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INCOMING WATER WAVES AGAINST A VERTICAL CLIFF IN AN OCEAN

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This paper deals with problems of incoming water waves against a vertical cliff in an ocean by employing an alternative mathematical technique. Some new results concerning finite uniform depth of the ocean and the effect of surface tension at the free surface are also obtained.

Key Words : **Incoming Waves; Vertical Cliff; Deep Ocean; Ocean of Finite Depth; Shore Line; Two and Three-Dimensional Waves; Surface Tension**

INTRODUCTION

IN a deep ocean bounded on one side by a vertical cliff, the solutions to the appropriate water wave equations within the framework of linearised theory, which behave as progressive waves moving towards the cliff, were obtained long ago by Stoker[1,2]. Since no reflection of waves at the cliff is assumed a source/sink type behaviour of the potential function there is necessary to account for this and as such the wave amplitude is logarithmically infinite at the shore line. A technique based on the complex variable theory was used to obtain the solution for the two-dimensional problem and a reduction procedure for the three-dimensional problem. However, if the effect of surface-tension at the free-surface is taken into account, the wave amplitude then remains finite at the shore line (cf. Packham[3]) so that the potential function is regular there. Packham[3] extended the two dimensional problem considered earlier by Stoker[1] to include the effect of surface-tension at the free-surface and employed a reduction procedure to solve this problem.

In the present paper, a simple and straightforward technique is used to solve the different problems of incoming waves against a vertical cliff considered earlier by Stoker[1,2] and Packham[3] and also extend their problems to an ocean of finite depth first neglecting the effect of surface tension at the free surface and later including it.

STATEMENT OF THE PROBLEM

A rectangular cartesian co-ordinate system is used in which the y-axis is taken vertically downwards so that the cliff lies in the plane $x = 0$ and the sea is

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contained in the region $x \geqslant 0$, $-\infty < z < \infty$ and $y \geqslant 0$ if it is infinitely deep or $0 \leqslant y \leqslant h$ if it is of uniform finite depth h. The origin is taken at a point on the shore line and $y = 0$, $x \geqslant 0$ is the undisturbed position of the free surface. Under the assumption of inviscid, incompressible sea water and irrotational motion under gravity only, a velocity potential $\varphi(x, y, z, t)$ exists which satisfies the Laplace's equation in the ocean region and other appropriate conditions at the free surface, the cliff, the bottom etc. We seek a solution for $\varphi(x, y, z, t)$ which behaves at infinity like a progressive wave moving towards the cliff. We consider both the two and three-dimensional waves moving towards the cliff in a deep ocean and then in an ocean of uniform finite depth in the following four sections. Solutions are given first neglecting the effect of surface tension at the free surface and later including it.

Two Dimensional Waves Against a Vertical Cliff in a Deep Ocean

In this case the potential function is independent of the z-coordinate. We set

$$\varphi(x, y, t) = Re\,\{\Phi(x, y) \exp(-i\sigma t)\}.$$

Then the equation of continuity gives

$$\nabla^2\Phi = 0 \text{ in the ocean region,} \qquad \ldots(1)$$

where ∇^2 is the two-dimensional Laplacian operator. The linearised free surface condition is

$$K\Phi + \Phi_y = 0 \text{ on } y = 0,\ x > 0 \qquad \ldots(2)$$

where $K = \dfrac{\sigma^2}{g}$, g being the gravity. The condition at the cliff is

$$\Phi_x = 0 \text{ on } x = 0,\ y > 0. \qquad \ldots(3)$$

Since at infinity, φ behaves as a progressive wave moving towards the cliff, we must have

$$\Phi \sim \exp(-Ky - iKx) \text{ as } x \to \infty. \qquad \ldots(4)$$

At the shore line (i.e., at $x = 0$, $y = 0$) Φ has a logarithmic singularity so that

$$\Phi \sim \ln r \text{ as } r = (x^2 + y^2)^{1/2} \to 0. \qquad \ldots(5)$$

Also at infinite depth of the ocean the condition of no motion gives

$$\nabla\Phi \to 0 \text{ as } y \to \infty. \qquad \ldots(6)$$

To solve for $\Phi(x, y)$, we set

$$\Phi(x, y) = 2\exp(-Ky)\cos Kx + \psi(x, y), \qquad \ldots(7)$$

where now $\psi(x, y)$ satisfies

$$\nabla^2\psi = 0 \text{ in the ocean region,} \qquad \ldots(8)$$

$$K\psi + \psi_y = 0 \quad \text{at } y = 0 \quad \text{for } x > 0, \qquad \ldots(9)$$

$$\psi_x = 0 \text{ at } x = 0 \text{ for } y > 0, \qquad \ldots(10)$$

$$\psi \to -\exp(-Ky + iKx) \text{ as } x \to \infty, \quad \ldots(11)$$

$$\psi \to \ln r \text{ as } r \to 0 \quad \ldots(12)$$

and $$\nabla\psi \to 0 \text{ as } y \to \infty. \quad \ldots(13)$$

An appropriate solution satisfying (8)-(10), (12) and (13) is

$$\psi(x, y) = c \int_0^\infty \frac{\exp(-ky)}{k - K} \cos kx \, dk, \quad \ldots(14)$$

where the contour is indented below the pole at $k = K$ and, c is a constant to be chosen such that the condition (11) is satisfied. We may note here that $\psi(x, y)$ has a logarithmic singularity at $r = 0$ (cf. Yu and Ursell[4]). Now an alternative representation of $\psi(x, y)$ is given by

$$\psi(x, y) = c \left[\int_0^\infty \frac{\exp(-kx)}{k^2 + K^2} (k \cos ky - K \sin ky) \, dk + \pi i \exp(-Ky + iKx) \right], \; x > 0. \quad \ldots(15)$$

Thus to satisfy (11), we must take

$$c = \frac{i}{\pi}. \quad \ldots(16)$$

Hence, from (7), using (15) and (16) we obtain

$$\Phi(x, y) = \exp(-Ky - iKx) + \frac{i}{\pi} \int_0^\infty \frac{\exp(-kx)}{k^2 + K^2} (k \cos ky - K \sin ky) \, dk, \quad \ldots(17)$$

so that

$$\varphi(x, y, t) = \exp(-Ky) \cos(Kx + \sigma t) + \pi \sin \sigma \cdot t \int_0^\infty \frac{\exp(-kx)}{k^2 + K^2} \times (k \cos ky - K \sin ky) \, dk \quad \ldots(18)$$

This result was obtained earlier by Stoker[1] using a different method.

If we now include the effect of surface tension at the free surface, then the conditions (2) and (5) are replaced respectively by (cf. Packham[3])

$$K\Phi + \Phi_y + M\Phi_{yyy} = 0 \; y = 0, \; x > 0 \quad \ldots(2')$$

and

$$\Phi \text{ remains finite as } r \to 0. \quad \ldots(5')$$

Also, in place of (4), we must take in this case

$$\Phi \to \exp(-k_0 y - ik_0 x) \text{ as } x \to \infty, \quad \ldots(4')$$

where k_0 is the real positive zero of the cubic equation

$$k(1 + Mk^2) - K = 0 \quad \ldots(19)$$

and $M = T/\rho g$, T being the co-efficient of surface tension and ρ being the density of the sea water.

As in (7), we set here

$$\Phi(x, y) = 2 \exp(-k_0 y) \cos k_0 x + \psi(x, y), \quad \ldots(20)$$

then ψ satisfies (8), (10), (13) together with

$$K\psi + \psi_y + M\psi_{yyy} = 0 \; y = 0, x > 0, \quad \ldots(9')$$

$$\psi \to -\exp(-k_0 y + ik_0 x) \text{ as } x \to \infty, \quad \ldots(11')$$

$$\psi \text{ remains finite as } r \to 0. \quad \ldots(12')$$

An appropriate solution for $\psi(x, y)$ satisfying (8), (9′), (10), (12′) and (13) is

$$\psi(x, y) = c \int_0^\infty \frac{\exp(-ky) \cos kx}{k(1 + Mk^2) - K} dk, \quad \ldots(21)$$

where the contour is indented below the pole at $k = k_0$ and c is a constant to be chosen such that (11′) is satisfied.

An alternative representation of $\psi(x, y)$ for $x > 0$ is given by

$$\psi(x, y) = c \left[\int_0^\infty \frac{\exp(-kx)}{k^2(1 - Mk^2)^2 + K^2} \{k(1 - Mk^2) \cos ky - K \sin ky\} dk + \frac{\pi i}{1 + 3Mk_0^2} \exp(-k_0 y + ik_0 x) \right]. \quad \ldots(22)$$

We may note here that so long as $M \neq 0$, $\psi(x, y)$ remains finite as $r \to 0$.

To satisfy (11′) we must take

$$c = \frac{i}{\pi}(1 + 3Mk_0^2). \quad \ldots(23)$$

Hence from (20), using (22) and (23) we obtain

$$\Phi(x, y) = \exp(-k_0 y - ik_0 x) + \frac{i}{\pi}(1 + 3Mk_0^2) \int_0^\infty \frac{\exp(-kx)}{k^2(1 - Mk^2)^2 + K^2} \times \{k(1 - Mk^2) \cos ky - K \sin ky\} dk, \quad \ldots(24)$$

so that

$$\varphi(x, y, t) = \exp(-k_0 y)\cos(k_0 x + \sigma t) + \frac{1 + 3Mk_0^2}{\pi}\sin\sigma t$$

$$\times \int_0^\infty \frac{\exp(-kx)\{k(1 - Mk^2)\cos ky - K\sin ky\}}{k^2(1 - Mk^2)^2 + K^2}\, dk. \quad ...(25)$$

This result was earlier obtained by Packham[3] employing a different method.

Three-Dimensional Waves Against a Vertical Cliff in a Deep Ocean

In this case we seek a solution which represents progressive waves moving towards the shore line (z-axis on the cliff) such that the wave crests at large distances from the shore tend to a straight line making an angle α with the shore line. We may then take

$$\varphi(x, y, z, t) = Re\,\{\Phi(x, y)\exp(-i\sigma t - iKz\sin\alpha)\}. \quad ...(26)$$

It may be noted here that if $\alpha = 0$, this problem reduces to the two-dimensional problem discussed earlier.

Now $\Phi(x, y)$ satisfies

$$(\nabla^2 - \nu^2)\Phi = 0 \text{ in the ocean region} \quad ...(27)$$

where

$$\nu = K\sin\alpha, \quad ...(28)$$

$$K\Phi + \Phi_y = 0 \text{ on } y = 0,\ x > 0. \quad ...(29)$$

$$\Phi_x = 0 \text{ on } x = 0,\ y > 0, \quad ...(30)$$

$$\Phi \to \exp(-Ky - iKx\cos\alpha) \text{ as } x \to \infty. \quad ...(31)$$

$$\Phi \to \ln r \text{ as } r \to 0, \quad ...(32)$$

$$\nabla\Phi \to 0 \text{ as } y \to \infty. \quad ...(33)$$

Setting

$$\Phi(x, y) = 2\exp(-Ky)\cos(Kx\cos\alpha) + \psi(x, y), \quad ...(34)$$

we obtain the following boundary value problem for $\psi(x, y)$ as

$$(\nabla^2 - \nu^2)\,\psi = 0 \text{ in the ocean region}, \quad ...(35)$$

$$K\psi + \psi_y = 0 \text{ for } y = 0,\ x > 0, \quad ...(36)$$

$$\psi_x = 0 \text{ for } x = 0,\ y > 0, \quad ...(37)$$

$$\psi \to -\exp(-Ky + iKx\cos\alpha) \text{ as } x \to \infty, \quad ...(38)$$

$$\psi \to \ln r \text{ as } r \to 0 \quad ...(39)$$

and

$$\nabla\psi \to 0 \text{ as } y \to \infty. \quad ...(40)$$

An appropriate solution for $\psi(x, y)$ in this case is

$$\psi(x, y) = c \int_{\nu}^{\infty} \frac{k \exp(-ky) \cos \zeta x}{(k-K)\zeta} \, dk, \qquad \ldots(41)$$

where $\zeta = (k^2 - \nu^2)^{1/2}$ and that branch of ζ is chosen for which $\zeta > 0$ for $k > \nu$, and the contour is indented below the pole at $k = K$.

For small r, $\psi(x, y)$ behaves as $\ln r$ (cf. Ursell[5], equation (2.19).

An alternative representation for $\psi(x, y)$ is

$$\psi(x, y) = c \left[\int_0^{\infty} \frac{k(k \cos ky - K \sin ky)}{(k^2+K^2)(k^2+\nu^2)^{1/2}} \exp\{-(k^2+\nu^2)^{1/2}x\} \, dk + \frac{\pi i}{\cos \alpha} \exp\{-Ky + iKx \cos \alpha\} \right], \; x > 0. \qquad \ldots(42)$$

To satisfy (38), we must take

$$c = \frac{i \cos \alpha}{\pi}. \qquad \ldots(43)$$

Thus from (34), using (42), (43) we obtain,

$$\Phi(x, y) = \exp(-Ky - iKx \cos \alpha) + \frac{i \cos \alpha}{\pi} \int_0^{\infty} \frac{k(k \cos ky - K \sin ky)}{(k^2+K^2)(k^2+\nu^2)^{1/2}} \times \exp\{-(k^2+\nu^2)^{1/2}x\} \, dk, \qquad \ldots(44)$$

so that

$$\varphi(x, y, z, t) = \exp(-Ky) \cos(Kx \cos \alpha + Kz \sin \alpha + \sigma t) + \frac{\cos \alpha}{\pi} \sin(Kz \sin \alpha + \sigma t) \int_0^{\infty} \frac{k(k \cos ky - K \sin ky)}{(k^2+K^2)(k^2+\nu^2)^{1/2}} \times \exp\{-(k^2+\nu^2)^{1/2} x\} \, dk. \qquad \ldots(45)$$

The result obtained by Stoker[1] using a different approach also leads to (45). The result there was given in the following form

$$\varphi(x, y, z, t) = \varphi_1(x, y) \cos(Kz \sin \alpha + \sigma t) + \frac{\cos \alpha}{2} \phi_2(x, y) \sin(Kz \sin \alpha + \sigma t), \qquad \ldots(46)$$

where $\varphi_1(x, y) = \exp(-Ky) \cos(Kx \cos \alpha)$.

$$\varphi_2(x, y) = \frac{2K}{\pi} \exp(-Ky) \int_{\infty}^{-y} \exp(-Ks) K_0\{\nu(x^2+s^2)^{1/2}\} \, ds + \frac{2}{\pi} K_0\{\nu(x^2+y^2)^{1/2}\} - \frac{2\exp(-Ky)}{\cos \alpha} \sin(Kx \cos \alpha),$$

where $K_0(z)$ is the modified Bessel function of second kind. Using the representation

$$K_0\{\nu(x^2+y^2)^{1/2}\} = \int_0^\infty \frac{\exp\{-(k^2+\nu^2)^{1/2}x\}}{(k^2+\nu^2)^{1/2}} \cos ky \, dk, \; y > 0$$

in $\varphi_2(x, y)$ and after simplifying we obtain

$$\varphi_2(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\exp\{-(k^2+\nu^2)^{1/2}x\}}{(k^2+\nu^2)^{1/2}} \cdot \frac{k(k\cos ky - K\sin ky)}{(k^2+K^2)}\, dk - \frac{2\exp(-Ky)}{\cos\alpha}\sin(Kx\cos\alpha).$$

Substituting these in (46), we recover the form (45).

If the effect of surface tension at the free surface is included, then in this case we seek solution in the form

$$\varphi(x, y, z, t) = Re\{\Phi(x, y)\exp(-i\sigma t - ik_0 z\sin\alpha)\} \quad \ldots(26')$$

where k_0 is the same in (19).

The equations (27)–(29), (31) and (32) are replaced respectively by

$$(\nabla^2 - \nu_0^2)\Phi = 0, \text{ for } y > 0, x > 0. \quad \ldots(27')$$

with $\nu_0 = k_0\sin\alpha,$ ...(28')

$$K\Phi + \Phi_y + M\Phi_{yyy} = 0 \text{ for } y = 0, x > 0, \quad \ldots(29')$$

$$\Phi \to \exp(-k_0 y - ik_0 x\cos\alpha) \text{ as } x \to \infty. \quad \ldots(31')$$

$$\Phi \text{ remains finite as } r \to 0. \quad \ldots(32')$$

Setting

$$\Phi(x, y) = 2\exp(-k_0 y)\cos(k_0 x\sin\alpha) + \psi(x, y), \quad \ldots(47)$$

we see that $\psi(x, y)$ satisfies (27′), (29′), (30), (32′) and (33) with Φ replaced by ψ and $\psi \to -\exp(-k_0 y + ik_0 x\cos\alpha)$ as $x \to \infty$. ...(38′)

An appropriate solution for $\psi(x, y)$ in this case (cf. Rhodes-Robinson[6]) is

$$\psi(x, y) = c\int_{\nu_0}^\infty \frac{k\exp(-ky)}{k(1+Mk^2)-K} \cdot \frac{\cos(k^2-\nu_0^2)^{1/2}x}{(k^2-\nu_0^2)^{1/2}}\, dk, \quad \ldots(48)$$

where the contour is indented below the pole at $k = k_0$. An alternative representation for $\psi(x, y)$ is given by

$$\psi(x, y) = c\left[\frac{\pi i}{(1+3Mk_0^2)\cos\alpha}\exp(-k_0 y + ik_0 x\cos\alpha\right.$$

equation contd.

$$+\int_0^\infty \frac{k}{k^2(1-Mk^2)^2+K^2}\cdot\frac{\exp\{-(k^2+\nu_0^2)^{1/2}x\}}{(k^2+\nu_0^2)^{1/2}}$$

$$\times\{k(1-Mk^2)\cos ky - K\sin ky\}\,dk. \qquad \ldots(49)$$

As in the previous case, we may note that so long as $M \neq 0$, ψ remains finite at $r = 0$. Noting the behaviour of ψ as $x \to \infty$, we must take

$$c = \frac{i\cos\alpha(1+3Mk_0^2)}{\pi}. \qquad \ldots(50)$$

Thus using (47), (49), (50) we obtain

$$\Phi(x, y) = \exp(-k_0y - ik_0x\cos\alpha) + \frac{i\cos\alpha\,(1+3Mk_0^2)}{\pi}$$

$$\times\int_0^\infty \frac{k\exp\{-(k^2+\nu^2)^{1/2}x\}\{k(1-Mk^2)\cos ky - K\sin ky\}}{\{k^2(1-Mk^2)^2+K^2\}(k^2+\nu_0^2)^{1/2}}\,dk,$$

$$\ldots(51)$$

so that

$$\varphi(x, y, z, t) = \exp(-k_0y)\cos(k_0x\cos\alpha + k_0z\sin\alpha + \sigma t)$$

$$+\frac{\cos\alpha(1+3Mk_0^2)}{\pi}\sin(\sigma t + k_0z\sin\alpha)$$

$$\times\int_0^\infty \frac{k\exp\{-(k^2+\nu_0^2)^{1/2}x\}\{k(1-Mk^2)\cos ky - K\sin ky\}}{\{k^2(1-Mk^2)^2+K^2\}(k^2+\nu_0^2)^{1/2}}\,dk$$

$$\ldots(52)$$

If we put $M = 0$, we recover the result (45).

Two-Dimensional Waves Against a Vertical Cliff in Ocean of Finite Depth

In this case we take the potential function as

$$\varphi(x, y, t) = Re\{\Phi(x, y)\exp(-i\sigma t)\}, \qquad \ldots(53)$$

where $\Phi(x, y)$ satisfies

$$\nabla^2\Phi = 0 \quad 0 \leqslant y \leqslant h,\ x \geqslant 0, \qquad \ldots(54)$$

$$K\Phi + \Phi_y = 0 \text{ for } y = 0,\ x > 0, \qquad \ldots(55)$$

$$\Phi_x = 0 \text{ for } x = 0,\ 0 < y < h, \qquad \ldots(56)$$

$$\Phi \to \ln r \text{ as } r \to 0, \qquad \ldots(57)$$

$$\Phi_y = 0 \text{ on } y = h, \qquad \ldots(58)$$

and

$$\Phi \to \frac{\cosh k_0(h-y)}{\cosh k_0h} \exp(-ik_0x) \text{ as } x \to \infty, \qquad \text{...(59)}$$

where now k_0 is the real positive zero of $k \tanh kh - K = 0$.
As before we write

$$\Phi(x, y) = \frac{2\cosh k_0(h-y)}{\cosh k_0h} \cos k_0x + \psi(x, y). \qquad \text{...(60)}$$

Then $\psi(x, y)$ satisfies (54)–(58) with Φ replaced by ψ and

$$\psi \to -\frac{\cosh k_0(h-y)}{\cosh k_0h} \exp(-ik_0x) \text{ as } x \to \infty. \qquad \text{...(61)}$$

Now an appropriate solution of $\psi(x, y)$ is given by

$$\psi(x, y) = c \int_0^\infty \frac{\cosh k(h-y)}{k \sinh kh - K \cosh kh} \cos kx \, dk \qquad \text{...(62)}$$

where the contour is indented below the pole at $k = k_0$. ψ has the alternative representation

$$\psi(x, y) = c \left[2\pi i \frac{\cosh k_0h \cosh k_0(h-y)}{2k_0h + \sinh 2k_0h} \exp(ik_0x) + 2\pi \sum_{n=1}^{\infty} \frac{\cos k_nh \cos k_n(h-y)}{2k_nh + \sin 2k_nh} \exp(-k_nx) \right], \qquad \text{...(63)}$$

where k_n's satisfy $k_n \tan k_nh + K = 0$.

This ψ has a logarithmic singularity as $r \to 0$ (cf. Yu and Ursell[4]). In order to satisfy (61), we must choose

$$c = \frac{i}{\pi} \frac{2k_0h + \sinh 2k_0h}{1 + \cosh 2k_0h}. \qquad \text{...(64)}$$

Thus $\psi(x, y)$ and hence $\Phi(x, y)$ is obtained. Using these we obtain finally

$$\varphi(x, y, t) = \frac{\cosh k_0(h-y)}{\cosh k_0h} \cos(k_0x + \sigma t) + 2 \frac{2k_0h + \sinh 2k_0h}{1 + \cosh 2k_0h} \sin \sigma t \times \sum_{n=1}^{\infty} \frac{\cos k_nh \cos k_n(h-y)}{2k_nh + \sin 2k_nh} \exp(-k_nx), \; x > 0. \qquad \text{...(65)}$$

If the effect of surface tension at the free surface is taken into account, we obtain similarly

$$\varphi(x, y, t) = \frac{\cosh \alpha_0 (h - y)}{\cosh \alpha_0 h} \cos(\alpha_0 x + \sigma t)$$

$$+ 2 \sin \sigma t \frac{2\alpha_0 h(1 + M\alpha_0^2) + (1 + 3M\alpha_0^2) \sinh 2\alpha_0 h}{1 + \cosh 2\alpha_0 h}$$

$$\times \sum_{n=1}^{\infty} \frac{\cos \alpha_n h \cos \alpha_n(h - y) \exp(-\alpha_n x)}{2\alpha_n h (1 - M\alpha_n^2) + (1 - 3M\alpha_n^2) \sin 2\alpha_n h}, \; x > 0, \quad \ldots(66)$$

where α_0 is the positive root of

$$k(1 + Mk^2) \text{ than } kh = K \quad \ldots(67)$$

and α_n's satisfy

$$\alpha_n(1 - M\alpha_n^2) \tan \alpha_n h + K = 0. \quad \ldots(68)$$

By making $M = 0$ in (66) we recover the corresponding result in the absence of surface tension, given by (65).

Three-Dimensional Waves Against a Vertical Cliff in Ocean of Finite Depth

Here we take the potential function as

$$\varphi(x, y, z, t) = \text{Re}\, \{\Phi(x, y) \exp(-i\sigma t - ik_0 z \sin \alpha)\}, \quad \ldots(69)$$

where k_0 is the real positive root of $k \tanh kh = K$.

The boundary value problem in $\Phi(x, y)$ is

$$(\nabla^2 - \nu_0^2)\, \Phi = 0,\; 0 \leqslant y, \leqslant h,\; x \geqslant 0 \quad \ldots(70)$$

with $\nu_0 = k_0 \sin \alpha$,

$$K\Phi + \Phi_y = 0 \;\; y = 0,\; x > 0, \quad \ldots(71)$$

$$\Phi_x = 0 \qquad x = 0,\; 0 < y < h, \quad \ldots(72)$$

$$\Phi \sim \ln r \;\text{ as }\; r \to 0, \quad \ldots(73)$$

$$\Phi_y = 0 \;\text{ on }\; y = h, \quad \ldots(74)$$

$$\Phi \to \frac{\cosh k_0(h - y)}{\cosh k_0 h} \exp(-ik_0\, x \cos \alpha) \text{ as } x \to \infty. \quad \ldots(75)$$

If we set

$$\Phi(x, y) = \frac{2 \cosh k_0(h - y)}{\cosh k_0 h} \cos(k_0 x \cos \alpha) + \psi(x, y), \quad \ldots(76)$$

then $\psi(x, y)$ satisfies (70)-(74) in which Φ is replaced by ψ and

$$\psi \to -\frac{\cosh k_0(h - y)}{\cosh k_0 h} \exp(ik_0\, x \cos \alpha) \text{ as } x \to \infty. \quad \ldots(77)$$

In this case an appropriate solution for $\psi(x, y)$ is (cf. Ursell[5])

$$\psi(x, y) = c \int_{\nu}^{\infty} \frac{k \cosh k(h - y)}{k \sinh kh - K \cosh kh} \cdot \frac{\cos(k^2 - \nu^2)^{1/2}x}{(k^2 - \nu^2)^{1/2}} \, \mathrm{d}k \qquad \ldots(78)$$

where the contour is indented below the pole at $k = k_0$. This has the alternative representation

$$\psi(x, y) = c \left[\frac{2\pi i}{\cos\alpha} \frac{\cosh k_0 h \cosh k_0(h - y)}{2k_0 h + \sinh 2k_0 h} \exp(ik_0 x \cos\alpha) + 2\pi \sum_{n=1}^{\infty} \frac{k_n \cos k_n h \cos k_n(h-y)}{2k_n h + \sin 2k_n h} \cdot \frac{\exp\{-(k_n^2 + \nu^2)^{1/2} x\}}{(k_n^2 + \nu^2)^{1/2}} \right],$$

$$x > 0. \qquad \ldots(79)$$

Noting the behaviour of $\psi(x, y)$ as $x \to \infty$, we obtain

$$c = \frac{i \cos\alpha}{\pi} \frac{2k_0 h + \sinh 2k_0 \mathrm{h}}{1 + \cosh 2k_0 h}. \qquad \ldots(80)$$

Thus $\psi(x, y)$ and hence $\Phi(x, y)$ is obtained. Hence

$$\varphi(x, y, z, t) = \frac{\cosh k_0(h - y)}{\cosh k_0 h} \cos(k_0 x \cos\alpha + k_0 z \sin\alpha + \sigma t) + 2\cos\alpha \frac{2k_0 h + \sinh 2k_0 h}{1 + \cosh 2k_0 h} \sin(k_0 z \sin\alpha + \sigma t) \times \sum_{n=1}^{\infty} \frac{k_n \cos k_n h \cos k_n(h - y)}{2k_n h + \sin 2k_n h} \frac{\exp\{-(k_n^2 + \nu^2)^{1/2} x\}}{(k_n^2 + \nu^2)^{1/2}},$$

$$x > 0. \qquad \ldots(81)$$

If the effect of surface tension is included then following a similar analysis we obtain,

$$\varphi(x, y, z, t) = \frac{\cosh\alpha_0(h - y)}{\cosh\alpha_0 h} \cos(\alpha_0 x \cos\alpha + \alpha_0 z \sin\alpha + \sigma t) + 2\cos\alpha \frac{2\alpha_0 h(1 + M\alpha_0^2) + (1 + 3M\alpha_0^2)\sinh 2\alpha_0 h}{1 + \cosh 2\alpha_0 h} \times \sin(\alpha_0 z \sin\alpha + \sigma t) \sum_{n=1}^{\infty} \frac{\alpha_n \cos\alpha_n (h - y) \exp\{-(\alpha_n^2 + \nu_0^2)^{1/2} x\}}{2\alpha_n h(1 - M\alpha_n^2) + (1 - 3M\alpha_n^2)\sin 2\alpha_n h} \times \frac{1}{(\alpha_n^2 + \nu_0^2)^{1/2}}, \qquad \ldots(82)$$

where $\nu_0 = \alpha_0 \sin\alpha$ and α_0, α_n satisfy (67) and (68) respectively. By the substitution of $M = 0$ in (82), (81) is recovered.

Conclusion

We have obtained solutions to the problems of incoming surface water waves against a vertical cliff for a number of situations. Both the two and three dimensional cases and the deep ocean and the ocean of uniform finite depth have been considered. The effect of surface tension is also included. Results already obtained by Stoker[1] and Packham[3] are deduced.

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References

1 J J Stoker *Quart Appl Maths* **5** (1947) 1
2 J J Stoker *Water Waves* (1957) Interscience
3 B A Packham *Proc Camb phil Soc* **64** (1968) 827
4 YS Yu and F Ursell *J Fluid Mech* **11** (1961) 529
5 F Ursell *J Fluid Mech* **14** (1962) 496
6 P F Rhodes-Robinson *Proc Camb Phil Soc* **75** (1974) 405