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GENERATION OF WATER WAVES BY AN OSCILLATING LINE SOURCE IN THE PRESENCE OF A NEARLY VERTICAL CLIFF

B N MANDAL

Physical and Earth Sciences Division, Indian Statistical Institute, 203 B.T. Road, Calcutta 700 035, *India*

AND

P K KUNDU

Department of Applied Mathematics, Vidyasagar University, Midnapore 721 101, *West Bengal, India*

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A technique based on perturbational analysis is employed to study the problem of an oscillating line source in the presence of a fixed nearly vertical cliff in deep water assuming linear theory. Analytical expressions for the first order corrections to the amplitude of the radiated waves at infinity as well as the velocity potential are derived in terms of integrals involving the shape of the cliff. Asymptotic expansion to the first order correction to the wave amplitude at infinity when the source is situated at a large distance from the cliff is calculated.

Key Words: Surface Water Waves; Nearly Vertical Cliff; Velocity Potential; Wave Amplitude; Perturbational Analysis; Linear Theory

Introduction

The problem of generation of water waves due to a harmonically oscillating line source in the presence of an obstacle was considered in the literature for some simple cases. For example, Evans[1] considered the case when the obstacle is in the form of a partially immersed thin vertical plate while Basu and Mandal[2] considered the corresponding complementary problem. Also the case of a submerged finite vertical plate was considered recently by Mandal[3]. In all these cases the wave amplitude at infinity was evaluated explicitly. Evans[4] also considered a submerged infinitely long horizontal circular cylinder and arrived at the remarkable result that it does not affect the line source.

Water wave scattering problems involving nearly vertical fixed barriers has attracted the attention of research workers recently (cf. Shaw[5], Mandal and Chakrabarti[6], Mandal and Kundu[7]), wherein the problems were solved approximately by a perturbational analysis. The problem of incoming water waves against a corrugated vertical cliff was considered by Chakrabarti[8], but his method cannot be used for any arbitrary shape of the barrier as the modified boundary condition used by him is valid only for small frequency of corrugation.

The present problem is concerned with the generation of water waves by a pulsating line source in the presence of a nearly vertical cliff. When the cliff is perfectly vertical, its effect on the source is equivalent to another source situated at the image point of the original source with respect to the vertical cliff. However, because of curved nature of the cliff, there will be other contributions. In the present paper, we use a perturbational analysis (cf. Mandal and Chakrabarti[6], Mandal and Kundu[7]) to find these contributions, at least the first term of the contributions to the potential function. Considering two particular shapes of the nearly vertical cliff, the first order corrections to the wave amplitude at infinity, and the velocity potential are obtained in terms of integrals involving the shape of the cliff. The integral representing the correction to the wave amplitude at infinity is evaluated asymptotically, when the source is situated at a large distance horizontally from the cliff. The problem for the second order correction is also formulated and can be solved in principle as the solution of the first order problem is known.

Statement and Formulation

A rectangular cartesian co-ordinate system is used in which the y-axis is taken vertically downwards into the fluid medium. The undisturbed free surface of the fluid being the line $y=0$ and the position of the nearly vertical cliff is taken as S: $x=\varepsilon c(y)$, $0<y<\infty$ where ε is a small dimensionless quantity and $c(y)$ is bounded for $0<y<\infty$ satisfying $c(0)=0$. We assume that a harmonically oscillating line source of unit strength and of circular frequency σ is present in the fluid at the point (ξ, η) with $\xi, \eta>0$. It is further assumed that the fluid is homogeneous, incompressible and inviscid, and the resulting motion is irrotational so that a velocity potential exists. As the motion of the fluid is produced by the harmonically oscillating line source, the motion will be everywhere harmonic of frequency σ so that the velocity potential is $\Phi(x, y, t)=Re\{\varphi(x, y)\exp(-i\sigma t)\}$. Within the framework of linearised theory of water waves, $\varphi(x, y)$ satisfies

$$\nabla^2\varphi=0 \text{ everywhere in the fluid region except at } (\xi, \eta), \quad \ldots (1)$$

$$K\varphi+\frac{\partial\varphi}{\partial y}=0 \text{ on } y=0, x>0, \quad \ldots (2)$$

with $K=\sigma^2/g$, g being gravity,

$$\frac{\partial\varphi}{\partial n}=0 \text{ on } S: x=\varepsilon c(y), y>0, \quad \ldots (3)$$

$\frac{\partial}{\partial n}$ denoting outward normal derivative to the surface of the cliff,

$$\varphi\sim\ln r \text{ as } r=\{(x-\xi)^2+(y-\eta)^2\}^{1/2}\to 0, \quad \ldots (4)$$

$$\varphi, \nabla\varphi\to 0 \text{ as } y\to\infty, \quad \ldots (5)$$

and as $x\to\infty$, φ represents outgoing wave so that

$$\varphi\sim A\exp(-Ky+iKx) \text{ as } x\to\infty,$$

where A is the amplitude of the radiated waves at infinity.

Assuming $\varepsilon\ll 1$ and neglecting $0(\varepsilon^3)$ terms, the boundary condition (3) can be expressed approximately as

$$\frac{\partial \varphi}{\partial x}(0, y) - \varepsilon \frac{\partial}{\partial y}\left\{c(y)\frac{\partial \varphi}{\partial y}(0, y)\right\} + \frac{\varepsilon^2}{2}\left[\{c(y)\}^2 \frac{\partial^3 \varphi}{\partial x^3}(0, y)\right.$$

$$\left. - 2c(y)c'(y)\frac{\partial^2 \varphi}{\partial x \partial y}(0, y) - \{c'(y)\}^2 \frac{\partial \varphi}{\partial x}(0, y)\right] + 0(\varepsilon^3) = 0$$

$$\text{for } 0 < y < \infty. \qquad \ldots (7)$$

This form of boundary condition suggests that the velocity potential $\varphi(x, y)$ and the unknown complex amplitude A may be expanded in terms of the small parameter ε as

$$\varphi(x, y, \varepsilon) = \varphi_0(x, y) + \varepsilon\varphi_1(x, y) + \varepsilon^2\varphi_2(x, y) + 0(\varepsilon^3) \qquad \ldots (8)$$

and

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + 0(\varepsilon^3). \qquad \ldots (9)$$

Substituting the expansions (8) and (9) into the original boundary value problem described by (1) to (6), equating co-efficients of like powers of ε from both sides of all the results derived thus, we find that φ_0, φ_1 and φ_2 must be the solution of the following three boundary value problems given by P_0, P_1 and P_2.

BVP-P_0: The problem is to determine the function $\varphi_0(x, y)$ satisfying

$\nabla^2\varphi_0 = 0$ in the fluid region except at (ξ, η),

$K\varphi_0 + \dfrac{\partial \varphi_0}{\partial y} = 0$ on $y = 0, x > 0$,

$\dfrac{\partial \varphi_0}{\partial x} = 0$ on $x = 0, 0 < y < \infty$,

$\varphi_0 \sim \ln r$ as $r \to 0$,

$\varphi_0, \nabla\varphi_0 \to 0$ as $y \to \infty$,

$\varphi_0 \sim A_0 \exp(-Ky + iKx)$ as $x \to \infty$.

BVP-P_1: To determine $\varphi_1(x, y)$ satisfying

$\nabla^2\varphi_1 = 0$ in the region $x > 0, y > 0$,

$K\varphi_1 + \dfrac{\partial \varphi_1}{\partial y} = 0$ on $y = 0, x > 0$,

$\dfrac{\partial \varphi_1}{\partial x}(0, y) = \dfrac{\partial}{\partial y}\left\{c(y)\dfrac{\partial \varphi_0}{\partial y}(0, y)\right\} = f(y)$, say, on $x = 0, 0 < y < \infty$,

$\varphi_1, \nabla\varphi_1 \to 0$ as $y \to \infty$,

$\varphi_1 \sim A_1 \exp(-Ky + iKx)$ as $x \to \infty$.

BVP-P$_2$: The problem is to determine $\varphi_2(x, y)$ which satisfies

$\nabla^2\varphi_2 = 0$ everywhere in the fluid region,

$$K\varphi_2 + \frac{\partial \varphi_2}{\partial y} = 0 \text{ on } y=0, x>0,$$

$$\frac{\partial \varphi_2}{\partial x}(0, y) = \frac{\partial}{\partial y}\left\{c(y)\frac{\partial \varphi_1}{\partial y}(0, y)\right\} - \frac{1}{2}\left[\{c(y)\}^2 \frac{\partial^3 \varphi_0}{\partial x^3}(0, y)\right.$$

$$\left. - 2c(y)\frac{\partial^2 \varphi_0}{\partial x \partial y}(0, y) - \{c'(y)\}^2 \frac{\partial \varphi_0}{\partial x}(0, y)\right] = g(y),$$

say, on $x=0, 0<y<\infty$,

$\varphi_2, \nabla\varphi_2 \to 0$ as $y \to \infty$,

$\varphi_2 \sim A_2 \exp(-Ky + iKx)$ as $x \to \infty$.

Solution of the Problem

As the source potential in the absence of the cliff is given by (cf. Thorne[9])

$$G(x, y; \xi, \eta) = 2\pi i \exp\{-K(y+\eta) + iK|x-\xi|\} + 2\int_0^\infty \frac{\exp(-k|x-\xi|)}{k(k^2+K^2)}$$

$$\cdot (k\cos ky - K\sin ky)(k\cos k\eta - K\sin k\eta)\,dk, \qquad \ldots (10)$$

$\varphi_0(x, y)$ is obtained as

$$\varphi_0(x, y) = G(x, y; \xi, \eta) + G(x, y; -\xi, \eta). \qquad \ldots (11)$$

Hence using (10), we find

$$\varphi_0(x, y) \sim 4\pi i \exp\{-k(y+\eta) + iKx\}\cos K\xi \text{ as } |x \to \infty. \qquad \ldots (12)$$

Thus A_0 is given by

$$A_0 = 4\pi i \exp(-K\eta)\cos K\xi. \qquad \ldots (13)$$

Solution for BVP-P$_1$: Using Havelock's[10] expansion of water wave potential we can represent $\varphi_1(x, y)$ as

$$\varphi_1(x, y) = A_1 \exp(-Ky + iKx) + \int_0^\infty A(l)(l\cos ly - K\sin ly)\exp(-lx)\,dl, \; x>0. \qquad \ldots (14)$$

Hence, using the third condition in BVP-P$_1$, we find

$$iKA_1 \exp(-Ky) - \int_0^\infty lA(l)(l\cos ly - K\sin ly)\,dl = f(y). \qquad \ldots (15)$$

Thus using Havelock's inversion theorem[10]

$$A_1 = -2i\int_0^\infty f(y)\exp(-Ky)\,\mathrm{d}y \qquad \ldots (16)$$

$$\text{and } A(l) = -\frac{2}{\pi l\Delta}\int_0^\infty f(y)(l\cos ly - K\sin ly)\,\mathrm{d}y, \qquad \ldots (17)$$

where $\Delta = l^2 + K^2$.

After simplification, (16) gives after utilizing (11) and (10)

$$A_1 = -8\pi K^2\exp(-K\eta + iK\xi)\int_0^\infty c(y)\exp(-2Ky)\,\mathrm{d}y$$

$$+8iK\int_0^\infty c(y)\left\{\int_0^\infty G_1(k, y)\,G_2(k, \eta)\mathrm{d}k\right\}\exp(-Ky)\mathrm{d}y, \qquad \ldots (18)$$

where $G_1(k, y) = k\sin ky + K\cos ky$

$$\text{and } G_2(k, \eta) = \frac{\exp(-k\xi)}{k^2+K^2}(k\cos k\eta - K\sin k\eta). \qquad \ldots (19)$$

Similarly, the general expression for $A(l)$ is given by (*see Appendix I*)

$$A(l) = \frac{8iK}{\Delta}\exp(-K\eta + iK\xi)\int_0^\infty c(y)\,G_1(l, y)\exp(-Ky)\,\mathrm{d}y$$

$$+\frac{8}{\pi\Delta}\int_0^\infty c(y)\,G_1(l, y)\left\{\int_0^\infty G_1(k, y)\,G_2(k, \eta)\,\mathrm{d}k\right\}\mathrm{d}y. \qquad \ldots (20)$$

Special Shapes of the Cliff

CASE-I: $c(y) = a\sin\lambda y$: a corrugated cliff.
In this case (*see Appendix-II*)

$$A_1 = -\frac{8a\pi\lambda K^2}{\lambda^2+4K^2}\exp(-K\eta + iK\xi)$$

$$+4ia\lambda K^2\int_0^\infty\left\{\frac{1}{K^2+(\lambda-k)^2}+\frac{1}{K^2+(\lambda+k)^2}\right\}G_2(k, \eta)\mathrm{d}k \qquad \ldots (21)$$

and

$$A(l) = -\frac{4ia\lambda K^2}{\Delta}\left\{\frac{1}{K^2+(\lambda+l)^2}-\frac{1}{K^2+(\lambda-l)^2}\right\}\exp(-K\eta + iK\xi)$$

$$+\frac{4a}{\Delta\pi}\int_0^\infty \left\{\frac{lk^2-K^2(\lambda-l)}{k^2-(\lambda-l)^2}-\frac{lk^2+K^2(\lambda+l)}{k^2-(\lambda+l)^2}\right\} G_2(k,\eta)\,\mathrm{d}k. \qquad \ldots (22)$$

CASE-II: $c(y)=y\exp(-\lambda y)$

In this case (*see Appendix-III*)

$$A_1=-\frac{8\pi K^2}{(\lambda+2K)^2}\exp(-K\eta+iK\xi)$$

$$+8iK\int_0^\infty \frac{k^2(2\lambda+K)+K(\lambda+K)^2}{\{(\lambda+K)^2+k^2\}^2}G_2(k,\eta)\,\mathrm{d}k, \qquad \ldots (23)$$

and

$$A(l)=\frac{8}{\Delta\pi}\left(\pi iK\frac{\exp(-K\eta+iK\xi)}{\{(\lambda+K)^2+l^2\}^2}[2l^2(\lambda+K)+K\{(\lambda+K)^2-l^2\}]\right.$$

$$+\int_0^\infty \frac{G_2(k,\eta)}{\{\lambda^2+(l+k)^2\}^2}\left[(K\lambda(l+k)-\frac{k}{2}\{\lambda^2-(l+k)^2\})\right.$$

$$\left.\left.+K(\lambda k(k+l)+\frac{K}{2}\{\lambda^2-(k+l)^2\}\right)\right]\mathrm{d}k$$

$$+\int_0^\infty \frac{G_2(k,\eta)}{\{\lambda^2+(l-k)^2\}^2}\left[(K\lambda(l-k)+\frac{k}{2}\{\lambda^2-(l-k)^2\})\right.$$

$$\left.\left.+K(\lambda k(k-l)+\frac{K}{2}\{\lambda^2-(k-l)^2\})\right]\mathrm{d}k\right). \qquad \ldots (24)$$

Asymptotic Results

When the source is situated at a large horizontal distance from the cliff (but of moderate depth below the free surface) then by using Watson's Lemma we find when $c(y)=a\sin\lambda y$

$$A_1\sim -\frac{8a\pi\lambda K^2}{\lambda^2+4K^2}\exp(-K\eta+iK\xi)+\frac{H_0(\eta)}{\xi^2}+\frac{H_1(\eta)}{\xi^4}+\frac{H_2(\eta)}{\xi^6}+0\left(\frac{1}{\xi^8}\right) \qquad \ldots (25)$$

and when $c(y)=y\exp(-\lambda y)$

$$A_1\sim -\frac{8\pi K^2}{(\lambda+2K)^2}\exp(-K\eta+iK\xi)+\frac{h_0(\eta)}{\xi^2}+\frac{h_1(\eta)}{\xi^4}+\frac{h_2(\eta)}{\xi^6}+0\left(\frac{1}{\xi^8}\right), \qquad \ldots (26)$$

where $H_0(\eta)=8C_0ia\lambda K(1-K\eta)$,

$$H_1(\eta)=48ia\lambda K^2\{F_1(\eta)-KF_2(\eta)\},$$
$$H_2(\eta)=960\ ia\lambda K\{F_3(\eta)-KF_4(\eta)\},$$
$$h_0(\eta)=8B_0iK(\lambda+K)(1-K\eta)\{1-\lambda(\lambda+K)B_0K^2\},$$
$$h_1(\eta)=48iK(\lambda+K)[\{f_1(\eta)-Kf_3(\eta)\}-\lambda(\lambda+K)\{f_5(\eta)-Kf_7(\eta)\}]$$
$$+\lambda B_0^2K^2(1-k\eta))$$

and $h_2(\eta)=960iK((\lambda+K)[\{f_2(\eta)-Kf_4(\eta)\}-\lambda(\lambda+K)\{f_6(\eta)-Kf_8(\eta)\}]$

$$+\lambda\{f_5(\eta)-Kf_7(\eta)\}\},$$

with

$$F_1(\eta)=C_0\{4C_0^2\lambda^2K^4-C_0K^2-g_0(\eta)\},\ F_2(\eta)=C_0\{4C_0^2\lambda^2\eta K^4-C_0\eta K^2-g_2(\eta)\},$$
$$F_3(\eta)=C_0\{16C_0^4\lambda^4K^8-12C_0^3\lambda^2K^6+C_0^2K^4$$
$$-(4C_0^2\lambda^2K^4-C_0K^2)g_0(\eta)+g_1(\eta)\},$$
$$F_4(\eta)=C_0\{(16C_0^2\lambda^4K^4-12C_0\lambda^2K^2+1)C_0^2\eta K^4$$
$$-(4C_0\lambda^2K^2-1)C_0K^2g_2(\eta)+g_3(\eta)\},$$
$$f_1(\eta)=-B_0\{B_0K^2+g_0(\eta)\},$$
$$f_2(\eta)=B_0[\{B_0K^2+g_0(\eta)\}B_0K^2+g_1(\eta)],$$
$$f_3(\eta)=-B_0\{B_0K^2\eta+g_2(\eta)\},$$
$$f_4(\eta)=B_0[\{\eta+g_2(\eta)\}B_0K^2+g_3(\eta)],$$
$$f_5(\eta)=-B_0^2K^2\{2B_0K^2+g_0(\eta)\},$$
$$f_6(\eta)=B_0^2K^2[\{3B_0K^2+2g_0(\eta)\}B_0K^2+g_1(\eta)],$$
$$f_7(\eta)=-B_0^2K^2\{2B_0K^2\eta+g_2(\eta)\},$$
$$f_8(\eta)=B_0K^2[\{3B_0K^2\eta+2g_2(\eta)\}B_0K^2+g_3(\eta)],$$

$$B_0=\frac{1}{K^2(\lambda+K)^2},\ C_0=\frac{1}{K^2(\lambda^2+K^2)},$$

$$g_0(\eta)=\frac{1}{K^2}+\frac{\eta^2}{2!},\ g_1(\eta)=\frac{1}{K^4}+\frac{\eta^2}{2!K^2}+\frac{\eta^4}{4!},$$

and $g_2(\eta)=\eta\left(\frac{1}{K^2}+\frac{\eta^2}{3!}\right),\ g_3(\eta)=\eta\left(\frac{1}{K^4}+\frac{\eta^2}{3!K^2}+\frac{\eta^4}{5!}\right).$

Discussion

A simplified perturbational analysis is employed to investigate the problem of generation of water waves by an oscillating line source in the presence of a nearly vertical cliff in deep water. General expression for the first order correction to the wave amplitude at infinity as well as $\varphi_1(x, y)$, the first order correction to the potential function are obtained in terms of some integrals involving the shape of the cliff. Assuming two different particular shape of the nearly vertical cliff, analytical expressions for these corrections are also obtained. The integral for the first order correction to the wave amplitude is asymptotically evaluated when the source is situated at a large horizontal distance from the cliff and at a moderate depth below the mean free surface of water, by using Watson's lemma.

Again using now the known expressions for A_1 and $A(l)$ in (14), the first order correction to the potential function, i.e. $\varphi_1(x, y)$ can be obtained, and thus the BVP-P_2 can be solved in principle by using an appropriate Havelock's expansion for $\varphi_2(x, y)$.

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Appendix-I

Calculation of A(l)

In order to calculate $A(l)$, we have to find

$$\int_0^\infty f(y)(l\cos ly - K\sin ly)\mathrm{d}y = I(l), \text{ say.} \qquad \ldots (\text{A.1.1})$$

Substituting for $f(y)$ from (P_1-3), $I(l)$ reduces to

$$I(l) = lI_1(l) - KI_2(l), \qquad \ldots (\text{A.1.2})$$

where $I_1(l) = \int_0^\infty \frac{\partial}{\partial y}\left\{c(y)\frac{\partial\varphi_0}{\partial y}(0, y)\right\}\cos ly\,\mathrm{d}y$

and $I_2(l) = \int_0^\infty \frac{\partial}{\partial y}\left\{c(y)\frac{\partial\varphi_0}{\partial y}(0, y)\right\}\sin ly\,\mathrm{d}y$.

Now, utilizing the known expression for $\varphi_0(0, y)$ obtained from (11) and exploiting $c(0) = 0$, we obtain finally

$$I_1(l) = -4\pi iKl\exp(-K\eta + iK\xi)\int_0^\infty c(y)\sin ly\exp(-Ky)\,\mathrm{d}y$$

$$-4l\int_0^\infty c(y)\sin ly\left\{\int_0^\infty G_1(k,y)\,G_2(k,\eta)\,\mathrm{d}k\right\}\mathrm{d}y, \qquad \ldots \text{(A.1.3)}$$

$$I_2(l)=4\pi iKl\exp(-K\eta+iK\xi)\int_0^\infty c(y)\cos ly\exp(-Ky)\,\mathrm{d}y$$

$$+4l\int_0^\infty c(y)\cos ly\left\{\int_0^\infty G_1(k,y)\,G_2(k,\eta)\,\mathrm{d}k\right\}\mathrm{d}y. \qquad \ldots \text{(A.1.4)}$$

Using (A.1.3) and (A.1.4) in (A.1.2), $I(l)$ is obtained and hence from (17) we find the general expression for $A(l)$ as given in (20).

Appendix-II

Explicit Calculation of Various Integrals for $c(y) = a\sin\lambda y$

Let us denote

$$\int_0^\infty c(y)\exp(-2Ky)\,\mathrm{d}y=I_1\,,$$

and $$\int_0^\infty c(y)\exp(-Ky)\left\{\int_0^\infty G_1(k,y)\,G_2(k,\eta)\,\mathrm{d}k\right\}\mathrm{d}y=I_2\,.$$

Taking $c(y) = a\sin\lambda y$, we find that

$$I_1=\frac{a\lambda}{\lambda^2+4K^2}\,, \qquad \ldots \text{(A.2.1)}$$

$$I_2=a\int_0^\infty G_2(k,\eta)\left\{\int_0^\infty \sin\lambda y\,G_1(k,y)\exp(-Ky)\,\mathrm{d}y\right\}\mathrm{d}k. \qquad \ldots \text{(A.2.2)}$$

The inner integral of (A.2.2) is equal to

$$\frac{\lambda K}{2}\left\{\frac{1}{K^2+(\lambda-k)^2}+\frac{1}{K^2+(\lambda+k)^2}\right\}.$$

Thus I_2 reduces to the form

$$I_2=\frac{\lambda aK}{2}\int_0^\infty G_2(k,\eta)\left\{\frac{1}{K^2+(\lambda-k)^2}+\frac{1}{K^2+(\lambda+k)^2}\right\}\mathrm{d}k. \qquad \ldots \text{(A.2.3)}$$

Hence, utilizing (A.2.1) and (A.2.3) in (19) we obtain the expression (21). Let us denote

$$\int_0^\infty c(y)\sin ly\exp(-Ky)\,\mathrm{d}y=I_3\,,$$

$$\int_0^\infty c(y)\sin ly\left\{\int_0^\infty G_1(k,y)G_2(k,\eta)\mathrm{d}k\right\}\mathrm{d}y=I_4,$$

$$\int_0^\infty c(y)\cos ly\exp(-Ky)\,\mathrm{d}y=I_5,$$

$$\int_0^\infty c(y)\cos ly\left\{\int_0^\infty G_1(k,y)\,G_2(k,\eta)\,\mathrm{d}k\right\}\mathrm{d}y=I_6.$$

Substituting $c(y)=a\sin\lambda y$, we obtain

$$I_3=\frac{K}{2}\left\{\frac{1}{K^2+(\lambda-l)^2}-\frac{1}{K^2+(\lambda+l)^2}\right\}. \qquad \ldots \text{(A.2.4)}$$

$$I_4=a\int_0^\infty G_2(k,\eta)\left\{\int_0^\infty \sin\lambda y\sin ly\,G_1(k,y)\,\mathrm{d}y\right\}\mathrm{d}k. \qquad \ldots \text{(A.2.5)}$$

$$I_5=\frac{1}{2}\left\{\frac{\lambda-l}{K^2+(\lambda-l)^2}+\frac{\lambda+l}{K^2+(\lambda+l)^2}\right\}. \qquad \ldots \text{(A.2.6)}$$

$$I_6=a\int_0^\infty G_2(k,\eta)\left\{\int_0^\infty \sin\lambda y\cos ly\,G_1(k,y)\,\mathrm{d}y\right\}\mathrm{d}k. \qquad \ldots \text{(A.2.7)}$$

Utilizing a convergence factor of the type used by Evans and Morris[11], the inner integrals of (A.2.5) and (A.2.7) are evaluated, and are given respectively by

$$\frac{k^2}{2}\left\{\frac{1}{k^2-(\lambda-l)^2}-\frac{1}{k^2-(\lambda+l)^2}\right\};$$

and

$$\frac{K}{2}\left\{\frac{\lambda-l}{(\lambda-l)^2-k^2}+\frac{\lambda+l}{(\lambda+l)^2-k^2}\right\}.$$

Using these we obtain finally

$$I_4=\frac{a}{2}\int_0^\infty k^2\,G_2(k,\eta)\left\{\frac{1}{k^2-(\lambda-l)^2}-\frac{1}{k^2-(\lambda+l)^2}\right\}\mathrm{d}k \qquad \ldots \text{(A.2.8)}$$

and

$$I_6=\frac{aK}{2}\int_0^\infty G_2(k,\eta)\left\{\frac{\lambda-l}{(\lambda-l)^2-k^2}+\frac{\lambda+l}{(\lambda+l)^2-k^2}\right\}\mathrm{d}k. \qquad \ldots \text{(A.2.9)}$$

Thus utilizing (A.2.4), (A.2.6), (A.2.8) and (A.2.9) in the expression (20) we obtain ultimately the expression for $A(l)$ given by (22).

Appendix-III

Explicit Calculations of Various Integrals for $c(y) = y\exp(-\lambda y)$

Putting $c(y) = y\exp(-\lambda y)$ in the integrals I_1 and I_2 defined in *Appendix-II*, we obtain

$$I_1 = \frac{1}{(\lambda + 2K)^2}, \qquad \ldots \text{(A.3.1)}$$

$$I_2 = \int_0^\infty G_2(k, \eta)\left[\int_0^\infty G_1(k, y)\, y\exp\{-(\lambda + K)y\}\, dy\right] dk. \qquad \ldots \text{(A.3.2)}$$

The inner integral of (A.3.2) can be evaluated and is given by

$$\frac{k^2(2\lambda + K) + K(\lambda + K)^2}{\{(\lambda + K)^2 + k^2\}^2}.$$

Thus using (A.3.2), we find

$$I_2 = \int_0^\infty \frac{G_2(k, \eta)}{\{(\lambda + K)^2 + k^2\}^2}\{k^2(2\lambda + K) + K(\lambda + K)^2\}\, dk. \qquad \ldots \text{(A.3.3)}$$

Utilizing the relations (A.3.1) and (A.3.3) into (19) we obtain the expression (23) for A_1.

It can be easily shown that when $c(y) = y\exp(-\lambda y)$, I_3 and I_5 defined in *Appendix-II* are given by

$$I_3 = \frac{2l}{\{(\lambda + K)^2 + l^2\}^2} \qquad \ldots \text{(A.3.4)}$$

and

$$I_5 = \frac{(\lambda + K)^2 - l^2}{\{(\lambda + K)^2 + l^2\}^2}. \qquad \ldots \text{(A.3.5)}$$

Also, $$I_4 = \int_0^\infty G_2(k, \eta)\left\{\int_0^\infty \sin ly\, G_1(k, y)\, y\exp(-\lambda y)\, dy\right\} dk \qquad \ldots \text{(A.3.6)}$$

and $$I_6 = \int_0^\infty G_2(k, \eta)\left\{\int_0^\infty \cos ly\, G_1(k, y)\, y\exp(-\lambda y)\, dy\right\} dk. \qquad \ldots \text{(A.3.7)}$$

Evaluating the inner integral of (A.3.6) and (A.3.7) I_4 and I_6 reduce to

$$I_4 = \int_0^\infty G_2(k, \eta)\left(\frac{k}{2}\left[\frac{\lambda^2 - (l - k)^2}{\{\lambda^2 + (l - k)^2\}^2} - \frac{\lambda^2 - (l + k)^2}{\{\lambda^2 + (l + k)^2\}^2}\right]\right.$$

$$+K\lambda\left[\frac{l+k}{\{\lambda^2+(l+k)^2\}^2}+\frac{l-k}{\{\lambda^2+(l-k)^2\}^2}\right]\Bigg)\,\mathrm{d}k, \qquad \ldots\ (\mathrm{A.3.8})$$

$$I_6=\int_0^\infty G_2(k,\eta)\left(\lambda k\left[\frac{k+l}{\{\lambda^2+(k+l)^2\}^2}+\frac{k-l}{\{\lambda^2+(k-l)^2\}^2}\right]\right.$$

$$\left.+\frac{K}{2}\left[\frac{\lambda^2-(k+l)^2}{\{\lambda^2+(k+l)^2\}^2}+\frac{\lambda^2-(k-l)^2}{\{\lambda^2+(k-l)^2\}^2}\right]\right)\mathrm{d}k\,. \qquad \ldots\ (\mathrm{A.3.9})$$

Using (A.3.4), (A.3.5), (A.3.8) and (A.3.9) in the general expression for $A(l)$ given by (20), we obtain finally the expression (24).