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THE PROBLEM OF INCLUSION IN ELASTO-PLASTIC NON-HOMOGENEOUS ANISOTROPIC SOLIDS

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The problems of spherical and circular inclusions in an elasto-plastic non-homogeneous anisotropic solids have been studied in this paper. Firstly, the problem has been solved by assuming that the elastic constants λ and μ are variable quantities. The variable nature of the elastic constants in this problem is characterised by the power law variation. Secondly, the problem of circular inclusion has been considered by taking the material constants to be variable quantities—the variable nature of the material constants is also characterised by the same type of power law variation. The elasto-plastic deformations are supposed to be small.

Key Words: Inclusions (Circular and Spherical); Anisotropy; Elasto-plastic Deformations

Introduction

Bhargava and Jaswon[1] studied different types of inclusions problems by complex variable formalism. Bhargava[2] solved the inclusion problems by applying the principle of minimum energy. The problems of inclusion in elasto-plastic solids have been considered by Bhargava[3]. Radially symmetric deformations and corresponding stresses in a spherical shell of isotropic solid under the influence of external and internal pressures were determined by Lame and given in Love[4]. The similar type of problem for a spherically anisotropic material was studied by Saint-Venant[5] and presented in the standard book of Lekhnitski[6]. Hopkins[7] solved the problem of dynamic expansion of spherical cavities in metals, taking into consideration the work-hardening of the material. The problem of expansion of a circular cavity under the action of internal pressure has been investigated by Hill[8]. Hill *et al.*[9] considered the problem of finite expansion of a circular cavity in an infinite medium. The flow in a hollow metallic cylinder considering strain-hardening of metal was presented by Nadai[10]. The note on problems of inclusion in an infinitely extended aeolotropic medium has been studied by Sengupta[11]. Also, he[12] solved the inclusion problem in elasto-plastic solids taking into consideration the work-hardening of the material. Sengupta and Kundu[13] investigated the same problem of inclusion in the finite case. Recently, inclusions in elasto-plastic anisotropic solids has been studied by Sengupta and Roy[14]. Sengupta and Basumallik[15] considered the problem of radial deformation of a non-homogeneous spherically anisotropic elastic sphere with a concentric spherical inclusion.

In this paper, the authors have considered the inclusion problems in elasto-plastic non-homogeneous anisotropic solids. Firstly, an elasto-plastic spherical inclusion in case of non-homogeneous anisotropic material has been studied. Secondly, the problem of circular inclusion in elasto-plastic non-homogeneous aniso-

tropic material has been investigated under the conditions of plane strain. The present authors have also discussed the same problems for the case of infinite medium.

Spherical Inclusion

Here a spherical inclusion of radius a in a finite elasto-plastic non-homogeneous anisotropic material has been considered, which tends to undergo a dimensional change to a sphere of radius $a(1+\delta)$, where δ falls in the range of plastic strain. The external boundary of the matrix is a spherical surface of radius b free from stresses. Due to the constraints of the matrix, stresses set in both the inclusion and the matrix.

The internal boundary of the matrix (a sphere of radius "a") be under a gradually increasing pressure p. At first, at a moderate pressure p the material of the matrix would behave elastically. The elastic constants A_{ij} of the material are assumed to vary according to the respective law given by

$$A_{ij} = \lambda_{ij}\, r^{n},$$

where λ_{ij} are positive constants and η is any positive rational number.

Introducing the spherical polar co-ordinates (r, θ, ϕ) and the corresponding components of the displacement (u, v, w), we suppose for the radially symmetric deformation

$$u = u(r),\ v = 0 \text{ and } w = 0\,. \qquad \ldots (1)$$

In this problem, the non-vanishing stress components are

$$\sigma_r = \lambda_{11}\, r^{n} \frac{\partial u}{\partial r} + 2\lambda_{12}\, r^{n-1}\, u \qquad \ldots (2)$$

and

$$\sigma_\theta = \sigma_\phi = \lambda_{12}\, r^{n} \frac{\partial u}{\partial r} + (\lambda_{22} + \lambda_{23})\, r^{n-1}\, u$$

The only equation of equilibrium in absence of body force is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{n+2}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{2[\lambda_{22} + \lambda_{23} - (n+1)\lambda_{12}]}{\lambda_{11}} \cdot \frac{u}{r^2} = 0\,. \qquad \ldots (3)$$

The solution of the above ordinary differential equation (3) is

$$u = Ar^{m-(1/2)(n+1)} + Br^{-m-(1/2)(n+1)}, \qquad \ldots (4)$$

where A and B are arbitrary constants and

$$m = \sqrt{\frac{(n+1)^2}{4} + \frac{2[\lambda_{22} + \lambda_{23} - (n+1)\lambda_{12}]}{\lambda_{11}}}$$

Therefore, the non-vanishing stress components are

$$\sigma_r = A\xi_m r^{m+(1/2)(n-3)} + B\bar{\xi}_m r^{-m+(1/2)(n-3)}$$

and

$$\sigma_\theta = A\eta_m r^{m+(1/2)(n-3)} + B\bar{\eta}_m r^{-m+(1/2)(n-3)} \quad \ldots (5)$$

where

$$\xi_m = \lambda_{11}\{m - 1/2(n+1)\} + 2\lambda_{12},$$
$$\bar{\xi}_m = \lambda_{11}\{-m - 1/2(n+1)\} + 2\lambda_{12}, \quad \ldots (6)$$
$$\eta_m = \lambda_{12}\{m - 1/2(n+1)\} + (\lambda_{22} + \lambda_{23})$$

and

$$\bar{\eta}_m = \lambda_{12}\{-m - 1/2(n+1)\} + (\lambda_{22} + \lambda_{23}).$$

The boundary conditions for this problem are

$$[\sigma_r]_{r=a} = -p_0$$

and

$$[\sigma_r]_{r=b} = 0, \quad \ldots (7)$$

p_0 being the yielding pressure.

Applying boundary conditions (7), we obtain

$$A\xi_m = \frac{p_0}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)}$$

and

$$B\bar{\xi}_m = -\frac{p_0 b^{2m}}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)}, \quad \ldots (8)$$

The stresses and displacement are, therefore

$$\sigma_r = \frac{p_0}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)} \times r^{m+(1/2)(n-3)}$$

$$- \frac{p_0 b^{2m}}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)} \times r^{-m+(1/2)(n-3)},$$

$$\sigma_\theta = \frac{\eta_m}{\xi_m} \times \frac{p_0}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)} \times r^{m+(1/2)(n-3)}$$

$$- \frac{\bar{\eta}_m}{\bar{\xi}_m} \times \frac{p_0 b^{2m}}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)} \times r^{-m+(1/2)(n-3)} \quad \ldots (9)$$

and

$$u = \frac{1}{\bar{\xi}_m} \times \frac{p_0}{b^{2m} - a^{2m}} \times a^{m-(1/2)(n-3)} \times r^{m-(1/2)(n+1)}$$

$$-\frac{1}{\xi_m}\times\frac{p_0\,b^{2m}}{b^{2m}-a^{2m}}\times a^{m-(1/2)(n-3)}\times r^{-m-(1/2)(n+1)}$$

Now the yielding commences when

$$\sigma_\theta-\sigma_r=\alpha(H+G)^{-1/2},\ (\alpha=\pm 1),\ \text{on } r=a\ (\textit{vide}\ \text{Hill}^{8})$$

Since the plastic loading is only considered, we take $\alpha=1$.

The corresponding pressure p_0 is given by

$$p_0=\frac{(H+G)^{-1/2}(b^{2m}-a^{2m})}{a^{m-(1/2)(n-3)}\left[\left(\dfrac{\eta_m-\xi_m}{\xi_m}\right)\times a^{m+(1/2)(n-3)}-\left(\dfrac{\bar{\eta}_m-\bar{\xi}_m}{\bar{\xi}_m}\right)\times b^{2m}\times a^{-m+(1/2)(n-3)}\right]}\quad\ldots(10)$$

and also the displacement at the inner boundary is

$$u_0(a)=(H+G)^{-1/2}\times\frac{1}{\left[\left(\dfrac{\eta_m-\xi_m}{\xi_m}\right]\times a^{m+(1/2)(n-3)}-\left(\dfrac{\bar{\eta}_m-\xi_m}{\xi_m}\right)\times b^{2m}\times a^{-m+(1/2)(n-3)}\right]}\times$$
$$\times\left[\frac{1}{\xi_m}\times a^{m-(1/2)(n+1)}-\frac{1}{\xi_m}\times b^{2m}\times a^{-m-(1/2)(n+1)}\right]\qquad\ldots(11)$$

With increasing pressure a plastic region spreads into the shell. For reasons of symmetry the plastic boundary in the non-homogeneous anisotropic material must be a spherical surface, its radius at any moment is denoted by c.

The stresses and displacement in the elastic region $c\leqslant r\leqslant b$ are still of the form

$${}^{e}\sigma_r=A_1\,r^{m+(1/2)(n-3)}+B_1\,r^{-m+(1/2)(n-3)},$$

$${}^{e}\sigma_\theta=\frac{\eta_m}{\xi_m}A_1\,r^{m+(1/2)(n-3)}+B_1\frac{\bar{\eta}_m}{\bar{\xi}_m}\,r^{-m+(1/2)(n-3)}\qquad\ldots(12)$$

and

$${}^{e}u=A_1\,r^{m-(1/2)(n+1)}+B_1\,r^{-m-(1/2)(n+1)},$$

where A_1 and B_1 are parameters to be determined.

Now the material just on the elastic side of the plastic boundary must be on the point of yielding and satisfy the condition $\sigma_\theta-\sigma_r=(H+G)^{-1/2}$, on $r=c$ (Hill[8]) and also the boundary condition $\sigma_r=0$, on $r=b$.

Therefore, the elastic stresses and displacement in the region $c\leqslant r\leqslant b$ are

$${}^{e}\sigma_r=-\frac{(H+G)^{-1/2}}{R_1}\times r^{m+(1/2)(n-3)}+\frac{(H+G)^{-1/2}\,b^{2m}}{R_1}\times r^{-m+(1/2)(n-3)},$$

$$^{e}\sigma_\theta = -\frac{(H+G)^{-1/2}}{R_1}\times\frac{\eta_m}{\xi_m}r^{m+(1/2)(n-3)}+\frac{(H+G)^{-1/2}\,b^{2m}}{R_1}\times\frac{\bar{\eta}_m}{\xi_m}r^{-m+(1/2)(n-3)} \quad \ldots (13)$$

and

$$^{e}u = -\frac{(H+G)^{-1/2}}{R_1}r^{m-(1/2)(n+1)}+\frac{(H+G)^{-1/2}\,b^{2m}}{R_1}\times r^{-m-(1/2)(n+1)},$$

where

$$R_1=\frac{\bar{\eta}_m-\bar{\xi}_m}{\bar{\xi}_m}b^{2m}c^{-m+(1/2)(n-3)}-\frac{\eta_m-\xi_m}{\xi_m}c^{m+(1/2)(n-3)}.$$

Now let us consider the plastic solid in the region $a \leqslant r \leqslant c$.

The equation of equilibrium is

$$\frac{\partial\sigma_r}{\partial r}+\frac{2(\sigma_r-\sigma_\theta)}{r}=0 \quad \ldots (14)$$

subject to the yield condition

$$\sigma_\theta-\sigma_r=(H+G)^{-1/2}\,(\text{Hill}^8). \quad \ldots (15)$$

The compressibility equation is

$$\sigma_r+2L\sigma_\theta=3K'\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right), \quad \ldots (16)$$

where

$$L=\frac{E'(1-\nu)-E\nu'}{E(1-2\nu')}\text{ and }K'=\frac{E'}{3(1-2\nu')},$$

E and E' being Young's modulli for the tension compression in the direction of the radius vector r and in the direction perpendicular to it, ν' being the Poisson co-efficient which characterises the transverse contraction in directions perpendicular to r for tension in the direction r and ν being the same which characterises the transverse compression in the plane normal to the radius vector for tension in this same plane.

Using (15) and (16), we have

$$^{p}\sigma_r=\frac{3K'}{1+2L}\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)-\frac{2L}{1+2L}(H+G)^{-1/2}$$

and $\quad \ldots (17)$

$$^{p}\sigma_\theta=\frac{3K'}{1+2L}\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+\frac{1}{1+2L}(H+G)^{-1/2}.$$

From (14) and (17) we get a differential equation satisfied by u

$$\frac{d}{dr}\left(\frac{du}{dr}+\frac{2u}{r}\right)=\frac{R_2}{r}, \qquad \ldots (18)$$

where

$$R_2=\frac{2(1+2L)}{3K'(H+G)^{-1/2}}.$$

Now solving the above differential equation (18), we obtain

$$^{p}u=A_2r+\frac{B_2}{r^2}+\frac{R_2r}{9}(\log r^3-1) \qquad \ldots (19)$$

With the help of (17) and (19), we obtain the non-vanishing components of stresses

$$^{p}\sigma_r=\frac{3K'}{1+2L}[3A_2+R_2\log r]-\frac{2L}{1+2L}(H+G)^{-1/2}$$

and

$$^{p}\sigma_\theta=\frac{3K'}{1+2L}[3A_2+R_2\log r]+\frac{1}{1+2L}(H+G)^{-1/2}, \qquad \ldots (20)$$

where A_2 and B_2 are constants can be calculated.

Now we have the following continuity conditions

$$[^{p}\sigma_r]_{r=c}=[^{e}\sigma_r]_{r=c}$$

and

$$[^{p}u]_{r=c}=[^{e}u]_{r=c}. \qquad \ldots (21)$$

Hence finally, the plastic stresses and displacement are

$$^{p}\sigma_r=(H+G)^{-1/2}\left[-2\log\frac{c}{r}+\frac{1}{R_1}\left\{-c^{m+(1/2)(n-3)}+b^{2m}\times c^{-m+(1/2)(n-3)}\right\}\right]$$

$$^{p}\sigma_\theta=(H+G)^{-1/2}\left[1-2\log\frac{c}{r}+\frac{1}{R_1}\left\{-c^{m+(1/2)(n-3)}+b^{2m}\times c^{-m+(1/2)(n-3)}\right\}\right] \qquad \ldots (22)$$

$$^{p}u=\frac{(H+G)^{-1/2}}{9K'}\left[\frac{9K'-(1+2L)c^n}{R_1\times c^{n/2}}\left(-c^{m+(3/2)}+b^{2m}c^{-m+(3/2)}\right)+\frac{2c^3(1-L)}{3}\right]\times\frac{1}{r^2}$$

$$-\frac{(H+G)^{-1/2}}{9K'}\left[\frac{2(1-L)}{3}+(1+2L)\left\{2\log\frac{c}{r}-\frac{1}{R_1}\left(-c^{m+(1/2)(n-3)}\right.\right.\right.$$

$$\left.\left.\left.+b^{2m}c^{-m+(1/2)(n-3)}\right)\right\}\right]\times r$$

If $^{p}\sigma_r=-p_1$ at $r=a$, then we have

$$p_1=(H+G)^{-1/2}\left[2\log\frac{c}{a}-\frac{1}{R_1}\left\{-c^{m+(1/2)(n-3)}+b^{2m}\times c^{-m+(1/2)(n-3)}\right\}\right] \quad \ldots (23)$$

The displacement at the inner boundary of the matrix is given by

$$u(a)=[^{p}u]_{r=a}=\frac{(H+G)^{-1/2}}{9K'}\left[\frac{9K'-(1+2L)c^{n}}{R_1\times c^{n/2}}(-c^{m+(3/2)}+b^{2m}c^{-m+(3/2)})\right.$$
$$+\frac{2c^3(1-L)}{3}\Bigg]\times\frac{1}{a^2}-\frac{(H+G)^{-1/2}}{9K'}\left[\frac{2(1-L)}{3}+(1+2L)\right.$$
$$\left.\left\{2\log\frac{c}{a}-\frac{1}{R_1}\ (-c^{m+(1/2)(n-3)}+b^{2m}c^{-m+(1/2)(n-3)})\right\}\right]\times a.$$

Now we consider the inclusion which is under a normal pressure p_1 on its external boundary and is therefore ${}^{i}\sigma_r=-p_1$, ${}^{i}\sigma_\theta=-p_1$, ${}^{i}\tau_{r\theta}=0$. It is not very difficult to mark that the material of the inclusion never yields and it is always in a state of elastic deformation. The displacement at the surface of the inclusion is, therefore

$${}^{i}u(a)=\frac{p_1}{\xi_m}\times a^{-(n-1)}. \quad \ldots (25)$$

Now, the radial displacement of the inclusion is $a(\delta-\varepsilon)$, which with the help of (25) can be written as

$$p_1=\xi_m(\delta-\varepsilon)\,a^{n} \quad \ldots (26)$$

By using the condition of continuity of normal stress at the inclusion and matrix, we have from (23) and (26)

$$\xi_m(\delta-\varepsilon)\,a^{n}=(H+G)^{-1/2}\left[2\log\frac{c}{a}-\frac{1}{R_1}\{-c^{m+(1/2)(n-3)}+b^{2m}c^{-m+(1/2)(n-3)}\}\right]\ldots (27)$$

The displacement of the inner boundary of the matrix is $\alpha\varepsilon$, which by equation (24) is

$$\alpha\varepsilon=\frac{(H+G)^{-1/2}}{9K'}\left[\frac{9K'-(1+2L)c^{n}}{R_1\times c^{n/2}}(-c^{n+(3/2)}+b^{2m}c^{-m+(3/2)})+\frac{2c^3(1-L)}{3}\right]\times\frac{1}{a^2}$$
$$-\frac{(H+G)^{-1/2}}{9K'}\left[\frac{2(1-L)}{3}+(1+2L)\left\{2\log\frac{c}{a}-\frac{1}{R_1}\left(-c^{m+(1/2)(n-3)}\right.\right.\right.$$
$$\left.\left.\left.+b^{2m}c^{-m+(1/2)(n-3)}\right)\right\}\right]\times a \quad \ldots (28)$$

Relation between ε and δ is

$$\varepsilon = \delta - \frac{(H+G)^{-1/2}}{\xi_m \times a^n}\left[2\log\frac{c}{a} - \frac{1}{R_1}\{c^{-m+(1/2)(n-3)} + b^{2m} \times c^{-m+(1/2)(n-3)}\}\right] \quad \ldots (29)$$

Circular Inclusion under Conditions of Plane Strain

Here in this case the circular inclusion in a non-homogeneous anisotropic material having co-axial cylindrical external free boundary has been discussed. Proceeding as in the case of spherical inclusion, let us suppose that the internal boundary of the matrix is under a pressure p with in the elastic limit. Our discussion is confined to the infinitesimal theory of strain.

The material constants are taken as

$$A = A_0 r^{n_1},\ C = C_0 r^{n_1},\ G = G_0 r^{n_1},\ H = H_0 r^{n_1},\ N = N_0 r^{n_1},$$

where A_0, C_0, G_0, H_0 and N_0 are positive constants and n_1 is any positive rational number.

We introduced here the cylindrical co-ordinates (r, θ, z) and the material is considered to be transversely isotropic.

In this problem the non-vanishing stress components are (Love[4])

$$\sigma_r = C_0 r^{n_1}\frac{\partial u}{\partial r} + F_0 r^{n_1-1} \times u$$

$$\sigma_\theta = A_0 r^{n_1-1} \times u + F_0 r^{n_1}\frac{\partial u}{\partial r} \quad \ldots (30)$$

and

$$\sigma_z = F_0 r^{n_1}\frac{\partial u}{\partial r} + H_0 r^{n_1-1} \times u.$$

The only equation of equilibrium in absence of body force is (Love[4])

$$\frac{d^2u}{dr^2} + \frac{(n_1+1)}{r}\frac{du}{dr} + \frac{(n_1F_0 - A_0)}{C_0} \times \frac{u}{r^2} = 0 \quad \ldots (31)$$

The solution of the above differential equation is

$$u = \alpha r^{m_1 - n_1/2} + \beta r^{-m_1 - n_1/2} \quad \ldots (32)$$

where

$$m_1 = \sqrt{\frac{n_1^2}{4} - \frac{(n_1F_0 - A_0)}{C_0}}$$

and α and β are arbitrary constants to be determined.

The stress components are, therefore

$$\sigma_r = \xi_{m_1}\alpha\, r^{m_1+(1/2)(n_1-2)} + \overline{\xi}_{m_1}\beta r^{-m_1+(1/2)(n_1-2)},$$

$$\sigma_\theta = \eta_{m_1}\alpha r^{m_1+(1/2)(n_1-2)} + \overline{\eta}_{m_1}\beta r^{-m_1+(1/2)(n_1-2)}$$

and ... (33)

$$\sigma_z = J_{m_1}\alpha r^{m_1+(1/2)(n_1-2)} + \bar{J}_{m_1}\beta r^{-m_1+(1/2)(n_1-2)},$$

where

$$\xi_{m_1} = C_0\left(m_1 - \frac{n_1}{2}\right) + F_0,$$

$$\bar{\xi}_{m_1} = C_0\left(-m_1 - \frac{n_1}{2}\right) + F_0,$$

$$\eta_{m_1} = A_0 + F_0\left(m_1 - \frac{n_1}{2}\right), \quad \ldots (34)$$

$$\bar{\eta}_{m_1} = A_0 + F_0\left(-m_1 - \frac{n_1}{2}\right),$$

$$J_{m_1} = F_0\left(m_1 - \frac{n_1}{2}\right) + H_0$$

and

$$\bar{J}_{m_1} = F_0\left(-m_1 - \frac{n_1}{2}\right) + H_0.$$

The constants α and β can be calculated by the boundary conditions

$[\sigma_r]_{r=a} = -p_0$

and $[\sigma_r]_{r=b} = 0$, ... (35)

p_0 being the yielding pressure.

Using boundary conditions (35) the values of α and β are calculated and the stresses and displacement are therefore

$$\sigma_r = p_0\left[\frac{b^{-m_1+(1/2)(n_1-2)} \times r^{m_1+(1/2)(n_1-2)} - b^{m_1+(1/2)(n_1-2)} \times r^{-m_1+(1/2)(n_1-2)}}{a^{-m_1+(1/2)(n_1-2)}\, b^{m_1+(1/2)(n_1-2)} - a^{m_1+(1/2)(n_1-2)}\, b^{-m_1+(1/2)(n_1-2)}}\right],$$

$$\sigma_\theta = \frac{p_0}{\left[a^{-m_1+(1/2)(n_1-2)}\, b^{m_1+(1/2)(n_1-2)} - a^{m_1+(1/2)(n_1-2)}\, b^{-m_1+(1/2)(n_1-2)}\right]} \times$$

$$\times\left[\frac{\eta_{m_1}}{\xi_{m_1}} \times b^{-m_1+(1/2)(n_1-2)} \times r^{m_1+(1/2)(n_1-2)} - \frac{\bar{\eta}_{m_1}}{\xi_{m_1}} \times b^{m_1+(1/2)(n_1-2)} \times r^{-m_1+(1/2)(n_1-2)}\right],$$

$$\sigma_z = \frac{p_0}{\left[a^{-m_1+(1/2)(n_1-2)}\, b^{m_1+(1/2)(n_1-2)} - a^{m_1+(1/2)(n_1-2)}\, b^{-m_1+(1/2)(n_1-2)}\right]} \times$$

$$\times\left[\frac{J_{m_1}}{\xi_{m_1}}\times b^{-m_1+(1/2)(n_1-2)}\times r^{m_1+(1/2)(n_1-2)}-\frac{\bar{J}_{m_1}}{\bar{\xi}_{m_1}}\times b^{m_1+(1/2)(n_1-2)}\times r^{-m_1+(1/2)(n_1-2)}\right]$$

and

$$u=\frac{p_0}{[a^{-m_1+(1/2)(n_1-2)}\times b^{m_1+(1/2)(n_1-2)}-a^{m_1+(1/2)(n_1-2)}\,b^{-m_1+(1/2)(n_1-2)}]}\times$$

$$\times\left[\frac{1}{\xi_{m_1}}\times b^{-m_1+(1/2)(n_1-2)}\times r^{m_1-(n_1/2)}-\frac{1}{\xi_{m_1}}\,b^{m_1+(1/2)(n_1-2)}\times r^{-m_1-(n_1/2)}\right]. \qquad \ldots (36)$$

Now yielding commences (Hill[8]), where $\sigma_\theta-\sigma_r=2T\sqrt{1-c}$, T being yied stress and

$$c=1-\frac{N_0(F_0+G_0)}{2(F_0G_0+G_0H_0+H_0F_0)}, \text{ on } r=a\,.$$

The corresponding pressure "p_0" is given by

$$p_0=\frac{2T\sqrt{1-c}[a^{-m_1+(1/2)(n_1-2)}\,b^{m_1+(1/2)(n_1-2)}-a^{m_1+(1/2)(n_1-2)}\,b^{m_1+(1/2)(n_1-2)}]}{\left[\left(\frac{\eta_{m_1}}{\xi_{m_1}}-1\right)a^{m_1+(1/2)(n_1-2)}b^{-m_1+(1/2)(n_1-2)}+\left(1-\frac{\bar{\eta}_{m_1}}{\bar{\xi}_{m_1}}\right)a^{-m_1+(1/2)(n_1-2)}b^{m_1+(1/2)(n_1-2)}\right]} \qquad \ldots (37)$$

and the displacement at the inner boundary is

$$u_0(a)$$

$$=\frac{2T\sqrt{1-c}}{\left[\left(\frac{\eta_{m_1}}{\xi_{m_1}}-1\right)a^{m_1+(1/2)(n_1-2)}b^{-m_1+(1/2)(n_1-2)}+\left(1-\frac{\bar{\eta}_{m_1}}{\bar{\xi}_{m_1}}\right)a^{-m_1+(1/2)(n_1-2)}b^{m_1+(1/2)(n_1-2)}\right]}\times$$

$$\times\left[\frac{1}{\xi_{m_1}}\times b^{-m_1+(1/2)(n_1-2)}\times a^{m_2-(n_1/2)}-\frac{1}{\bar{\xi}_{m_1}}\times b^{m_1+(1/2)(n_1-2)}\times a^{-m_1-(n_1-(n_1/2)}\right] \qquad \ldots (38)$$

For an increasing pressure beyond p_0 (given by eq. (37)), a plastic zone would develop in the matrix and if c be the radius of the elasto-plastic interface, proceeding as in the case of spherical inclusion under pressure, the elastic field in the matrix beyond c is given by

$${}^e\sigma_r=\frac{2T\sqrt{1-c}}{R_1'}[-r^{m_1+(1/2)(n_1-2)}+b^{2m_1}r^{-m_1+(1/2)(n_1-2)}],$$

$${}^e\sigma_r=\frac{2T\sqrt{1-c}}{R_1'}\left[-\frac{\eta_{m_1}}{\xi_{m_1}}\times r^{m_1+(1/2)(n_1-2)}+\frac{\bar{\eta}_{m_1}}{\bar{\xi}_{m_1}}\times b^{2m_1}r^{-m_1+(1/2)(n_1-2)}\right],$$

$$^{e}\sigma_z = \frac{2T\sqrt{1-c}}{R'_1}\left[-\frac{J_{m_1}}{\xi_{m_1}} \times r^{m_1+(1/2)(n_1-2)} + \frac{\overline{J}_{m_1}}{\overline{\xi}_{m_1}} \times b^{2m_1} r^{-m_1+(1/2)(n_1-2)}\right]$$

and

$$^{e}u = \frac{2T\sqrt{1-c}}{R'_1}\left[-r^{m_1-(n_1/2)} + b^{2m_1} r^{-m_1-(n_1/2)}\right], \qquad \ldots (39)$$

where

$$R'_1 = \frac{\overline{\eta}_{m_1} - \overline{\xi}_{m_1}}{\overline{\xi}_{m_1}} b^{2m_1} c^{-m_1+(1/2)(n_1-2)} - \frac{\eta_{m_1} - \xi_{m_1}}{\xi_{m_1}} c^{m_1+(1/2)(n_1-2)}.$$

Let us now consider the plastic solid in the region $a \leqslant r \leqslant c$.

The equilibrium equation is

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \qquad \ldots (40)$$

subject to the yield condition

$$\sigma_\theta - \sigma_r = 2T\sqrt{1-c} \quad \text{(Hill}^{8}\text{)}. \qquad \ldots (41)$$

The plastic material is under plane strain and satisfies the compressibility equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0. \qquad \ldots (42)$$

Thus, in the region $a \leqslant r \leqslant c$ the component of stresses and displacement are

$$^{p}\sigma_r = 2T\sqrt{1-c}\log r + \alpha_1,$$

$$^{p}\sigma_\theta = 2T\sqrt{1-c}\,(1+\log r) + \alpha_1,$$

$$^{p}\sigma_z = \frac{1}{G_0+F_0}(G_0\,^{p}\sigma_r + P_0\,^{p}\sigma_\theta)$$

$$= \frac{1}{G_0+F_0}[G_0\{2T\sqrt{1-c}\log r + \alpha_1\} + F_0\{2T\sqrt{1-c}(1+\log r) + \alpha_1\}] \qquad \ldots (43)$$

and $^{p}u = \dfrac{\overline{\beta}_1}{r}$;

where α_1 and β_1 are constants can be calculated with the help of the following continuity conditions

$$[^{p}\sigma_r]_{r=c} = [^{e}\sigma_r]_{r=c}$$

and $[^{p}u]_{r=c} = [^{e}u]_{r=c}$. $\qquad \ldots (44)$

Therefore, we finally obtain the plastic stress components and displacement as follows:

$$^{p}\sigma_r = \frac{2T\sqrt{1-c}}{R_1'}\left[-c^{m_1+(1/2)(n_1-2)} + b^{2m_1}c^{-m_1+(1/2)(n_1-2)}\right] - 2T\sqrt{1-c}\log\frac{c}{r},$$

$$^{p}\sigma_\theta = \frac{2T\sqrt{1-c}}{R_1'}\left[-c^{m_1+(1/2)(n_1-2)} + b^{2m_1}c^{-m_1+(1/2)(n_1-2)}\right] + 2T\sqrt{1-c}\left(1-\log\frac{c}{r}\right),$$

$$^{p}\sigma_z = \frac{1}{G_0+F_0}\left[G_0\left\{\frac{2T\sqrt{1-c}}{R_1'}\left(-c^{m_1+(1/2)(n_1-2)} + b_1^{2m}c^{-m_1+(1/2)(n_1-2)}\right)\right.\right.$$

$$\left. - 2T\sqrt{1-c}\log\frac{c}{r}\right\} + F_0\left\{\frac{2T\sqrt{1-c}}{R_1'}\left(-c^{m_1+(1/2)(n_1-2)} + b^{2m_1}c^{-m_1+(1/2)(n_1-2)}\right)\right.$$

$$\left.\left. + 2T\sqrt{1-c}\left(1-\log\frac{c}{r}\right)\right\}\right] \qquad \ldots (45)$$

$$^{p}u = \frac{2T\sqrt{1-c}}{R_1'}\left[-c^{m_1-(1/2)(n_1-2)} + b^{2m_1}c^{-m_1-(1/2)(n_1-2)}\right]\times\frac{1}{r}$$

If $^{p}\sigma_r = -p_1$ at $r=a$, then we have

$$p_1 = 2T\sqrt{1-c}\log\frac{c}{a} - \frac{2T\sqrt{1-c}}{R_1'}\left[-c^{m_1+(1/2)(n_1-2)} + b^{2m_1}c^{-m_1+(1/2)(n_1-2)}\right] \qquad \ldots (46)$$

The displacement at the inner boundary of the matrix is, therefore

$$u(a) = [^{p}u]_{r=a} = \frac{2T\sqrt{1-c}}{R_1'}\left[-c^{m_1-(1/2)(n_1-2)} + b^{2m_1}c^{-m_1-(1/2)(n_1-2)}\times\frac{1}{a}\right. . \qquad \ldots (47)$$

Now let us consider the inclusion which is under a normal pressure p_1, on its external boundary and is therefore

$$^{i}\sigma_r = -p_1\,,\ ^{i}\sigma_\theta = -p_1,\ ^{i}\sigma_z = -\frac{F_0m_1+H_0}{C_0m_1+F_0}\times p_1\,. \qquad \ldots (48)$$

Then the displacement at the surface of the inclusion is

$$^{i}u(a) = \frac{p_1}{\xi_{m_1}}\times a^{-n_1+1} \qquad \ldots (49)$$

Now the radial displacement of the inclusion is $a(\delta-\varepsilon)$, which with the help of (49) can be written as

$$\delta-\varepsilon = \frac{p_1}{\xi_{m_1}}\times a^{-n_1}\,. \qquad \ldots (50)$$

We have from (46) and (50)

$$2T\sqrt{1-c}\log\frac{c}{a}-\frac{2T\sqrt{1-c}}{R_1'}[-c^{m_1+(1/2)(n_1-2)}$$

$$+b^{2m_1}c^{-m_1+(1/2)(n_1-2)}]=(\delta-\varepsilon)\,\xi_{m_1}\times a^{n_1} \qquad \ldots (51)$$

The displacement of the inner boundary of the matrix shall be $a\varepsilon$, where

$$a\varepsilon=\frac{2T\sqrt{1-c}}{R_1'}[-c^{m_1-(1/2)(n_1-2)}+b^{2m_1}c^{-m_1-(1/2)(n_1-2)}]\times\frac{1}{a} \qquad \ldots (52)$$

Also ε and δ are connected by the following relation

$$\varepsilon=\delta-\frac{2T\sqrt{1-c}}{\xi_{m_1}\times a^{n_1}}\left[\log\frac{c}{a}-\frac{1}{R_1'}\left\{-c^{m_1+(1/2)(n_1-2)}+b^{2m_1}\,c^{-m_1+(1/2)(n_1-2)}\right\}\right] \qquad \ldots (53)$$

In the case of homogeneous anisotropic material, we take $n=n_1=0$, then we have

$$m=\sqrt{\frac{1}{4}+\frac{2(\lambda_{22}+\lambda_{23}-\lambda_{12})}{\lambda_{11}}},\ m_1=\sqrt{\frac{A_0}{C_0}}$$

and

$$\xi_m=\lambda_{11}(m-\tfrac{1}{2})+2\lambda_{12},\ \bar{\xi}_m=\lambda_{11}(-m-\tfrac{1}{2})+2\lambda_{12}$$

$$\eta_m=\lambda_{12}(m-\tfrac{1}{2})+(\lambda_{22}+\lambda_{23}),\ \bar{\eta}_m=\lambda_{12}(-m-\tfrac{1}{2})+(\lambda_{22}+\lambda_{23})$$

$$\xi_{m_1}=C_0m_1+F_0,\ \bar{\xi}_{m_1}=F_0-C_0m_1$$

$$\eta_{m_1}=A_0+F_0m_1,\ \bar{\eta}_{m_1}=A_0-F_0m_1$$

$$J_{m_1}=F_0m_1+H_0,\ \bar{J}_{m_1}=H_0-F_0m_1$$

$$R_1=\frac{\bar{\eta}_m-\bar{\xi}_m}{\bar{\xi}_m}\,b^{2m}c^{-m-(3/2)}-\frac{\eta_m-\xi_m}{\xi_m}\,c^{m-(3/2)}$$

$$R_1'=\frac{\bar{\eta}_{m_1}-\bar{\xi}_{m_1}}{\bar{\xi}_{m_1}}\,b^{2m_1}c^{-m_1-1}-\frac{\eta_{m_1}-\xi_{m_1}}{\xi_{m_1}}\,c^{m_1-1}$$

With the above substitutions the expressions (10), (11), (13), (22) and also the expressions (37), (38), (39), (45) tally with the results obtained by Roy[14].

Deduction: Infinite Cases

(*a*) *Spherical Inclusion*

This problem can be solved by letting $b\to\infty$. Then in this case the yielding pressure p_0 is

$$p_0=-(H+G)^{-1/2}\times\frac{\bar{\xi}_m}{\bar{\eta}_m-\bar{\xi}_m} \qquad \ldots (54)$$

The stresses and displacement in the region $c\leqslant r\leqslant\infty$ are

$$^{e}\sigma_r = -\frac{(H+G)^{-1/2}\,\xi_m \times r^{-m_1+(1/2)(n_1-3)}}{(\bar{\xi}_m - \bar{\eta}_m)\times c^{-m_1+(1/2)(n_1-3)}}$$

$$^{e}\sigma_\theta = -\frac{(H+G)^{-1/2}\times \eta_m r^{-m_1+(1/2)(n_1-3)}}{(\bar{\xi}_m - \eta_m)\times c^{-m_1+(1/2)(n_1-3)}}$$

and

$$^{e}u = -\frac{(H+G)^{-1/2}+\bar{\xi}_m \times r^{-m-(1/2)(n+1)}}{(\bar{\xi}_m - \bar{\eta}_m)\, c^{m_1+(1/2)(n_1-3)}} \qquad \ldots (55)$$

The stresses and displacement in the region $a \leqslant r \leqslant c$ are

$$^{p}\sigma_r = -(H+G)^{-1/2}\left[2\log\frac{c}{r}+\frac{\bar{\xi}_m}{\bar{\xi}_m-\bar{\eta}_m}\right]$$

$$^{p}\sigma_\theta = (H+G)^{-1/2}\left[1-2\log\frac{c}{r}-\frac{\bar{\xi}_m}{\bar{\xi}_m-\bar{\eta}_m}\right] \qquad \ldots (56)$$

$$^{p}u = \frac{(H+G)^{-1/2}}{9K'}\left[-\frac{\{9K'-(1+2L)c^{n}\{\times \bar{\xi}_m \times c^{-m-(1/2)(n-3)}}{(\bar{\xi}_m-\bar{\eta}_m)\times c^{-m_1+(1/2)(n_1-3)}}+\frac{2c^3(1-L)}{3}\right]\times\frac{1}{r^2}$$

$$-\frac{(H+G)^{-1/2}}{9K'}\left[\frac{2(1-L)}{3}+(1+2L)\left\{2\log\frac{c}{r}+\frac{\bar{\xi}_m}{\bar{\xi}_m-\bar{\eta}_m}\right\}\right]\times r$$

Now repeating the similar discussion as in the case of spherical inclusion when the external boundary is a sphere of radius b, we have the relation between ε and δ as

$$\varepsilon = \delta - \frac{H+G)^{-1/2}}{\xi_m \times a^n}\left[2\log\frac{c}{a}+\frac{\bar{\xi}_m}{\bar{\xi}_m-\bar{\eta}_m}\right] \qquad \ldots (57)$$

(*b*) *Circular Inclusion*

This problem can also be solved by making external radius of the circular cylinder to extend up to inifinity.

In this case the yielding pressure p_0 is given by

$$p_0 = \frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})} \qquad \ldots (58)$$

The components of stresses and displacement in the region $c \leqslant r \leqslant \infty$ are

$$^{e}\sigma_r = -\frac{2T\sqrt{1-c\times\xi_{m_1}}\times r^{-m_1+(1/2)(n_1-2)}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})\times c^{-m_1+(1/2)(n_1-2)}},$$

$$^{e}\sigma_{\theta}=\frac{2T\sqrt{1-c}\times r^{-m_1+(1/2)(n_1-2)}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})\times c^{-m_1+(1/2)(n_1-2)}},$$

$$^{e}\sigma_{z}=-\frac{2T\sqrt{1-c}\times\bar{J}_{m_1}\times r^{-m_1+(1/2)(n_1-2)}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})\times c^{-m_1+(1/2)(n_1-2)}}$$

and

$$^{e}u=-\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}\times r^{-m_1-(n_1/2)}}{(\bar{\eta}_{m_1}-\bar{\xi}_{m_1})\times c^{-m_1+(1/2)(n_1-2)}}. \qquad \ldots (59)$$

Those in the region $a\leqslant r\leqslant c$ are

$$^{p}\sigma_{r}=-\left[\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})}+2T\sqrt{1-c}\log\frac{c}{r}\right],$$

$$^{p}\sigma_{\theta}=-\left[\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})}-2T\sqrt{1-c}\left(1-\log\frac{c}{r}\right)\right],$$

$$^{p}\sigma_{z}=-\frac{1}{G_0+F_0}\left[G_0\left\{\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})}+2T\sqrt{1-c}\log\frac{c}{r}\right\}\right. \qquad \ldots (60)$$

$$\left.+F_0\left\{\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})}-2T\sqrt{1-c}\left(1-\log\frac{c}{r}\right)\right\}\right]$$

and

$$^{p}u=-\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})}\times\frac{c^{-(n_1-2)}}{r}.$$

The displacement of the inner boundary of the matrix of infinite extent is given by

$$a\varepsilon=-\frac{2T\sqrt{1-c}\times\bar{\xi}_{m_1}}{(\bar{\xi}_{m_1}-\bar{\eta}_{m_1})}\times\frac{c^{-(n_1-2)}}{a} \qquad \ldots (61)$$

We have the relation between ε and δ given by

$$\varepsilon=\delta-\frac{2T\sqrt{1-c}}{\xi_{m_1}\times a^{n_1}}\left[\log\frac{c}{a}+\frac{\bar{\xi}_{m_1}}{\bar{\xi}_{m_1}-\bar{\eta}_{m_1}}\right] \qquad \ldots (62)$$

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