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PROBLEMS OF TWO-DIMENSIONAL ELECTROSTRICTION OF ELASTICALLY ANISOTROPIC DIELECTRICS

G H MAIKAP* *and* P R SENGUPTA**

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A theory has been derived to describe the distribution of stresses in an elastically anisotropic as well as electrically isotropic dielectric material medium under the influence of an electric field. Using complex variable technique a two-dimensional form of the theory has been derived and it is applied to find out the distribution of stresses in an infinite anisotropic, elastic, dielectric plane with an elliptical hole being filled up with air.

Key Words : 2-Dimensional Electrostriction; Elastically Anisotropic Dielectrics; Complex Variable Technique

INTRODUCTION

IT has been observed that when an elastic solid dielectric medium is subjected to an electric field the deformation caused by it is observed to be proportional to even power of electric intensity vector. This phenomenon is called *electrostriction.* The classical development dealing with electrostriction[1,2,3] is to derive the field equations governing electrostriction from thermodynamical considerations. These equations are then applied to find out the solutions of the problems relating to spheres and cylinders. Knops[4] first investigated the problems of electrostriction from a separate standpoint arising out of the consideration of stress equilibrium condition using complex variable formalism. He succeeded in deriving the solutions of equations in complex variable structure governing the electrostriction field equations. This theory of Knops[4] is highly expository in character and admits wider applications in solving various two dimensional problems. In recent years several investigators[5,6,7] had applied this method in solving two-dimensional problems in a similar approach as discussed by Muskhelisvilli,[8] Savin,[9] and Green and Zerna.[10]

In Knops theory, the dielectric was assumed to be initially isotropic in elastic and electric properties. However, when the dielectric being homogeneous and initially anisotropic in its elastic properties is also of greater interest and worth for further investigation. In this paper following the work of Chakraborty and Sengupta[6,7] an interesting problem of two-dimensional electrostriction of elastically anisotropic dielectrics with elliptical hole has been investigated. This is done in terms of complex variable formalism. It is believed that the problem has not been investigated previously. The present authors propose to investigate further various problems of two-dimensional electrostriction of elastically anisotropic dielectrics with various boundaries.

*C/o Uday Chand De, Vill. Narayantala (East), P.O. Ashininagar, Calcutta-700 059 W.B., India.
**Department of Mathematics, University of Kalyani, (W.B.), India.

BASIC EQUATIONS AND THE STRESS FUNCTION

An elastically anisotropic dielectric is considered with permeability ϵ_0 bounded by a cylindrical surface which is under influence of an electric field. Let us consider that the plane of an arbitrary cross section of the cylinder contains the x-y axes and the z-axis be parallel to the generator. It is considered that the electric field does not vary along the generator. All the above conditions be satisfied for an isotropic body of a two-dimensional deformation but which is not the case for an anisotropic body. For this purpose an additional assumption is to be made that the body has at each point a plane of elastic symmetry normal to the generators and also to the z-axis. Thus, we are in a position to have a pure two-dimensional deformation. This deformation will give rise to a displacement vector $\mathbf{U} = (u_1)$. We consider the displacement vector is single valued function in (x, y) only. Hence, in this case, the displacement component along the z-axis is $w = 0$. Now we are to find the strains in the usual way and they are satisfying the following relation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y} . \qquad \text{...(1)}$$

The cylinder be in equilibrium under the stresses σ_{xx}, σ_{yy}, τ_{xy}. So ignoring the body forces, the equilibrium equations can be written as following

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \text{and} \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \end{aligned}\right\} \qquad \text{...(2)}$$

The stresses are considered to be linear function of strains and quadratic functions of the electric intensity vector $\mathbf{E} = (\mathbf{E}_x, \mathbf{E}_y)$. Now we can form a system of general relations between the stresses, strains and component of electric intensity vectors in the following way

$$\begin{aligned} \sigma_{xx} &= A_{11} e_{xx} + A_{12} e_{yy} + A_{16} v_{xy} + A_{16} v_{xy} + aE_x^2 + b(\mathbf{E}_x^2 + \mathbf{E}_y^2) \\ \sigma_{yy} &= A_{21} e_{xx} + A_{22} e_{yy} + A_{26} v_{xy} + a\mathbf{E}_y^2 + b(\mathbf{E}_x^2 + \mathbf{E}_y^2) \\ \tau_{xy} &= A_{16} e_{xx} + A_{26} e_{yy} + A_{26} v_{xy} + a\mathbf{E}_x\mathbf{E}_y \end{aligned} \qquad \text{...(3)}$$

After deformation, the dielectric also becomes anisotropic in its electric properties. The electric displacement vector $\mathbf{D} = (D_x, D_y)$ is related to the intensity vector by the following relations

$$\left.\begin{aligned} \mathbf{D_x} &= \mathcal{E}_{xx}\mathbf{E}_x + \mathcal{E}_{xy}\mathbf{E}_y \\ \text{and} \qquad \mathbf{D_y} &= \mathcal{E}_{xy}\mathbf{E}_x + \mathcal{E}_{yy}\mathbf{E}_y \end{aligned}\right\}, \qquad \text{...(4)}$$

where $\mathcal{E}_{xx}$, $\mathcal{E}_{yy}$, $\mathcal{E}_{xy}$ are components of the dielectric tensor. Since the state of a slightly deformed body is described by the strain tensor and as the strain components are small, only the first order of e_{xx}, e_{xy} e_{yy} etc. need be retained in the variation of the components $\mathcal{E}_{xx}$, $\mathcal{E}_{yy}$, $\mathcal{E}_{xy}$. Therefore, we can write

$$\left.\begin{aligned} \mathcal{E}_{xx} &= \epsilon_0 + \epsilon_1 e_{xx} + \epsilon_2(e_{xx} + e_{yy}), \\ \mathcal{E}_{yy} &= \epsilon_0 + \epsilon_1 e_{yy} + \epsilon_2(e_{xx} + e_{yy}) \\ \text{and} \qquad \mathcal{E}_{xy} &= \epsilon_1 e_{xy} \end{aligned}\right\}, \qquad \ldots(5)$$

where ϵ_1 and ϵ_2 are the scalers which are to be determined experimentally and are delineated.[3] Substituting $\mathcal{E}_{xx}$, $\mathcal{E}_{yy}$, $\mathcal{E}_{xy}$ in to the relation (4) and considering the strains are proportional to the even powers of the components of the electric intensity vector, we have, on neglecting the terms of third and heigher orders, the relation $\mathbf{D} = \epsilon_0\mathbf{E}$ determining the electric field in the underformed body.

The relation (3) expresses the stresses in terms of strains and components of the electric intensity vector **E**. Using some modifications these equations can be written in an alternative way, by expressing the strains in terms of stresses and components of electric intensity vector. Now we can write these relations in the following manner

$$e_{xx} = a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\tau_{xy} - \{(a + b)\,a_{11} + ba_{12}\}\,\mathbf{E}_x^2$$
$$- \{(a + b)\,a_{12} + ba_{11}\}\,\mathbf{E}_y^2 - aa_{16}\mathbf{E}_x\mathbf{E}_y,$$

$$e_{yy} = a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\tau_{xy} - \{(a + b)\,a_{12} + ba_{22}\}\,\mathbf{E}_x^2$$
$$- \{(a + b)\,a_{22} + ba_{12}\}\,\mathbf{E}_y^2 - aa_{26}\mathbf{E}_x\mathbf{E}_y$$

and

$$e_{xy} = a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\tau_{xy} - \{(a + b)\,a_{16} + ba_{26}\}\,\mathbf{E}_x^2$$
$$- \{(a + b)\,a_{26} + ba_{16}\}\,\mathbf{E}_y^2 - aa_{66}\mathbf{E}_x\mathbf{E}_y, \qquad \ldots(6)$$

where

$$a_{11} = \frac{1}{\Delta}\,(A_{22}A_{66} - A_{26}^2),$$

$$a_{12} = \frac{1}{\Delta}\,(-A_{12}A_{66} + A_{16}A_{26}),$$

$$a_{16} = \frac{1}{\Delta}\,(A_{12}A_{26} - A_{16}A_{22}),$$

$$a_{22} = \frac{1}{\Delta}\,(A_{11}A_{66} - A_{16}^2),$$

$$a_{66} = \frac{1}{\Delta}(A_{11}A_{22} - A_{21}A_{12}),$$

$$a_{26} = \frac{1}{\Delta}\,(A_{16}A_{21} - A_{11}A_{26})$$

and

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{vmatrix} \qquad \text{...(7)}$$

Very quickly we observed that the relations (2) are identically satisfied if we put

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \qquad \text{...(8)}$$

where χ is a function of (x, y) and is known as Airy's stress function. In the equations (6), we write for σ_{xx}, σ_{yy} τ_{xy} the expression obtained in (8) and the final expressions for the strains are substituted in the compatibility equation (1) yielding the following relation :—

$$a_{22}\frac{\partial^4 \chi}{\partial x^4} - 2a_{26}\frac{\partial^4 \chi}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 \chi}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 \chi}{\partial x \partial y^3} + a_{11}\frac{\partial^4 \chi}{\partial y^4}$$
$$= \frac{\partial^2}{\partial x^2}[\{(a+b)\, a_{12} + ba_{22}\}\, E_x^2$$
$$+ \{(a+b)\, a_{22} + ba_{12}\}\, E_y^2 + aa_{26}E_xE_y]$$
$$+ \frac{\partial^2}{\partial y^2}[\{(a+b)\, a_{11} + ba_{12}\}\, E_x^2 + \{(a+b)\, a_{12} + ba_{11}\}\, E_y^2$$
$$+ aa_{16}E_xE_y]$$
$$- \frac{\partial^2}{\partial x \partial y}[\{(a+b)\, a_{16} + ba_{16}\}\, E_x^2$$
$$+ \{(a+b)\, a_{26} + ba_{16}\}\, E_y^2 + aa_{66}E_xE_y] \qquad \text{...(9)}$$

The solution of equation (9) will determine the function χ and the function χ so obtained, when substituted in (8) will determine the stresses.

Body Immersed in a Uniform Field

The solution of the partial differential equation (9) becomes impossible in case of a general electric field in which the dielectric is immersed. The general solution is possible if we take certain assumptions regarding the properties of some dielectrics which we often come across in nature.

In this section it is assumed that the whole body is immersed in a uniform electric field E^0 acting in the $X - Y$ plane. The outer field will cause a new field E inside the cylinder and is given by

$$E = E^0 + e, \qquad \text{...(10)}$$

where e is the self component of the field intensity generated within the dielectric medium. In this part of the field, intensity is always smaller than the original electric field in which the dielectric is immersed. For a certain dielectric having small permeabilities, the self component part e is so small in comparison to the original field $\mathbf{E}^0$ that the squares and products of the components of e can be neglected. Using the above assumption the equation (9) can be written in the following way

$$a_{22}\frac{\partial^4\chi}{\partial x^4} - 2a_{26}\frac{\partial^4\chi}{\partial x^3\partial y} + (2a_{12} + a_{66})\frac{\partial^4\chi}{\partial x^2\partial y^2} - 2a_{16}\frac{\partial^4\chi}{\partial x\partial y^3} + a_{11}\frac{\partial^4\chi}{\partial y^4}$$

$$= \frac{\partial^2}{\partial x_2}\,[2\{(a+b)\,a_{12} + ba_{22}\}\,\mathbf{E}_x^0 e_x + 2\{(a+b)\,a_{22} + b_{12}\}\,\mathbf{E}_y^0 e_y$$

$$+ aa_{26}\,\mathbf{E}_x^0 e_y + aa_{26}\,\mathbf{E}_x^0 e_x] + \frac{\partial^2}{\partial y^2}\,[2\{(a+b)\,a_{11} + ba_{12}\}\,\mathbf{E}_x^0\, e_x$$

$$+ 2\{(a+b)\,a_{12} + ba_{11}\}\,\mathbf{E}_y^0 e_y + aa_{16}\,\mathbf{E}_x^0 e_y + aa_{16}\,\mathbf{E}_y^0 e_x]$$

$$- \frac{\partial^2}{\partial x\partial y}\,[2\{(a+b)\,a_{16} + ba_{26}\}\,\mathbf{E}_x^0\, e_x + 2\{(a+b)\,a_{26}$$

$$+ ba_{16}\}\,\mathbf{E}_y^0 e_y + aa_{66}\,\mathbf{E}_x^0 e_y + aa_{66}\,\mathbf{E}_y^0\, e_x] \qquad \ldots(11)$$

Using the complex potential function w(z), we can write

$$f_1(z) = w'(z) = -(\mathbf{E}_x^0 + e_x) + i(\mathbf{E}_y^0\, e_y) \qquad \ldots(12)$$

So that

$$\overline{f_1(z)} = \overline{w'(z)} = -(\mathbf{E}_x^0 + e_x) - i(\mathbf{E}_y^0 + e_y) \qquad \ldots(13)$$

From (12) and (13) we finally get,

$$e_x = -\frac{1}{2}\{f_1(z) + \overline{f_1(z)}\} - \mathbf{E}_x^0 \qquad \ldots(14)$$

and

$$e_y = -\frac{i}{2}\{f_1(z) - \overline{f_1(z)}\} - \mathbf{E}_y^0\,. \qquad \ldots(15)$$

Putting the values of e_x and e_y from the relations (14) and (15) in the equation (11), then the equation (11) reduces to the following form

$$a_{22}\frac{\partial^4\chi}{\partial x^4} - 2a_{26}\frac{\partial^4\chi}{\partial x^3\partial y} + (2a_{12} + a_{66})\frac{\partial^4\chi}{\partial x^2\partial y^2} - 2a_{16}\frac{\partial^4\chi}{\partial x\partial y^3} + a_{11}\frac{\partial^4\chi}{\partial y^4}$$

$$= \frac{\partial^2}{\partial x^2}\,[A_1 f_1(z) + \overline{A_1 f_1(z)}] + \frac{\partial^2}{\partial y^2}\,[B_1 f_1(z) + \overline{B_1 f_1(z)}]$$

$$- \frac{\partial^2}{\partial x\partial y}\,[C_1 f_1(z) + \overline{C_1 f_1(z)}], \qquad \ldots(16)$$

where $A_1 = -\{(a+b)\,(a_{12}\,\mathbf{E}_x^0 + ia_{22}\,\mathbf{E}_y^0) + b(a_{22}\,\mathbf{E}_x^0 + ia_{12}\,\mathbf{E}_y^0)$

$$+ \frac{a}{2}\,(a_{26}\,\mathbf{E}_y^0 + ia_{26}\,\mathbf{E}_p^0)\}$$

$$B_1 = -\{(a+b)(a_{11}\mathbf{E}_x^0 + ia_{12}\mathbf{E}_y^0) + b(a_{12}\mathbf{E}_x^0 + ia_{11}\mathbf{E}_y^0) + \frac{a}{2}(a_{16}\mathbf{E}_y^0 + ia_{16}\mathbf{E}_x^0)\}$$

and

$$C_1 = -\{(a+b)(a_{16}\mathbf{E}_x^0 + ia_{26}\mathbf{E}_y^0) + b(a_{26}\mathbf{E}_x^0 + ia_{16}\mathbf{E}_y^0) + \frac{a}{2}(a_{66}\mathbf{E}_y^0 + ia_{66}\mathbf{E}_x^0)\}. \qquad ...(17)$$

The solution of (16), which will determine the stress function is given as

$$\chi = \chi_1 + \chi_2, \qquad ...(18)$$

where χ_1 is the solution of

$$a_{22}\frac{\partial^4\chi}{\partial x^4} - 2a_{26}\frac{\partial^4\chi}{\partial x^3\partial y} + (2a_{12} + a_{96})\frac{\partial^4\chi}{\partial x^2\partial y^2} - 2a_{16}\frac{\partial^4\chi}{\partial x\partial y^3} + a_{11}\frac{\partial^4\chi}{\partial y^4} = 0 \qquad ...(19)$$

and χ_2 is the particular integral. The solution of (19) is known to be

$$\chi_1 = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}, \qquad ...(20)$$

where $F_1(z_1)$, $F_2(z_2)$ are analytic functions of their arguments and bar denotes the conjugate function.

Also

$$z_1 = x + s_1 y,\ z_2 = x + s_2 y, \qquad ...(21)$$

s_1, s_2, $\bar{s}_1$, $\bar{s}_2$ are the roots of the equation

$$a_{11}s^4 - 2a_{16}s^3 + (2a_{12} + a_{66})s^2 - 2a_{26}s + a_{22} = 0 \qquad ...(22)$$

and $s_1 = \alpha_1 + i\beta_1$, $s_2 = \alpha_2 + i\beta_2$

To find the particular integral we express equation (16) in terms of the complex variable $z = x + iy$ and $\bar{z} = x - iy$ in the following form

$$A\frac{\partial^4\chi}{\partial z^4} + B\frac{\partial^4\chi}{\partial z^3\partial\bar{z}} + C\frac{\partial^4\chi}{\partial z^2\partial\bar{z}^2} + \bar{B}\frac{\partial^4\chi}{\partial z\partial\bar{z}^3} + \bar{A}\frac{\partial^4\chi}{\partial\bar{z}^4} = (A_1 - B_1 - iC_1)\frac{\partial^2}{\partial z^2}f_1(z) + (\bar{A}_1 - \bar{B}_1 + i\bar{C}_1)\frac{\partial^2}{\partial\bar{z}^2}\overline{f_1(z)}, \qquad ...(23)$$

where

$$A = \{(a_{11} + a_{22} - 2a_{12} - a_{66}) - 2i(a_{26} - a_{16})\},$$

$$B = \{4(a_{22} - a_{11}) - 4i(a_{26} + a_{16})\} \qquad ...(24)$$

and

$$C = \{6(a_{11} + a_{22}) + 2(a_{12} + a_{66}),$$

where the bar represents conjugation.

It is found that the solution of (23) can be taken as

$$\chi_2 = Kf_3(z) + \bar{K}\overline{f_3(z)}, \qquad \ldots(25)$$

where

$$K = \frac{A_1 - B_1 - iC_1}{A} \text{ and } f_3(z) = \int f_2(z)\, dz; f_2(z) = \int f_1(z)\, dz \qquad \ldots(26)$$

So the general solution of (16) is

$$\chi = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)} + Kf_3(z) + \bar{K}\, \overline{f_3(z)} \qquad \ldots(27)$$

Using the earlier notations we can calculate the stresses from (27) with the help of relations (8). So we get

$$\sigma_{xx} = 2Re\,[s_1^2\, \varphi'(z_1) + s_2^2\, \psi'(z_2)] - Kf_1(z) - \bar{K}\, \overline{f_1(z)},$$

$$\sigma_{yy} = 2Re\,[\varphi'(z_1) + \psi'(z_2)] + Kf_1(z) + \bar{K}\, \overline{f_1(z)}$$

and $$\tau_{xy} = -\, 2Re\,[s_1\varphi'(z_1) + s_2\psi'(z_2)] + i[Kf_1(z) + \bar{K}\, \overline{f_1(z)}, \qquad \ldots(28)$$

where

$$\varphi(z_1) = F'(z_1),\ \psi(z_2) = F'(z_2).$$

First Boundary Value Problem of an Infinite Plane with an Elliptic Hole

We take an infinite anisotropic homogeneous elastic dielectric plane of permeability ϵ containing an elliptical hole filled up with air, the permiability is known to be unity. The entire dielectric plane is placed in presence of an electric field E^0 at infinity. It is assumed that the hole is free from external stresses. However, at infinity there acts a uniaxial tension T which is assumed firstly to display in the direction of the X-axis and secondly obliquely to the X-axis. As the hole is free from external stresses we assume the following boundary conditions at the boundary of the hole :

$$\sigma_{xx} \cos\,(n, x) + \tau_{xy} \cos\,(n, y) = 0; \qquad \ldots(29)$$

and $$\tau_{xy} \cos\,(n, x) + \sigma_{yy} \cos\,(n, y) = 0. \qquad \ldots(30)$$

Substituting in (29) and (30) the expressions for σ_{xx}, σ_{yy}, τ_{xy} interms of the stress function χ given by (9), we get after performing some routine calculations, the boundary equations

$$2Re\,[\varphi(z_1) + \psi(z_2)]\, \chi K f_2(z) + \bar{K}\overline{f_2(z)} = 0 \qquad \ldots(31)$$

and $$2Re\,[s_1\varphi(z_1) + s_2\, \psi(z_2)] + iKf_2(z) - i\bar{K}\, \overline{f_2(z)} = 0 \qquad \ldots(32)$$

The two planes z_1 and z_2 may be obtained from the z-plane by the affiine transformations

and
$$\left.\begin{aligned} z_1 &= x_1 + iy_1 = x + s_1 y \\ z_2 &= x_2 + iy_2 = x + s_2 y \end{aligned}\right\} \qquad \ldots(33)$$

Consequently, the hole in the z-plane will be transformed to the corresponding holes in the z_1, z_2 planes by the transformation (33).

The exterior regions of the holes in the planes are respectively denoted by S, S_1, S_2.

The region S can now be mapped on to the exterior of an ellipse in ζ plane by the transformation

$$z = g(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right) \qquad \ldots(34)$$

It can easily be verified that the regions S_1, S_2 may also be mapped on to the exterior of the same ellipse in the ζ-plane by the transformations

$$\begin{aligned} Z_1 = g_1(\zeta) &= \frac{1}{2} R\left\{\left(\frac{1}{\zeta} + \zeta\right) - i\left(\frac{1}{\zeta} - \zeta\right) s_1 + m\left(\zeta + \frac{1}{\zeta}\right)\right. \\ &\qquad \left. - im\left(\zeta - \frac{1}{\zeta}\right) s_1\right\} \\ &= \frac{1}{2} R\left\{(m+1)\zeta - (m-1) is_1\zeta\right. \\ &\qquad \left. + (m+1)\frac{1}{\zeta} + (m-1) is_1 \frac{1}{\zeta}\right\} \\ &= \frac{1}{2} R\left[\{(m+1) - (m-1) is_1\}\zeta + \{(m+1)\right. \\ &\qquad \left. + (m-1) is_1\} \frac{1}{\zeta}\right] \end{aligned} \qquad \ldots(35)$$

and

$$\begin{aligned} Z_2 = g_2(\xi) &= \frac{1}{2} R\left[\left\{\left(\frac{1}{\zeta} + \zeta\right) - i\left(\frac{1}{\zeta} - \zeta\right) s_2 + m\left(\zeta + \frac{1}{\zeta}\right)\right.\right. \\ &\qquad \left.\left. - im\left(\zeta - \frac{1}{\zeta}\right) s_2\right\}\right) \\ &= \frac{1}{2} R\left[\{(m+1) - (m-1) is_2\}\zeta\right. \\ &\qquad \left. + \{(m+1) + (m-1) is_2\}\frac{1}{\zeta}\right]. \end{aligned} \qquad \ldots(36)$$

Paying attention to the fact that the stresses are single valued and finite, we can split the function $\varphi(z_1)$ and $\psi(z_2)$ as

$$\varphi(z_1) = \mathbf{B}^* Z_1 + \varphi_0(z_1) \qquad \ldots(37)$$

and

$$\psi(z_2) = (\mathbf{B}'^* + ic'^*) Z_2 + \psi_0(z_2), \qquad \ldots(38)$$

where $\varphi_0(z_1)$ and $\psi_0(z_2)$ are holomorphic in the regions S_1 and S_2 respectively. Also the constants B^*, B'^*, C'^* can be determined by invoking the conditions at infinity.

The expressions of $\varphi(z_1)$ and $\psi(z_2)$ given by (37) and (38) when substituted in (31) and (32) reveal that $\varphi_0(z_1)$ and $\psi_0(z_2)$ also satisfy the same boundary equations except that, in that case, there appears the right-hand sides (31) and (32) two new expressions f_0^1 and f_0^2 respectively, where

$$f_0^1 = -2Re\,[B^* Z_1 + (B'^* + iC'^*)\,Z_2] \qquad \ldots(39)$$

and

$$f_0^2 = -2Re\,[B^* s_1 z_1 + (B'^* + iC'^*)\,s_2 z_2]\cdot \qquad \ldots(40)$$

Inserting for Z, Z_1, Z_2, there values $g(\zeta), g_1(\zeta), g_2(\zeta)$ respectively and denoting $\Phi_0(\zeta) = \varphi_0\{g_1(\zeta)\}$, $\psi_0(\zeta) = \psi_0\{g_2(\zeta)\}$, $F_2(\zeta) = f_2\{g(\zeta)\}$, we can write boundary conditions for the functions $\Phi_0(\zeta)$ and $\psi_0(\zeta)$ at the interface $\bar{\zeta} = \dfrac{1}{\zeta} = \sigma = e^{i\theta}$ as follows :

$$2Re\,[\Phi_0(\sigma) + \psi_0(\sigma)] = -2Re\left[L_1\sigma + L_2\frac{1}{\sigma}\right] - KF_2(\zeta) - \bar{K}\overline{F_2(\zeta)}; \qquad \ldots(41)$$

and

$$2Re\,[s_1\Phi_0(\sigma) + s_2\psi_0(\sigma)] = -2Re\left[L_3\sigma + L_4\frac{1}{\sigma}\right] - iKF_2(\zeta) + i\bar{K}\overline{F_2(\zeta)}, \qquad \ldots(42)$$

where

$$L_1 = \frac{R}{2}[B^*\{(m+1) - (m-1)\,is_1\} + (B'^* + iC'^*)\{(m+1) - (m-1)\,is_2\}],$$

$$L_2 = \frac{R}{2}[B^*\{(m+1) + (m-1)\,is_1\} + (B'^* + iC'^*)\{(m+1) + (m-1)\,is_2\}]$$

$$L_3 = \frac{R}{2}[B^* s_1\{(m+1) - (m-1)\,is_1\} + (B'^* + iC'^*)\,s_2\{(m+1) - (m-1)\,is_2\}]$$

$$L_4 = \frac{R}{2}[B^* s_1\{(m+1) + (m-1)\,is_1\} + (B'^* + iC'^*)\,s_2\{(m+1) + (m-1)\,is_2\}] \qquad \ldots(43)$$

Under the boundary condition that the tangential component of the electrical intensity vector and the normal component of the electrical displacement vector are continuous at the interface, the complex electrical potential functions assumes the form

$$\left.\begin{aligned} [W(\zeta)]_{|\zeta|>1} &= -\overline{E^0\zeta} + P E^0 \zeta^{-1} \\ \text{and} \quad [W(\zeta)]_{|\zeta|<1} &= -Q\,\overline{E^0\zeta} \end{aligned}\right\} \quad \ldots(44)$$

where

$$P = \frac{1-\epsilon}{1+\epsilon}$$
$$\ldots(45)$$
$$Q = \frac{2\epsilon}{1+\epsilon}$$

Thus we get from (44)

$$F_2(\zeta) = -\overline{E^0\zeta} + P E^0 \frac{1}{\zeta}\,. \quad \ldots(46)$$

Hence, (41) (42) and can be finally reduced to

$$\begin{aligned} 2Re\,[\Phi_0(\sigma) + \psi_0(\sigma)] = &- 2Re\left[L_1\sigma + L_2\frac{1}{\sigma}\right] \\ &- K\left[-\overline{E^0\sigma} + P E^0 \frac{1}{\sigma}\right] \\ &- \bar{K}\left[-E^0\frac{1}{\sigma} + P\,\overline{E^0\sigma}\right] \end{aligned} \quad \ldots(47)$$

and

$$\begin{aligned} 2Re\,[s_1\Phi_0(\sigma) + s_2\psi_0(\sigma) = &- 2Re\left[L_3\sigma + L_4\frac{1}{\sigma}\right] \\ &- iK\left[-\overline{E^0\sigma_4} + P E^0 \frac{1}{\sigma}\right] \\ &+ i\bar{K}\left[-E^0\frac{1}{\sigma} + P\overline{E^0}\sigma\right] \end{aligned} \quad \ldots(48)$$

respectively. Both sides of (47) and (48) are now multiplied by $\frac{1}{2\pi i}\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma}$ and using the Schwartz formula[8] we get on integrating around the unit circle

$$\Phi_0(\zeta) + \psi_0(\zeta) = -(L_2 + \bar{L}_1)\frac{1}{\zeta} - (KP - \bar{K})E^0\frac{1}{\zeta} \quad \ldots(49)$$

and

$$s_1\Phi_0(\zeta) + s_2\psi_0(\zeta) = -(L_4 + \bar{L}_3)\frac{1}{\zeta} - i(KP + \bar{K})E^0\frac{1}{\zeta} \quad \ldots(50)$$

At infinity there acts only the tension T in the direction of the X-axis. This condition at infiinity gives rise the values of the constants

$$B^* = \frac{T - 2(1 + \alpha_2^2 - \beta_2^2)\, Re\left(\frac{K\bar{E}^0}{R}\right) - 4\alpha_2 \mathrm{Im}\left(\frac{K\bar{E}^0}{R}\right)}{2\{(\alpha_2 - \alpha_1)^2 + (\beta_2^2 - \beta_1^2)\}},$$

$$B'^* = \frac{-T + 2(1 + \alpha_1^2 - \beta_1^2 - 2\alpha_1\alpha_2)\, Re\left(\frac{K\bar{E}^0}{R}\right) + 4\alpha_2 \mathrm{Im}\left(\frac{K\bar{E}^0}{R}\right)}{2\{(\alpha_2 - \alpha_1)^2 + (\beta_2^2 - \beta_1^2)\}},$$

and

$$C'^* = \frac{(\alpha_1 - \alpha_2)T + 2(\alpha_1^2\alpha_2 - \alpha_1\alpha_2^2 + \alpha_1\beta_2^2 - \alpha_2\beta_1^2 + \alpha_2 + \alpha_1)\, Re\left(\frac{K\bar{E}^0}{R}\right) - 2\{(\alpha_1^2 - \alpha_2^2) + (\beta_2^2 - \beta_1^2)\, \mathrm{Im}\left(\frac{K\bar{E}^0}{R}\right)}{2\beta_2\{(\alpha_2 - \alpha_1)^2 + (\beta_2^2 - \beta_1^2)\}}. \quad \ldots(51)$$

Solving the equations (49) and (50), we have for $\Phi_0(\zeta)$ and $\psi_0(\zeta)$

$$\Phi_0(\zeta) = \frac{\{s_2(L_2 + \bar{L}_1) - (L_4 + \bar{L}_3)\}\frac{1}{\zeta}}{s_1 - s_2} + \frac{E^0\{s_2(KP - \bar{K}) - i(KP + \bar{K})\}\frac{1}{\zeta}}{s_1 - s_2}. \quad \ldots(52)$$

and

$$\psi_0(\zeta) = \frac{\{s_1(L_2 + \bar{L}_1) - (L_4 + \bar{L}_3)\}\frac{1}{\zeta}}{s_2 - s_1} + \frac{E^0\{s_1(KP - \bar{K}) - i(KP + \bar{K})\}\frac{1}{\zeta}}{s_2 - s_1} \quad \ldots(53)$$

Inverting the relations (35) and (36), we get

$$\zeta = \frac{Z_1 + [Z_1^2 - R^2\{(m+1)^2 + (m-1)^2 s_1^2\}]^{1/2}}{R\{(m+1) - (m-1)\, is_1\}} \quad \ldots(54)$$

and

$$\zeta = \frac{Z_2 + [Z_2^2 - R^2\{(m+1)^2 + (m-1)^1 s_2^2\}]^{1/2}}{R\{(m+1) - (m-1)\, is_2\}} \quad \ldots(55)$$

Inserting for ζ its value from (54) into the function $\Phi_0(\zeta)$ in (52) and the value from (55) into the function $\psi_0(\zeta)$ in (53) and in both cases using the value of the constants given by (51) we get

$$\Phi_0(z_1) = \frac{R\{(m+1) - (m-1)\, is_1\}\{s_2(L_2 + \bar{L}_1) - (L_4 + \bar{L}_3)\}}{(s_1 - s_2)[z_1 + \sqrt{z_1^2 - R^2\{(m+1)^2 + (m-1)^2 s_1^2\}}]}$$

$$+ \frac{R\{(m+1) - (m-1)\, is_1\} E^0\{s_2(KP - \bar{K}) - i(KP + \bar{K})\}}{(s_1 - s_2)[z_1 + \sqrt{z_1^2 - R^2\{(m+1) + (m-1)^2 s_1^2\}})} \quad \ldots(56)$$

and

$$\psi_0(z_2) = \frac{R\{(m+1)-(m-1)is_2\}\{s_1(L_2+\bar{L}_1)^2-(L_4+\bar{L}_3)\}}{(s_2-s_1)[z_2+\sqrt{z_2^2-R^2\{(m+1)^2+(m-1)^2s_2^2\}}]}$$
$$+\frac{R\{(m+1)-(m-1)is_2\}\mathbf{E}^0\{s_1(KP-\bar{K})-i(KP+\bar{K})\}}{(s_2-s_1)[z_2+\sqrt{z_2^2-R^2\{(m+1)^2+(m-1)^2s_2^2\}}]}. \quad \ldots(57)$$

The functions $\Phi(z)_1$ and $\psi(z_2)$ are thus determined and hence the problem is solved.

At infinity there acts the tension T, at an angle α to the X-axis. This condition at infinity gives rise to the values of the constants.

$$B^* = \frac{T[\cos^2\alpha+(\alpha_2^2+\beta_2^2)\sin^2\alpha+\alpha_2\sin 2\alpha] - 2(1+\alpha_2^2-\beta_2^2)\,Re\left(\frac{K\bar{\mathbf{E}}^0}{R}\right) - 4\alpha_2\,\mathrm{Im}\left(\frac{K\bar{\mathbf{E}}^0}{R}\right)}{2\{(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)\}}$$

$$B'^* = \frac{T[\{(\alpha_1^2-\beta_1^2)-2\alpha_1\beta_1\}\sin^2\alpha-\cos^2\alpha-\alpha_2\sin 2\alpha] + 2(1+\alpha_1^2-\beta_1^2-2\alpha_1\alpha_2)\,Re\left(\frac{K\bar{\mathbf{E}}^0}{R}\right)+4\alpha_2\,\mathrm{Im}\left(\frac{K\bar{\mathbf{E}}^0}{R}\right)}{2\{(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)\}}$$

$$C'^* = \frac{\begin{aligned}&T[(\alpha_1-\alpha_2)\cos^2\alpha+\{\alpha_2(\alpha_1^2-\beta_1^2)-\alpha_1(\alpha_2^2-\beta_2^2)\}\sin^2\alpha \\ &+\{(\alpha_1^2-\beta_1^2)-(\alpha_2^2-\beta_2^2)\}\sin\alpha\cos\alpha] \\ &+2(\alpha_1^2\alpha_2-\alpha_1\alpha_2^2+\alpha_1\beta_2^2-\alpha_2\beta_1^2+\alpha_2-\alpha_1)\,Re\left(\frac{K\bar{\mathbf{E}}^0}{R}\right) \\ &-2\{(\alpha_1^2-\alpha_2^2)+(\beta_2^2-\beta_1^2)\}\,\mathrm{Im}\left(\frac{K\bar{\mathbf{E}}^0}{R}\right)\end{aligned}}{2\beta_2\{(\alpha_2-\alpha_1)^2+(\beta_2^2-\beta_1^2)\}} \quad \ldots(58)$$

Solving the equations (49) and (50) we have for $\Phi_0(\zeta)$ and $\psi_0(\zeta)$,

$$\Phi_0(\zeta) = \frac{\{s_2(L_2+\bar{L}_1)-(L_4+\bar{L}_3)\}\frac{1}{\zeta}}{s_1-s_2} + \frac{\mathbf{E}^0\{s_2(KP-\bar{K})-i(KP+\bar{K})\}\frac{1}{\zeta}}{s_1-s_2} \quad \ldots(59)$$

and

$$\psi_0(\zeta) = \frac{\{s_1(L_2+\bar{L}_1)-(L_4+\bar{L}_3)\}\frac{1}{\zeta}}{s_2-s_1} + \frac{\mathbf{E}^0\{s_1(KP-\bar{K})-i(KP+\bar{K})\}\frac{1}{\zeta}}{s_2-s_1}. \quad \ldots(60)$$

Inverting the relations (35) and (36), we get

$$\zeta = \frac{Z_1 + [Z_1^2 - R^2\{(m+1)^2 + (m-1)^2 s_1^2\}]^{1/2}}{R\{(m+1) - (m-1)\, is_1\}} \qquad ...(61)$$

and

$$\zeta = \frac{Z_2 + [Z_2^2 - R^2\{(m+1)^2 + (m-1)^2 s_2^2\}]^{1/2}}{R\{(m+1) - (m-1)\, is_2\}}. \qquad ...(62)$$

Inverting for ζ its value from (61) into the function $\varphi_0(\zeta)$ in (59) and the value from (62) into the function $\psi_0(\zeta)$ in (60) and in both cases using the value of the constants given by (58), we get

$$\varphi_0(z_1) = \frac{R\{(m+1) - (m-1)\, is_1\}\{s_2(L_2 + \bar{L}_1) - (L_4 + \bar{L}_3)\}}{(s_1 - s_2)[Z_1 + \sqrt{Z_1^2 - R^2\{(m+1)^2 + (m-1)^2 s_1^2\}}]}$$

$$= \frac{R\{(m+1) - (m-1)\, is_1\}\, E^0\{s_2(KP - \bar{K}) - i(KP + \bar{K})\}}{(s_1 - s_2)[Z_1 + \sqrt{Z_1^2 - R^2\{(m+1)^2 + (m-1)^2 s_1^2\}}]} \qquad ...(63)$$

and

$$\psi_0(z_2) = \frac{R\{(m+1) - (m-1)\, is_2\}\{s_1(L_2 + \bar{L}_1) - (L_4 + \bar{L}_3)\}}{(s_2 - s_1)[Z_2 + \sqrt{Z_2^2 - R^2\{(m+1)^2 + (m-1)^2 s_2^2\}}]}$$

$$+ \frac{R\{(m+1) - (m-1)\, is_2\}\, E^0\{s_1(KP - \bar{K}) - i(KP + \bar{K})\}}{(s_2 - s_1)[Z_2 + \sqrt{Z_2^2 - R^2\{(m+1)^2 + (m-1)^2 s_2^2\}}]} \qquad ...(64)$$

The function $\varphi(z_1)$ and $\psi(z_2)$ are thus determined and hence, the problem is solved.

DISCUSSION

It is observed that when an elastic solid dielectric having anisotrophic properties is exposed to an electric field it undergoes a small deformation. This small deformation is governed by the constitutive relations and the differential equations similar to the small deformation theory of elasticity. In mechanics of solid, the two-dimensional form of the theory, constitutive relations and the differential equations presented in the foregoing articles will lead to the analysis of stress and deformation for relevant type of problems. In complex variable framework the said theory has been developed with due consideration for practical problems associated with the theory. The final results are highly expository in character as derived in various equations for the distribution of stress and deformation. It will be of great importance in its applications as well as in further development of two-dimensional theory of stress analysis in dielectrics with elliptical holes.

It is noteworthy that the stresses are derived in terms of stress function satisfying the partial differential equation of two-dimensional form. The solution of this equation is derived in the framework of complex variable. The electric field in which the dielectric being exposed is of two-dimensional nature and acts in the $X - Y$ plane. This electric field has caused a new field in the dielectric in addition to orginal field which is called the self component of the field intensity generated within the dielectric medium. This part of the field being always smaller than the original electric field leads to an approximation of equation which enables one to derive the solution in the present form. Finally to derive explicit results for the stresses the analytic functions $\varphi(z_1)$ and $\psi(z_2)$ are to be expressed in the relevant co-ordinate systems such as cartesian, curvilinear, plane-polar and so on and enables one to derive the stress system in the appropriate co-ordinate system. Lastly, we find that the results presented in the foregoing articles are amenable to the corresponding results when the electric field is absent. The present authors propose in their further investigations to derive a similar set of results in the case when the dielectric properties of material is also anisotropic.

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