f_2(D)\}\right]D^2+\frac{e}{2c_0}$$

$$\times D\{\bar{f}_3(D)f_4(D)+f_3(D)\bar{f}_4(D)\}+\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)^2$$

$$\times\{\bar{f}_3(D)f_2(D)+f_3(D)\bar{f}_2(D)\}-\left\{\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)\right.$$

$$\left.\times\frac{\omega_1c_1(\bar{\alpha})}{2c_0}(k_1-\gamma)\right\}\cdot\{f_3(D)\bar{f}_4(D)+\bar{f}_3(D)f_4(D)\}$$

$$\bar{F}(D)=\left[-\frac{\omega_1c_1(\bar{\alpha})}{e_0}\cdot\frac{e}{2c_0k_1}\{f_3(D)\bar{f}_4(D)+\bar{f}_3(D)f_4(D)\}\right.$$

$$\left.-\frac{e^2}{4c_0^2k_1^2}(\bar{f}_4(D)f_1(D)+f_4(D)\bar{f}_1(D))\right]D^2-\frac{e}{2c_0}$$

$$\times\{f_3(D)\bar{f}_4(D)+\bar{f}_3(D)f_4(D)\}D-(k_1-\gamma)$$

$$\times\frac{c_1(\bar{\alpha})}{2c_0}\omega_1\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)\{\bar{f}_3(D)f_4(D)+f_3(D)\bar{f}_4(D)\}$$

$$-\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)^2\{\bar{f}_4(D)f_1(D)+f_4(D)\bar{f}_1(D)\}$$

$$\bar{\varphi}(D)=\left[-\frac{e}{2c_0k_1}\{\bar{f}_3(D)f_4(D)+f_3(D)\bar{f}_4(D)\}\right.$$

$$\left.+\frac{e^2}{4c_0^2k_1^2}\{\bar{f}_4(D)f_2(D)-f_4(D)\bar{f}_2(D)\}\right]D^2$$

$$-\left\{\frac{\omega_1c_1(\bar{\alpha})}{c_0}\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)+\frac{\omega_1c_1(\bar{\alpha})}{c_0}(k_1-\gamma)\frac{e}{2c_0k_1}\right\}$$

$$\times D\{\bar{f}_3(D)f_4(D)+f_3(D)\bar{f}_4(D)\}+\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)^2$$

$$\times\{\bar{f}_4(D)f_2(D)-f_4(D)\bar{f}_2(D)\}-\frac{\gamma}{2}\left(\frac{e\gamma}{4c_0k_1}-\frac{e}{2c_0}\right)$$

$$\times\{\bar{f}_3(D)f_4(D)+f_3(D)\bar{f}_4(D)\}\qquad\ldots(37)$$

Using (37), (35) and (29) we get from (36) the following differential equation in α or β

$$(\lambda^* D^2 + \mu^* D)(\alpha, \beta) = 0, \qquad \ldots(38)$$

where

$$\lambda^* = \lambda_1 + \lambda_2 (c_1(\bar{\alpha}))^2 \qquad \ldots(39)$$

and

$$\mu^* = \mu_1 + \mu_2 (c_1(\bar{\alpha})) \qquad \ldots(40)$$

neglecting all higher order terms in $D\left(= \frac{\partial}{\partial x} \right)$ since the amplitudes vary slowly with x. Moreover, while simplifying the differential equation (36) to the above form (38) we have retained only the dominant terms in λ_1, λ_2, μ_1 and μ_2 subject to the approximations that

$$\left(\gamma, \frac{e^2}{c_0^2}, u_3^2(0), \epsilon u_3(0), \epsilon^2, \left(\frac{K_B T}{q}\right)^2, e^2 u_3(0), \frac{K_B T}{q} u_3(0), \frac{K_B T}{q} \epsilon, \frac{K_B T}{q} \gamma, \gamma\epsilon \right)$$

are small quantities and therefore their products and other higher order terms can be neglected,

$$\begin{aligned} \lambda_1 = \Bigg[& \frac{e^8 \epsilon \omega_1^2 E_0^2 k_1^6 k_3^4 \sigma(\sigma - 1)}{8 c_0^3} + \frac{e^6 \sigma^3 E_0^3 k_1^7 \epsilon \omega_1 k_3^3 (1 + k_3)}{2 c_0^2} \\ & - \frac{\sigma^2 E_0^2 k_1^6 \epsilon \omega_1^2 e^8 k_3^4}{2 c_0^3} - \frac{e^8 \sigma^2 E_0^2 \epsilon \omega_1^2 k_1^5 k_3^2 (k_1 + k_3)}{8 c_0^3} \\ & + \frac{e^8 \sigma^3 E_0^3 k_3^4 k_1^6 (k_1 + 1)}{8 c_0^3} + \frac{e^8 \sigma^3 k_1^7 k_3^4 \omega_1 \sigma^2 (1 - \sigma)}{4 c_0^3} \\ & + \frac{\omega_1^3 \sigma E_0 k_1^5 k_3^4 e^8 \epsilon (1 + 2\epsilon\omega_1)}{8 c_0^3} - \frac{e^6 \sigma^4 E_0^4 k_1^8 k_3^4}{4 c_0^3} \Bigg] \end{aligned} \qquad \ldots(41)$$

$$\lambda_2 = -\frac{2 \omega_1 k_3^5 e^6 \sigma^2 E_0^2 k_1^9 \mu_0 \omega_3 u_3(0)}{c_0^2} \qquad \ldots(42)$$

$$\mu_1 = -\frac{e^8 \sigma^3 E_0^3 k_1^6 k_3^4 k_B T}{8 q c_0^3} \qquad \ldots(43)$$

$$\begin{aligned} \mu_2 = \Bigg[& -\frac{e^6 \gamma \sigma^4 k_1^9 k_3^5 \mu_0 u_3(0)\, \omega_3}{c_0} - \frac{\sigma^3 E_0^3 k_1^9 e^8 k_3^5 \omega_1 \mu_0 \omega_3 u_3(0)}{c_0^2} \\ & - \frac{e^6 \sigma^4 E_0^4 k_1^9 k_3^4 \omega_1}{2 c_0^3} + \frac{e^6 \sigma^3 E_0^3 k_1^8 k_3^4 \epsilon \omega_1^2}{c_0^3} \Bigg] \end{aligned} \qquad \ldots(44)$$

Solving the second order differential equation (38) we get

$$\alpha = \bar{c}_1 + \bar{c}_2 \exp\left(-\frac{\mu^*}{\lambda^*} x \right) \qquad \ldots(45)$$

and

$$\beta = \bar{D}_1 + \bar{D}_2 \exp\left(-\frac{\mu^*}{\lambda^*} x\right) \qquad ...(46)$$

Using (45) and (46) we find $\bar{u}_1$ $(= \alpha + i\beta)$ to be the following

$$\bar{u}_1 = (\bar{c}_1 + i\bar{D}_1) + (\bar{c}_2 + i\bar{D}_2) \exp\left(-\frac{\mu^*}{\lambda^*} x\right) \qquad ...(47)$$

Thus we find $u_1 \left(= \bar{u}_1 \exp\left(\frac{i\gamma x}{2}\right)\right)$ in the following form

$$u_1 = (\bar{c}_1 + i\bar{D}_1) \exp\left(\frac{i\gamma x}{2}\right) + (\bar{c}_2 + \bar{D}_2) \exp\left(\left(\frac{i\gamma}{2} - \frac{\mu^*}{\lambda^*}\right) x\right) \qquad ...(48)$$

The values of $\bar{c}_1, \bar{c}_2, \bar{D}_1, \bar{D}_2$ in (48) are obtained from the following two suitable boundary conditions

$$u_1(x)\Big|_{x=0} = u_1(0) \qquad ...(49)$$

and

$$\frac{\partial u_1(x)}{\partial x}\Big|_{x=0} = i\gamma' u_1(0) \qquad ...(50)$$

where in the conditions (50) we have assumed the contribution of non-linearity to $\frac{\partial u_1}{\partial x}\Big|_{x=0}$ to be negligibly small as in Sinha *et al.*[5] The constant γ' is given by the following

$$\gamma' = \frac{k_1}{2} k^2 \frac{\frac{\omega_1}{\omega_D}}{\left(\frac{\omega_c}{\omega_1} + \frac{\omega_1}{\omega_D}\right)}$$

Using the two boundary conditions given by equations (49) and (50) we obtain $u_1(x)$ as follows :

$$u_1(x) = \left\{\left[u_1(0) - \frac{i\lambda^*}{\mu^*} u_1(0)\left(\frac{\gamma}{2} - \gamma'\right)\right] + \frac{\lambda}{\mu} iu_1(0)\left(\frac{\gamma}{2} - \gamma'\right) \exp\left(-\frac{\mu^*}{\lambda^*} x\right\}\right) \exp\left(\frac{i\gamma x}{2}\right) \qquad ...(51)$$

Now the strain amplitude $e\omega_1$ of the frequency ω_1 is given by, vide, Shiren[3]

$$e\omega_1 = 2k_1 \mid u_1(x) \mid$$

which in the present case becomes

$$e\omega_1 = 2k_1 u_1(0) \left\{1 + \frac{\lambda^{*2}}{\mu^{*2}}\left(\frac{\gamma}{2} - \gamma'\right)^2 \left(\exp\left(-\frac{\mu^*}{\lambda^*} x\right) - 1\right)^2\right\}^{1/2} \qquad ...(52)$$

This leads to the energy $E\omega_1$ of the component at ω_1 given by

$$E\omega_1 = \frac{1}{2}\rho u_s^2 e^2\omega_1$$

$$= 2\rho\omega_1^2 u_1^2(0)\left\{1 + \frac{\lambda^{*2}}{\lambda^{*2}}\left(\frac{\Upsilon}{2} - \Upsilon'\right)^2\left(\exp\left(-\frac{\mu^*}{\lambda^*}x\right) - 1\right)^2\right\} \quad \ldots(53)$$

vide Shiren[3]

This enables us to obtain the fractional change in energy over the propagation path x which is the determining factor in energy gain/loss and is given by

$$\frac{\delta E_1(x)}{E_1(0)} = \frac{\lambda^{*2}}{\mu^{*2}}\left(\frac{\Upsilon}{2} - \Upsilon'\right)^2\left(\exp\left(-\frac{\mu^*}{\lambda^*}x\right) - 1\right)^2 \quad \ldots(54)$$

From (54) we can conclude that there is always a gain of energy of the signal wave for all propagation path x. Moreover, the gain increases with increasing values of x and the maximum value of the gain is given by

$$\frac{\lambda^{*2}}{\mu^{*2}}\left(\frac{\Upsilon}{2} - \Upsilon'\right)^2$$

Statistics of the Problem

Since the displacement field, strain amplitude, energy component and fractional change in energy etc. given by equations (51), (52), (53) and (54) involve λ^*/μ^*, which in turn contains $c_1(\bar{\alpha})$ they all become random. So we calculate the mean-square displacement field, mean-square strain amplitude, mean energy component and mean fractional energy change by evaluating the following integrals

$$\langle \mid u_1(x)\mid^2 \rangle = \int_S \mid u_1(x)\mid^2 p(\bar{\alpha})\, d\bar{\alpha} \quad \ldots(55)$$

$$\langle e^2\omega_1 \rangle = 4k_1^2 \langle \mid u_1(x)\mid^2 \rangle = 4k_1^2 \int_S \mid u_1(x)\mid^2 p(\bar{\alpha})\, d\bar{\alpha} \quad \ldots(56)$$

$$\langle E\omega_1 \rangle = 2\rho\omega_1^2 \langle \mid u_1(x)\mid^2 \rangle$$

$$= 2\rho\omega_1^2 \int_S \mid u_1(x)\mid^2 p(\bar{\alpha})\, d\bar{\alpha} \quad \ldots(57)$$

$$\left\langle \frac{\delta E_1(x)}{E_1(0)} \right\rangle = \int_S \frac{\delta E_1(x)}{E_1(0)}\, p(\bar{\alpha})\, d\bar{\alpha} \quad \ldots(58)$$

For simplicity, we assume the random function

$$c_1(\bar{\alpha}) = c_1^* + \bar{\alpha}c_1^{**} \quad \ldots(59)$$

where c_1^*, c_1^{**} are constants and the corresponding probability distribution

$$p(\bar{\alpha}) = \begin{cases} \frac{1}{2}(\bar{\alpha} + 1) & \text{for } -1 < \bar{\alpha} < 1 \\ 0 & \text{otherwise.} \end{cases} \quad \ldots(60)$$

Therefore,

$$\int_S p(\bar{\alpha})\, d\bar{\alpha} = 1.$$

Using (39), (40) and (41) we obtain from equation (55)

$$\langle \mid u_1(x) \mid^2 \rangle = u_1^2(0) + \frac{u_1^2(0)\left(\frac{\Upsilon}{2} - \Upsilon'\right)^2}{2}$$

$$\times \int_{-1}^{1} \left(\frac{\lambda_1 + \lambda_2(c_1(\bar{\alpha}))^2}{\mu_1 + \mu_2 c_1(\bar{\alpha})}\right)\left(\exp\left(-\frac{\mu_1 + \mu_2 c_1(\bar{\alpha})}{\lambda_1 + \lambda_2(c_1(\bar{\alpha}))^2}\, x\right) - 1\right)^2$$

$$\times (\bar{\alpha} + 1)\, d\bar{\alpha}. \qquad \ldots(61)$$

The mean fractional change of energy which is the other most important parameter of the problem is given by

$$\left\langle \frac{\delta E_1(x)}{E_1(0)} \right\rangle = \frac{1}{u_1^2(0)} \langle \mid u_1(x) \mid^2 \rangle - 1 \qquad \ldots(62)$$

Since the mean fractional change of energy depends on the mean square displacement, we restrict our analysis only on $\langle \mid u_1(x) \mid^2 \rangle$. Moreover, it is clear from the expression of $\langle \mid u_1(x) \mid^2 \rangle$ that the integrand becomes much complicated when the value of $c_1(\bar{\alpha})$ from equation (59) is substituted in it. We, therefore, evaluate the integral for the following two extreme cases :

Case 1 (when X is Small)

In this case we assume the propagation path x is sufficiently small and retain terms upto $O(x^3)$. Thus we find from (61)

$$\langle \mid u_1(x) \mid^2 \rangle \Big|_{\text{small}\, x} = u_1^2(0) + \frac{u_1^2(0)\left(\frac{\Upsilon}{2} - \Upsilon'\right)^2}{2} I. \qquad \ldots(63)$$

Here

$$I = 2x^2 - x^3(I_1 + I_2), \qquad \ldots(64)$$

$$I_1 = \int_{-1}^{1} \frac{a_0 + b_0\bar{\alpha}}{(\bar{a}_0 + \bar{b}_0\bar{\alpha} + \bar{c}_0\bar{\alpha}^2)}\, \bar{\alpha}\, d\bar{\alpha} \qquad \ldots(65)$$

and

$$I_2 = \int_{-1}^{1} \frac{a_0 + b_0\bar{\alpha}}{(\bar{a}_0 + \bar{b}_0\bar{\alpha} + \bar{c}_0\bar{\alpha}^2)}\, d\bar{\alpha}, \qquad \ldots(66)$$

where

$$\bar{a}_0 = \mu_1 + \mu_2 c_1^*,$$

$$b_0 = \mu_2 c_1^{**},$$

$$\bar{a}_0 = \lambda_1 + \lambda_2 c_1^{*2},$$

$$\bar{b}_0 = 2\lambda_2 c_1^* c_1^{**}$$

and

$$\bar{c}_0 = \lambda_2 (c_1^{**})^2. \qquad \ldots(67)$$

Evaluating the above integrals we find

$$I_1 = \frac{2b_0}{\bar{c}_0} + \left(a_0 - \frac{b_0\bar{b}_0}{\bar{c}_0}\right)\frac{1}{2\bar{c}_0}\cdot\log\frac{(\bar{c}_0+\bar{b}_0+\bar{a}_0)}{(\bar{c}_0-\bar{b}_0+\bar{a}_0)}$$
$$-\frac{2\left\{\left(a_0 - \frac{b_0\bar{b}_0}{\bar{c}_0}\right)\frac{\bar{b}_0}{2\bar{c}_0} + \frac{\bar{a}_0 b_0}{\bar{c}_0}\right\}}{\sqrt{4\bar{c}_0\bar{a}_0 - \bar{b}_0}}\cdot\tan^{-1}\left\{\frac{4\bar{c}}{\sqrt{4\bar{c}_0\bar{a}_0 - \bar{b}_0^2}}\right\} \qquad \ldots(68)$$

and

$$I_2 = \frac{b_0}{2\bar{c}_0}\left\{\log\frac{(\bar{c}_0+\bar{b}_0+\bar{a}_0)}{(\bar{c}_0-\bar{b}_0+\bar{a}_0)} + \frac{2(2\bar{c}_0 a_0 - b_0\bar{b}_0)}{b_0\sqrt{4\bar{c}_0\bar{a}_0 - \bar{b}_0^2}}\right.$$
$$\left.\times\tan^{-1}\left(\frac{4\bar{c}_0}{\sqrt{4\bar{c}_0\bar{a}_0 - \bar{b}_0^2}}\right)\right\} \qquad \ldots(69)$$

Substituting the values of I_1 and I_2 from (68) and (69) in (64), the value of I can be found out and so finally $\langle\, |u_1(x)|^2 \,\rangle$ for small values of x can be determined by using equation (63). The values of $\langle\, |u_1(x)|^2 \,\rangle$ and I so obtained enable us to determine $\langle\, e^2\omega_1 \rangle$, $\langle\, E\omega_1 \,\rangle$ and $\left\langle \frac{\delta E_1(x)}{E_1(0)} \right\rangle$ for small values of x by using equations (56), (57) and (62).

Case 2 (when X is Large)

In this case we assume the propagation path X to be sufficiently large say $O(\infty)$. Thus we find from equation (61).

$$\langle\, |u_1(x)|^2 \,\rangle \Big|_{\text{large } x} = u_1^2(0) + \frac{u_1^2(0)\left(\frac{\Upsilon}{2} - \Upsilon'\right)^2}{2}\, I \qquad \ldots(70)$$

where

$$I = I_3 + I_4, \qquad \ldots(71)$$

$$I_3 = \int_{-1}^{1} \frac{(\bar{c}_0^2)\,\bar{\alpha}^5 + (2\bar{c}_0\bar{b}_0)\,\bar{\alpha}^4 + (2\bar{a}_0\bar{c}_0 + \bar{b}_0^2)\bar{\alpha}^3 + (2\bar{a}_0\bar{b}_0)\,\bar{\alpha}^2 + \bar{a}_0^2\bar{\alpha}}{b_0^2\bar{\alpha}^2 + 2a_0b_0\bar{\alpha} + a_0^2}\,d\bar{\alpha} \quad \ldots(72)$$

and

$$I_4 = \int_{-1}^{1} \frac{(\bar{c}_0^2)\,\bar{\alpha}^4 + (2\bar{c}_0\bar{b}_0)\,\bar{\alpha}^3 + (2\bar{a}_0\bar{c}_0 + \bar{b}_0^2)\,\bar{\alpha} + (2\bar{a}_0\bar{b}_0)\,\bar{\alpha} + \bar{a}_0^2}{b_0^2\bar{\alpha}^2 + 2a_0b_0\bar{\alpha} + a_0^2}\,d\bar{\alpha} \quad \ldots(73)$$

Evaluating the above integrals, we find

$$\begin{aligned}
I_3 = \frac{2}{3}\cdot\frac{2\bar{c}_0\bar{b}_0 - \dfrac{2a_0\bar{c}_0^2}{b_0}}{b_0^2} &+ \left\{\left[2\bar{a}_0\bar{b}_0 - \frac{a_0^2}{b_0^2}\right.\right.\\
&\times\left.\left(2\bar{c}_0\bar{b}_0 - \frac{2a_0\bar{c}_0^2}{b_0}\right)\right] - \frac{2a_0}{b_0}\left[2\bar{a}_0\bar{c}_0 + \bar{b}_0^2\right.\\
&-\left.\left.\frac{a_0\bar{c}_0^2}{b_0^2} - \frac{2a_0}{b_0}\left(2\bar{c}_0\bar{b}_0 - \frac{2\bar{c}_0^2a_0}{b_0}\right)\right]\right\}.\\
&\times\left\{\frac{2}{b_0^2} - \frac{a_0}{b_0^3}\log\frac{(b_0+a_0)^2}{(b_0-a_0)^2} + \frac{6a_0^2}{b_0^2(a_0^2-b_0^2)}\right\}\\
&+\left\{a_0^2 - \frac{a_0^2}{b_0^2}\left[2\bar{a}_0\bar{c}_0 + \bar{b}_0^2 - \frac{a_0\bar{c}_0^2}{b_0^2} - \frac{2a_0}{b_0}\right.\right.\\
&\times\left.\left.\left(2\bar{c}_0\bar{b}_0 - \frac{2a_0\bar{c}_0^2}{b_0}\right)\right]\right\}\left[\frac{1}{2b_0^2}\log\frac{(b_0+a_0)^2}{(b_0-a_0)^2} - \frac{2a_0}{b_0(a_0^2-b_0^2)}\right]
\end{aligned} \quad \ldots(74)$$

and

$$\begin{aligned}
I_4 = \frac{2\bar{c}_0^2}{3b_0^2} &+ \frac{2}{b_0^2}\left\{2\bar{a}_0\bar{c}_0 + \bar{b}_0^2 - \frac{a_0^2\bar{c}_0^2}{b_0^2}\right.\\
&-\left.\frac{2a_0}{b_0}\left(2\bar{c}_0\bar{b}_0 - \frac{2a_0\bar{c}_0^2}{b_0}\right)\right\} + \frac{1}{2b_0^2}\left\{\left[2\bar{a}_0\bar{b}_0\right.\right.\\
&-\left.\frac{a_0^2}{b_0^2}\left(2\bar{c}_0\bar{b}_0 - \frac{2a_0\bar{c}_0^2}{b_0}\right)\right] - \left[\frac{2a_0}{b_0}\left(2\bar{a}_0\bar{c}_0 + \bar{b}_0^2\right.\right.\\
&-\left.\left.\left.\frac{a_0\bar{c}_0^2}{b_0^2} - \frac{2a_0}{b_0}\left(2\bar{b}_0\bar{c}_0 - \frac{2a_0\bar{c}_0^2}{b_0}\right)\right)\right]\right\}\log\frac{(b_0+a_0)^2}{(b_0-a_0)^2}\\
&+\frac{2}{a_0^2-b_0^2}\left\{\left[\bar{a}_0^2 - \frac{a_0^2}{b_0^2}\left(2\bar{a}_0\bar{c}_0 + \bar{b}_0^2 - \frac{a_0^2\bar{c}_0^2}{b_0^2}\right.\right.\right.\\
&-\left.\left.\frac{2a_0}{b_0}\left(2\bar{c}_0\bar{b}_0 - \frac{2a_0\bar{c}_0^2}{b_0}\right)\right] - \frac{a_0}{b_0}\left(\left[2\bar{a}_0 - \frac{a_0^2}{b_0^2}\right.\right.
\end{aligned}$$

$$\times\left(2\bar{c}_0\bar{b}_0-\frac{2a_0\bar{c}_0^2}{b_0}\right)\Bigg]-\left[\frac{2a_0}{b_0}\left(2\bar{a}_0\bar{c}_0+\bar{b}_0^2-\frac{a_0^2\bar{c}_0^2}{b_0^2}\right)\right.$$

$$\left.-\frac{2a_0}{b_0}\left(2\bar{b}_0\bar{c}_0-\frac{2a_0\bar{c}_0^2}{b_0}\right)\right]\Bigg)\Bigg\} \qquad \ldots(75)$$

Substituting the values of I_3 and I_3 from (74) and (75) in (71) the value of I can be found out and ultimately $\langle\, |u_1(x)|\,\rangle$ can be determined for large values of x. With these values of $\langle\, |u_1(x)|^2\,\rangle$ and I we can determine $\langle\, e^2\omega_1\,\rangle$ $\langle\, E\omega_1\,\rangle$ and $\left\langle \frac{\delta E_1(x)}{E_1(0)} \right\rangle$ for large x using (56), (57) and (62).

Numerical Calculation and Discussion

Analysing the expressions for $\langle\, |u_1(x)|^2\,\rangle$, $\langle\, e^2\omega_1\,\rangle$ $\langle\, E\omega_1\,\rangle$ and $\left\langle \frac{\delta E_1(x)}{E_1(0)} \right\rangle$ for the two cases it can be seen that we would get parabolic curves for the above

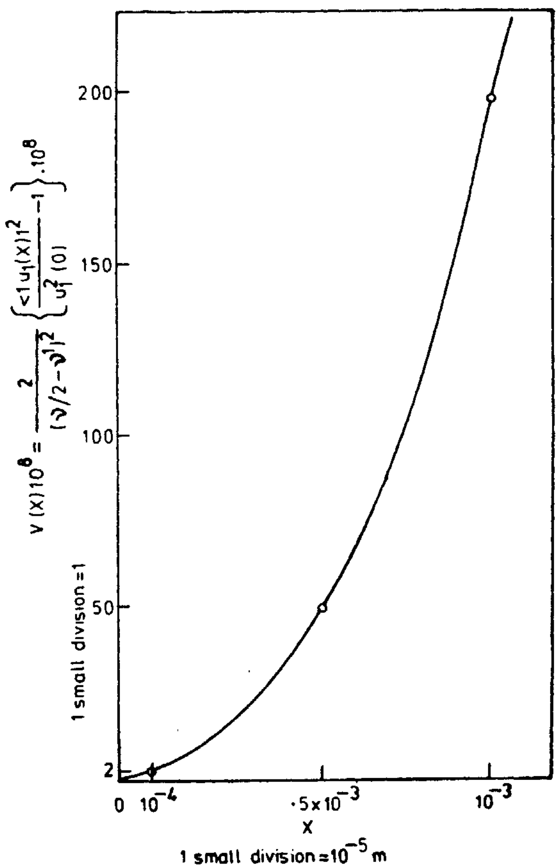

FIG 1 $V(X)\cdot10^8$ *vs* X (Propagation path).

mentioned statistics in the first case where as in the second case we find them to be constants independent of x.

We plot the mean square displacement $\langle |u_1(x)|^2 \rangle$ for small x given by equation (63) against the propagation path x taking Cadmium Sulphide (Cds) as the medium of propagation, writing equation (63) in the following form.

$$\frac{2}{(\gamma/2 - \gamma')^2} \left\{ \frac{\langle |u_1(x)|^2 \rangle}{u_1^2(0)} - 1 \right\} = I = V(x)$$

Substituting the values of the different meterial constants for CdS, *vide*, Steel and Vural[13] and assuming $c_1^* = c_1^{**} = 1$, $u_3(0) = 10^3\, u_1(0)$ where $u_1(0) = 0.0005\, m$, the equation has been plotted graphically. The graph is found to be a parabola as in Fig. 1.

If the elastic stiffness constant c is independent of time ($c_1 = 0$) then we have

$$\lambda^* = \lambda_1, \mu^* = \mu_1 < 0.$$

Therefore,

$$\frac{\delta E_1(x)}{E_1(0)} = \frac{\lambda_1^2}{\mu_1^2} \left(\frac{\gamma}{2} - \gamma' \right)^2 \left(\exp\left(\frac{\mu_1}{\lambda_1} x \right) - 1 \right)^2.$$

In this case also we find a gain of energy for all x but unlike the situation considered in the problem, the gain of energy never reaches a maximum. Under these circumstances we need not calculate the mean displacement, mean strain amplitude etc. and the expression of the relevant quantities are given by equations (51) to (54) by substituting λ_1 and μ_1 for λ^* and μ^*.

References

1. O K Rudenko and S I Soluyan *Theoretical Foundation of Nonlinear Acoustics* Consultants Bureau New York (1975)
2. H R Carleton H Krager and E W Prohofsky *Proc IEEE SU* **53** (1965) 1462
3. N S Shiren *Proc IEEE* **53** (1965) 1541
4. R K Pal and M Gupta *J Indian Acad Math* **3** (1981) 27
5. R K Pal M Gupta and D K Sinha *J appl Phys* **54** (1983) 5651
6. J E Ecanomoe and H N Spector *J appl Phys* **50** (1979) 6969
7. G Paria *Indian J theory Phys* **26** (1978) 79
8. F Ziegler *Acta Mechanica* **2** (1966) 307
9. H J Wintle *J appl Phys* **43** (1972) 2927
10. A Vander Ziel *Solid State Physical Electronics* Prentice Hall India (1971)
11. D L White *J appl Phys* **33** (1962) 2547
12. K Seegar *Semiconductor Physics* 3rd edn Springer Verlag Berlin New York (1985)
13. M C Steele and B Vural *Wave Introduction in Solid State Plasmas* McGraw Hill Publ Co New York (1969)