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PROPAGATION OF PLANE SHEAR WAVES THROUGH PERIODICALLY LAYERED COMPOSITE MEDIA

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Propagation of horizontally polarised shear harmonic waves corresponding to an antiplane strain motion through a periodically layered elastic composite body has been studied here. A general dispersion relation for waves in periodically layered composite has been deduced. It has been proved that the rigidity and density with respect to the depth of the layer affects the dispersion relation and the effect appears to be a linear superposition of a function of phase velocity and wave number. The roots of the frequency equation are either real or purely imaginary, the former implying a progressive type of waves in the elastic medium and the later implying locally standing waves.

Key Words : Propagation; Plane Shear Waves; Composite Media; Frequency Equation

Introduction

Wave propagation in layered media has been studied extensively by many authors, specially in the last two decades. Wave propagation in a heterogeneous medium has received more attention in connection with the development of composite materials. The medium is composed of two or more homogeneous layers of material with obvious discontinuities in the elastic coefficients at the interface. Various approximate theories have been proposed to predict the dynamical response in a layered medium. Some approximate theories are the stiffness theory by Sun, Achenbach and Herrmann[1], the theory of mixture by Stern, Bedford and Yew[2] and the theory of interacting continua by Hegemier and Bache[3]. Although most of the approximate theories can predict satisfactorily the dispersion relation due to a harmonic wave, the availability of exact solutions will be helpful in assessing the domains of validity.

Wave motion in a layered medium may be of considerable importance in problems involving non-destructive testing of composite material. Understanding of the dispersion effects of inhomogeneity in a single coordinate direction can help one to predict the elastic nature of composite layers. Also this will help in the theoretical determination of effective elastic property of composite materials, which is still an important problem in the field of materials science.

In a well-known paper, Rytov[4] obtained exact dispersion relations for the propagation of harmonic waves in directions normal and parallel to the layers of a bilaminated periodic composite. The problem herein deals with the propagation of horizontally polarized shear harmonic waves corresponding to an antiplane strain motion through a periodically layered elastic composite body. Each layer has been taken to be homogeneous with perfectly bonded interfaces with the adjacent lay-

ers. The main objective of the present analysis is to determine a general dispersion relation for waves in a periodically layered composite and to find the infuence of the basic cell upon the overall dispersion relations for the composite.

Formulation of the problem

Let us consider an elastic medium of infinite horizontal extent and finite thickness so that $|x| < \infty$, $|y| < \infty$ and $0 \leqq z \leqq z_{mn}$. It is further assumed to be consisting of n identical cells and each cell of m different layers. Therefore, the composite structure consists of $m \times n$ layers. Perfect bonding is assumed between the layers and cells. The wave motion is assumed to be of the antiplane-strain type with displacement field specified by $u_x = u_z = 0$ and $u_y = v(x, z, t)$. The jth layer in the rth fundamental cell is characterized by its thickness h_j, the rigidity $\mu_j \exp[\varepsilon\{z - z_{m,r-1}\}]$ and the density $\rho_j \exp[\varepsilon\{z - z_{m,r-1}\}]$, where $j = m(r-1)+1,\ m(r-1)+2, \ldots, mr$, $r = 1, 2, \ldots, n$ and ε is a parameter (Fig. 1).

Keeping in mind the above mentioned patterns of displacement, rigidity and density, in the absence of body force, the only existing equation of motion for the jth layer is

$$R_j\left(\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial z^2}\right) + \frac{dR_j}{dz}\frac{\partial v_j}{\partial z} = S_j \frac{\partial^2 v_j}{\partial t^2}, \qquad \ldots(1)$$

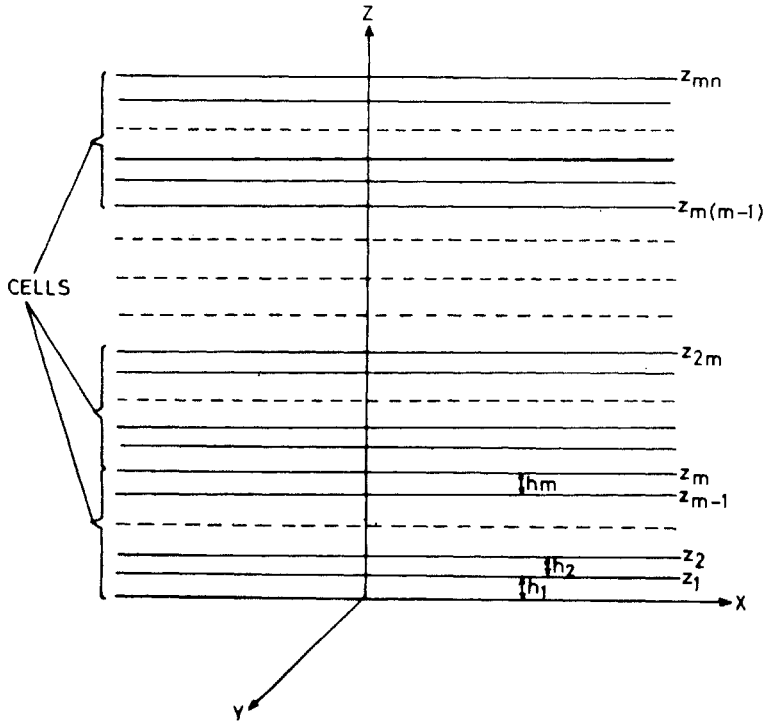

Fig. 1

where

$$R_j = \mu_j \exp\left[\varepsilon\{z - z_{m(r-1)}\}\right] \qquad \ldots(2)$$

and

$$S_j = \rho_j \exp\left[\varepsilon\{z - z_{m(r-1)}\}\right]$$

The upper and the lower faces of the composite body are stress free, a condition which implies the following type of non-mixed boundary conditions

$$\sigma_{yz}(x, 0, t) = 0, \quad \sigma_{yz}(x, z_{mn}, t) = 0, \qquad \ldots(3)$$

where

$$z_{mn} = n \sum_{j=1}^{m} h_j .$$

Solution of the Problem

For waves in the plate, the harmonic nature of displacement field $v(x, z, t)$ will be specified by the following assumption

$$v(x, z, t) = u(z) \exp\{ik(x - ct)\} \qquad \ldots(4)$$

where k represents the wave number and c, the phase velocity. Using (4), the equation of motion for the jth layer can be written as

$$\frac{d^2 u_j}{dz^2} + \varepsilon \frac{du_j}{dz} + p_j^2 u_j = 0, \qquad \ldots(5)$$

where

$$p_j^2 + k^2(1 - c^2/\beta_j^2) = 0, \beta_j^2 = \mu_j/\rho_j, 1 \leqslant j \leqslant mn. \qquad \ldots(6)$$

From (4) and (5), the displacement and the stress field in the jth layer of the fundamental cell are lgiven by

$$v_j(x, z, t) = \{P_j \cos s_j z + Q_j \sin s_j z\} \exp\{ik(x - ct) - \varepsilon z/2\}, \qquad \ldots(7)$$

and

$$(\sigma_{yz})_j(x, z, t) = \mu_j \left[s_j\{-P_j \sin s_j z + Q_j \cos s_j z\} - \frac{\varepsilon}{2}\{p_j \cos s_j z + Q_j \sin s_j z\}\right]$$

$$\times \exp\left[ik(x - ct) - \varepsilon\{z/2 - z_{m(r-1)}\}\right], \qquad \ldots(8)$$

where

$$s_j^2 = p_j^2 - \frac{\varepsilon^2}{4} \qquad \ldots(8a)$$

and P_j, Q_j are constants.

Let us consider the elastodynamic state connected with the interface $z = z_j$ as a column matrix with displacement and stress as its elements and denote it by V_j.

Thus

$$V_j^{\pm} = \begin{bmatrix} v \\ \sigma_{yz} \end{bmatrix}_{z=z_j^{\pm}}, \quad j = 0, 1, 2, \ldots \quad mn, \qquad \ldots(9)$$

where the approach of z to z_j from above or below is indicated by the superscripts + and − respectively. Following Haskell[5] the two elastodynamic states corresponding to the two surfaces of the jth layer are related through

$$V_j^- = A_j V_{j-1}^+, \quad j = 1, 2, \ldots, mn, \qquad \ldots(10)$$

where A_j is the transformation matrix related to the jth layer. Now in order to find A_j, we write V_j for $z_{j-1} < z < z_{j+1}$ in the following convenient form

$$V_j = \begin{bmatrix} e^{-\varepsilon z/2} & 0 \\ 0 & e^{\varepsilon(z/2 - z_{m(r-1)})} \end{bmatrix} \left[\begin{bmatrix} P_j \cos s_j z + Q_j \sin s_j z \\ \mu_j s_j(-p_j \sin s_j + Q_j \cos s_j z) \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 & 0 \\ P_j & Q_j \end{bmatrix} \begin{bmatrix} 0 & \mu_j \cos s_j z \\ 0 & \mu_j \sin s_j s \end{bmatrix} \begin{bmatrix} 0 \\ -\varepsilon/2 \end{bmatrix} \right] \qquad \ldots(11)$$

$$= L_j^{-1} \{L_j^{-2} + L_j^3 L_j^{-4} L_5\},$$

where

$$L_j^{-1} = \begin{bmatrix} e^{-\varepsilon z/2} & 0 \\ 0 & e^{\varepsilon(z/2 - z_{m(r-1)})} \end{bmatrix},$$

$$L_j^{-2} = \begin{bmatrix} P_j \cos s_j z + Q_j \sin s_j z \\ \mu_j s_j(-P_j \sin s_j z + Q_j \cos s_j z) \end{bmatrix}, \qquad \ldots(12)$$

$$L_j^3 = \begin{bmatrix} 0 & 0 \\ P_j & Q_j \end{bmatrix}, \quad L_j^{-4} = \begin{bmatrix} 0 & \mu_j \cos s_j z \\ 0 & \mu_j \sin s_j z \end{bmatrix} \text{ and } L_5 = \begin{bmatrix} 0 & 0 \\ -\varepsilon/2 & 0 \end{bmatrix}.$$

This type of decomposition of V_j will permit us to visualize the number and form of transformation matrices required to relate the elastodynamic conditions of two faces of an elementary layer.

We assume that the two elastodynamic states for the jth layer are related through

$$V_j^- = f(R_j^1, R_j^2, \ldots, R_j^r, L_{j-1}^1, L_{j-1}^2, L_{j-1}^3, L_{j-1}^4, L_5), \qquad \ldots(13)$$

where

$$\lim_{z \to z_{j-1}^{+0}} L_j^{-r} = L_{j-1}^r, \quad r = 1, 2, 4$$

and R_j^m are different transformation matrices required to transform different constituent matrices. A simple calculation gives

$$V_j^- = R_j^1 L_{j-1}^1 \{R_j^2 L_{j-1}^2 + L_j^3 (R_j^3 L_{j-1}^4) L_5\}, \qquad \ldots(14)$$

where

$$R_j^1 = \begin{bmatrix} e^{-\varepsilon h_j/2} & 0 \\ 0 & e^{\varepsilon h_j/2} \end{bmatrix}$$

$$R_j^2 = \begin{bmatrix} \cos s_j h_j & \sin s_j h_j \\ -\mu_j s_j \sin s_j h_j & \mu_j \cos s_j h_j \end{bmatrix} \qquad \ldots(15)$$

$$R_j^3 = \begin{bmatrix} \mu_1 \cos s_j h_j & -\mu_j \sin s_j h_j \\ \mu_j \sin s_j h_j & -\mu_j \cos s_j h_j \end{bmatrix}, \qquad j = 1, 2, \ldots, mn.$$

In the same way the elastodynamic states at the top and the bottom of the $(j-1)$th layer are related through

$$V_{j-1}^- = R_{j-1}^1 L_{j-2}^1 \{R_{j-1}^2 L_{j-2}^2 + L_{j-1}^3 R_{j-1}^3 L_{j-2}^4 L_5\} \qquad \ldots(16)$$

Since layers are assumed in welded contact, the continuity of the stress and displacement field at the jth interface gives

$$V_j^+ = V_j^- . \qquad \ldots(17)$$

Using this in the $(j-1)$th interface, we have from V_j^- and V_{j-1}^-,

$$V_j^- = R_j^1 R_{j-1}^1 L_{j-2}^1 \{R_j^2 R_{j-1}^2 L_{j-2}^3 + L_{j-1}^3 (R_j^3 R_{j-1}^3 L_{j-2}^4) L_5\}, \qquad j = 1, 2, \ldots, mn \quad \ldots(18)$$

In this way the elastodynamic states at the top and bottom of the composite medium are related through

$$V_{mn}^- = \left\{ \prod_{i=0}^{mn-1} \right\} R_{mn-1}^1 L_0^1 \left[\left\{ \prod_{i=0}^{mn-1} R_{mn-i}^2 \right\} L_0^2 \right.$$

$$\left. + L_1^3 \left\{ \left(\prod_{i=0}^{mn-1} R_{mn-i}^3 \right) L_0^4 \right\} L_5 \right]. \qquad \ldots(19)$$

As the composite structure is periodic, transformation matrices $R_j = R_{j+rm}$, $r = 1, 2, \ldots, n$, $l = 1, 2, 3$ and hence, Eq. (19) can be replaced by

$$V_{mn}^{-} = \left(\prod_{i=0}^{m-1} R_{m-i}^{1}\right)^{n} L_0^1 \left[\left(\prod_{i=0}^{m-1} R_{m-i}^{2}\right)^{n} L_0^2 + L_1^3 \left\{\left(\prod_{i=0}^{m-1} R_{m-i}^{3}\right)^{n} L_0^4\right\} L_5\right] \qquad \ldots(21)$$

The set S, defined by

$$S = \left\{\left(\prod_{i=0}^{m-1} R_{m-i}^{l}\right), l = 1, 2, 3\right\}. \qquad \ldots(22)$$

gives the characteristics of a fundamental cell. It might be noted here that the elements of the set S are all 2×2 matrices. Denoting the entries of the matrix $\mathbf{B}$, where

$$\mathbf{B} = \left[\prod_{i=0}^{m-1} R_{m-1}^{l}\right]^{n}, l = 1, 2, 3$$

by b_{ij}, $i, j = 1, 2$, $l = 1, 2, 3$ and utilizing the boundary conditions (3), it follows from (21) that the equation

$$b_{21}^{2} - \frac{\varepsilon}{2} b_{21}^{3} = 0 \qquad \ldots(23)$$

gives the corresponding dispersion equation. Now denoting the elements of the matrix $\mathbf{A}_l = \prod_{i=0}^{m-1} R_{m-i}^{l}$ by a_{ij}, $i, j = 1, 2$, $l = 1, 2, 3$, the dispersion eq. (23) can be written as

$$a_{2i_1}^{2} a_{i_1 i_2}^{2} a_{i_2 i_3}^{2} \ldots a_{i_{n-3} i_{n-2}}^{2} a_{i_{n-2} i_{n-1}}^{2} a_{i_{n-1} l}^{2}$$

$$- \frac{\varepsilon}{2} a_{2i_1}^{3} a_{i_1 i_2}^{3} a_{i_2 i_3}^{3} \ldots a_{i_{n-3} i_{n-2}}^{3} a_{i_{n-2} i_{n-1}}^{3} a_{i_{n-1} l}^{3} = 0, \qquad \ldots(24)$$

where summation convention over the repeated indices has been exploited, each index taking two values only, viz., 1 and 2.

For example, for a sandwiched plate consisting of one, two, three etc. of fundamental cells, the dispersion equations are:

$$n = 1, a_{21}^{2} - \frac{\varepsilon}{2} a_{21}^{2} = 0,$$

$$n = 2, a_{2i_1}^{2} a_{i_1 1}^{2} - \frac{\varepsilon}{2} a_{2i_1}^{3} a_{i_1 1}^{3} = 0,$$

$$n = 3, a_{2i_1}^{2} a_{i_1 i_2}^{2} a_{i_2 1}^{2} - \frac{\varepsilon}{2} a_{2i_1}^{3} a_{i_1 i_2}^{3} a_{i_2 1}^{3} = 0$$

and

$$n=4,\ a^2_{2i_1}a^2_{i_1i_2}a^2_{i_2i_3}a^2_{i_31}-\frac{\varepsilon}{2}a^3_{2i_1}a^3_{i_1i_2}a^3_{i_2i_3}a^3_{i_31}=0.$$

These equations can be rewritten as

$$n=1,\ a^2_{21}-\frac{\varepsilon}{2}a^3_{21}=0, \qquad \ldots(25)$$

$$n=2,\ a^2_{21}a^2_{i_1i_1}-\frac{\varepsilon}{2}a^3_{21}a^3_{i_1i_1}=0 \qquad \ldots(26)$$

and

$$n=3,\ a^2_{21}(a^2_{1i_1}a^2_{i_11}+a^2_{22}a^2_{i_1i_1})-\frac{\varepsilon}{2}a^3_{21}(a^3_{1i_1}a^3_{i_11}+a^3_{22}a^3_{i_1i_1})=0 \qquad \ldots(27)$$

$$n=4,\ a^2_{21}[a^2_{i_11}a^2_{i_1i_2}a^2_{i_21}+a^2_{22}(a^2_{1i_1}a^2_{i_11}+a^2_{22}a^2_{i_1i_1})]$$

$$-\frac{\varepsilon}{2}a^3_{21}[a^3_{i_11}a^3_{i_1i_2}a^3_{i_21}+a^3_{22}(a^3_{1i_1}a^3_{i_11}+a^3_{22}a^3_{i_1i_1})]=0 \qquad \ldots(28)$$

Discussion

We note the following salient features of the dispersion equations (25)–(28):

(*i*) A variation in rigidity and density with respect to the depth of the layer affects the dispersion relation and the effect appears to be a linear supperposition of a function of phase velocity and wave number on the corresponding dispersion equation for the case by Dravinski[6], where the composite has been considered to be layerwise homogeneous.

(ii) The two parts of the dispersion relation of a fundamental cell can be separately factored out which facilitates numerical computation. Therefore the roots of the dispersion equation can conveniently be found out by evaluating roots for the single cell first and then proceeding with the roots associated with the additional cells.

For every choice of frequency, there are an infinite number of roots of the frequency equation, which represent the wave number of an infinite number of modes of wave propagation in the layers.

The roots of the frequency equation are assumed to be either real or purely imaginary, the former implying a progressive type of waves in the elastic medium, while the latter locally standing waves, which exponentially decay with increasing x.

Example

Now let us examine, in particular, the dispersion equation corresponding to the case $m=2$, $n=1$, which, in the explicit form, appears as

$$\mu_2 s_2 \cos s_1 h_1 \sin s_2 h_2 + \mu_1 s_1 \sin s_1 h_1 \cos s_2 h_2 + \frac{\varepsilon}{2} \mu_1 \mu_2 \sin (s_1 h_1 + s_2 h_2) = 0. \quad \ldots(29)$$

The top layer is taken to be of unit thickness, with rigidity, density all equal to one. Rigidity, density and thickness for the bottom layer of the cell are assumed to be 2, 2 and 1 respectively.

Substitution of these into the dispersion relation (27) leads to the following three different cases:

(i) $\sin \sqrt{w^2 - k^2 - (\varepsilon/2)^2} = 0;$

(ii) $\cos \sqrt{w^2 - k^2 - (\varepsilon/2)^2} = 0;$ and

(iii) $3\sqrt{w^2 - k^2 - (\varepsilon/2)^2} + 2\varepsilon = 0,$

where we have replaced s_1 and s_2 by either explicit expressions from (6) and (8a).

For the dispersion relation to be fulfilled any one of the above three conditions must be satisfied. The first two conditions are satisfied if

$$\sqrt{w^2 - k^2 - (\varepsilon/2)^2} = \frac{n\pi}{2},$$

where

$n = 0, 2, 4, 6, \ldots$. for condition (i)

and

$n = 1, 3, 5, \ldots$. for condition (ii).

Here different values of n correspond to different modes.

The condition (iii) gives

$$w^2 - \left(\frac{5\varepsilon}{6}\right)^2 = k^2$$

which shows that besides the above mentioned modes the wave will propagate with another mode given by $n = 5\varepsilon/6$, which is due only to the inhomogeneity parameter.

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