Proc. Indian natn. Sci. Acad., **58**, A, No. 5, 1992, pp. 397-409.
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PANDIT JAWAHARLAL NEHRU BIRTH CENTENARY LECTURE—1992

PHASE SPACE METHODS IN QUANTUM MECHANICS AND OPTICS

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(*Delivered* on* 27 *March* 1992)

An account is given of the many ways in which methods based on phase space and the real symplectic groups can be fruitfully exploited in quantum mechanics and optics. These include the concept of generalised rays and action of optical systems on them; the properties of variance matrices for general n-dimensional quantum systems; a coset-space analysis of Gaussian pure states; a global approach to the concept of squeezing for multimode systems; and an intriguing instance of complementarity between Bose statistics for generalized rays and action of optical systems on them.

Key Words: Phase Space Methods; Quantum Mechanics; Optics; Gaussian States; Bose Statistics

Introduction

I would like to describe here some work done over the past few years in the general areas of quantum mechanics and optics, exploiting phase space methods and properties of the family of real symplectic groups. As it will become clear as we go along, it has been a case of successful transfer of ideas and techniques from one area to another. Anyone working in theoretical high energy physics or related fields gets acquainted with a variety of methods based on mechanics, group theory, operator techniques and ideas of symmetry. We have found it possible to exploit such methods in theoretical optics quite effectively, and it is always a pleasure to see that what one has learnt in one field can be used to illuminate another.

Let me begin with a brief historical introduction. As is well known, when one makes the transition from classical to quantum mechanics, classical commuting dynamical variables get replaced by noncommuting quantum mechanical operators. One can ask whether it is possible to set up in any reasonable way a one-to-one correspondence between these two sets of quantities. It turns out that there is no natural, intrinsic or preferred correspondence of this kind, but there are sever-

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al, each based on some convention. The most convenient of these, in many ways, is that due to Weyl, which was suggested by him already around 1930[1]. Let us recall the details by looking first at systems with one degree of freedom. For such a classical system we have a two-dimensional phase space with basic variables q and p, and a general classical dynamical variable is some numerical function of them. For a quantum system with one degree of freedom, we have operators $\hat{q}$ and $\hat{p}$ obeying the Heisenberg commutation relation

$$[\hat{q}, \hat{p}] = i\hbar, \qquad \ldots (1)$$

and a general dynamical variable is an operator function of these basic ones. The Weyl rule of association is completely specified by saying that it is linear and that, for all real numbers λ and μ we have the correspondence:

classical variable $e^{i(\lambda q - \mu p)} \rightarrow$

$$\text{quantum operator } e^{i(\lambda \hat{q} - \mu \hat{p})} \qquad \ldots (2)$$

Therefore, for a general classical quantity $f(q,p)$ the quantum operator image is given by performing a Fourier transformation:

$$f(q,p) = \int d\lambda\, d\mu\, \tilde{f}(\lambda, \mu)\, e^{i(\lambda q - \mu p)} \rightarrow$$

$$\hat{F} = \int d\lambda\, d\mu\, \tilde{f}(\lambda, \mu)\, e^{i(\lambda \hat{q} - \mu \hat{p})} \qquad \ldots (3)$$

A nice feature of this Weyl rule is that real classical quantities get mapped to hermitian quantum operators. Of course, from $\hat{F}$ one can recover $f(q, p)$.

Somewhat later, in 1932, Wigner introduced an interesting way to represent states of a quantum system, somewhat different from their description using wave functions[2]. This was in the context of studying departures of quantum statistical mechanics from classical statistical mechanics. While the representation of quantum states *via* state vectors or wave functions emphasizes the superposition principle of quantum mechanics, the Wigner representation is at the level of the density operator, and so the superposition principle is in a sense hidden and not manifest. Again for a system of one-degree of freedom if the wave function $\Psi(q)$ is known, then the Wigner distribution is a function $W(q, p)$ on the classical phase space defined by

$$W(q,p) = \frac{1}{2\pi\hbar} \int dq'\, \Psi\left(q - \frac{1}{2}q'\right) \Psi^*\left(q + \frac{1}{2}q'\right) e^{ipq'/\hbar} \qquad \ldots (4)$$

It is easy to see that this is a real function on phase space, but it is not always or everywhere nonnegative in general. And what ties together the Weyl and Wigner associations is this: if a state $|\Psi>$ and a dynamical variable $\hat{F}$ are given for a quantum mechanical system, and the classical variable corresponding to $\hat{F}$ is the function $f(q, p)$, then

$$<\Psi|\hat{F}|\Psi> = \int dq\, dp\, f(q,p)\, W(q,p). \qquad \ldots (5)$$

So quantum expectation values appear like classical phase space averages.

One can appreciate that even the noncommutative multiplication of quantum operators, and the taking of their commutators, can be given a classical phase space description. This was actually done in explicit form by Moyal some years later[3], so we shall refer to this entire technology or machinery as the WWM method in quantum mechanics.

One important feature of all this which, however, seems not to have been so clearly expressed, or at least not much emphasized, is the very special role played by the family of real symplectic groups $Sp(2n, R)$[4]. This group already appears when one has one degree of freedom, $n=1$. In most applications of group theory to problems of symmetry in quantum mechanics, the groups one often encounters are the unitary or real orthogonal ones, $SU(n)$ or $SO(n)$. As a result most of us have an intuitive appreciation or understanding of their structures and properties. In comparison the symplectic groups seem to have some strangeness about them. This is rather unfortunate since they are so closely connected to the basic structures of phase space and canonical Hamiltonian mechanics, which are at the base of both classical and quantum mechanics. I shall introduce and briefly describe this family of groups, and the connection to the WWM method, after generalising the latter to an arbitrary number, n, of degrees of freedom.

To simplify matters hereafter let us set $\hbar=1$ when dealing with quantum mechanics. Now consider a classical system with a $2n$-dimensional phase space and canonical coordinates q_r, p_r, $r=1, 2, \ldots, n$. To deal with them and express Poisson Brackets (PB) in a compact manner we collect them into a $2n$-component column vector (ξ_a):

$$\xi=(\xi_a)=\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{2n}\end{pmatrix}=\begin{pmatrix}q_1\\ \vdots\\ q_n\\ p_1\\ \vdots\\ p_n\end{pmatrix} \qquad \ldots (6)$$

We list the q's first and then the p's. The basic PB's can be written as

$$\{\xi_a, \xi_b\}=\beta_{ab}, \beta=\begin{pmatrix}0 & 1_{n\times n}\\ -1_{n\times n} & 0\end{pmatrix} \qquad \ldots (7)$$

This matrix β-antisymmetric and even dimensional—will turn out to be fundamental for $Sp(2n, R)$ as a "symplectic metric". For the quantum mechanical case we have operators $\hat{q}_r$, $\hat{p}_r$ put together to form a column vector $\hat{\xi}$ obeying canonical commutation relations:

$$\hat{\xi}=\begin{pmatrix}\hat{q}_1\\ \vdots\\ \hat{q}_n\\ \hat{p}_1\\ \vdots\\ \hat{p}_n\end{pmatrix} : [\hat{\xi}_a, \hat{\xi}_b]=i\beta_{ab} \qquad \ldots (8)$$

Now let $\hat{\Gamma}$ be any operator pertaining to the quantum system. It could for instance be the density operator for a pure or mixed state. We define the Wigner distribution corresponding to $\hat{\Gamma}$ to be a function on the corresponding classical phase space by

$$W(\xi)=(2\pi)^{-n}\int \mathrm{d}^n q' < \underline{q}-\frac{1}{2}\underline{q}'|\hat{\Gamma}|\underline{q}+\frac{1}{2}\underline{q}' > e^{i\underline{p}\cdot\underline{q}'} \qquad \ldots (9)$$

It is simple to check that one can recover the coordinate-space kernel of $\hat{\Gamma}$ from $W(\xi)$; so $\hat{\Gamma}$ and $W(\xi)$ contain the same information. Then the n-dimensional version of (5) holds and we may express it as

$$Tr(\hat{\Gamma}\exp\{i(\underline{\lambda}\cdot\hat{\underline{q}}-\underline{\mu}\cdot\hat{\underline{p}})\})=\int \mathrm{d}^{2n}\xi W(\xi)\exp\{i(\lambda\cdot q-\mu\cdot p)\} \qquad \ldots (10)$$

By linearity and Fourier transformation this kind of relation holds for expectation values of general quantum mechanical operators.

The $2n$-dimensional real symplectic group $Sp(2n, R)$ can now be introduced. It consists of all real linear invertible transformations on the q's and p's such that classically the PB's and quantum mechanically the commutation relations are preserved:

$$\xi'_a=S_{ab}\xi_b \;:\; \{\xi'_a, \xi'_b\}=\beta_{ab},$$

and

$$\hat{\xi}'_a=S_{ab}\hat{\xi}_b \;:\; [\hat{\xi}'_a, \hat{\xi}'_b]=i\beta_{ab}\,. \qquad \ldots (11)$$

The condition on $\boldsymbol{S}$-the defining property of $Sp(2n, R)$-is

$$\boldsymbol{S}\in Sp(2n, R)\Leftrightarrow \boldsymbol{S\beta S}^T=\boldsymbol{\beta}\,. \qquad \ldots (12)$$

In other words the symplectic metric must be preserved. Because $\boldsymbol{\beta}$ is antisymmetric non-singular, these groups exist in even dimensions only.

The preservation of the PB's or commutation relations, as the case may be, means that classically we have a canonical transformation, and in quantum mechanics a unitary one. The latter means that for any symplectic matrix $\boldsymbol{S}$, we have a unitary operator $U(S)$ which implements the above transformation through conjugation:

$$\boldsymbol{S}\in Sp(2n, R)\colon U(S)^{-1}\,\xi U(S)=S\xi,$$
$$U(S_1)U(S_2)=\pm U(S_1S_2) \qquad \ldots (13)$$

There is an unavoidable sign ambiguity here, similar to the sign ambiguity in the spin half representation of angular momentum. So one says that one has here a faithful representation, not of $Sp2n, R)$ but of its two-fold covering metaplectic group $Mp(2n)$[5].

Now comes the fundamental connection to and significance of the Wigner distribution, indeed of the entire WWM technique. It is that when an operator $\hat{\boldsymbol{\Gamma}}$ undergoes the operator symplectic transformation, its Wigner function changes in a very simple way by a point transformation in phase space:

$$\hat{\boldsymbol{\Gamma}}'=U(S)\Gamma U(S)^{-1} \leftrightarrow W'(\xi)=W(S^{-1}\xi) \qquad \ldots (14)$$

Indeed one can view the WWM procedure as a way to make this operator action of $Sp(2n, R)$ as simple and classical looking as possible. We can take this as the defining characteristic of this procedure. The infinitesimal aspect of this relationship is also worth exhibiting. Since we are considering linear canonical transformations, the generators are quadratic in the q's and p's. That is, the unitary operator $U(S)$ is essentially the exponential of a symmetric quadratic expression in the operators $\hat{q}$, $\hat{p}$. Pursuing this, one finds that the processes of taking the anticommutators or commutators of an operator $\hat{\Gamma}$ with the $\hat{q}$'s and $\hat{p}$'s has a nice appearance in terms of the Wigner function[6]:

$$\frac{1}{2}\{\hat{\xi}_a, \hat{\Gamma}\} \leftrightarrow \xi_a W(\xi),$$

and

$$[\hat{\xi}_a, \hat{\Gamma}] \leftrightarrow i\beta_{ab}\frac{\partial}{\partial \xi_b} W(\xi). \qquad \ldots (15)$$

It is this intimate relationship between the symplectic groups and the WWM method in quantum mechanics that somehow seems to have been not as much emphasized and exploited as one might have expected. Already for one degree of freedom, $n=1$, we have something nontrivial. The group $Sp(2, R)$ is easy to define as a group of real 2×2 unimodular matrices:

$$Sp(2, R) = \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | ad - bc = 1 \right\} \qquad \ldots (16)$$

And there are many uses of it in various contexts.

After this brief exposure to these concepts, let me turn for a while to some problems in statistical optics. As is well known, for the theoretical treatment of all classical optical experiments, and for a first understanding of coherence and partial coherence, the basic object is the so-called two-point correlation function whose study in a systematic way was pioneered by Wolf long ago[7]:

$$\Gamma(x; x') \sim \langle A(x) A(x')^* \rangle \qquad \ldots (17)$$

I leave unspecified the dimensionality of the spatial or space-time arguments x and x'. Here, $A(x)$ is the vector potential or the electric field as one wishes; more precisely it is the positive frequency (analytic signal) part of the full real field. And the angular brackets denote an ensemble or statistical average. This two-point function can be defined either classically or quantum mechanically, though at this level one cannot distinguish between the two theories. The intensity of light at a space-time point is a particular case of this function, when the arguments coincide. And what we mean by classical optical experiments are all those which only involve measurements of intensity of light at one space-time point at a time. The two-point function is hermitian in the obvious sense, and regarded as the kernel of an operator it is also positive semi-definite.

Now Wolf in 1976 posed the following problem[8]: is there a way to obtain the phenomenological quantities and equations of the theory of radiative transfer

from a starting point based on Maxwell's field equations and statistical ideas? He came to the conclusion that this was possible provided one worked with the 2-point function, and the ensemble was time-stationary. The latter means that one can deal with radiation of different frequencies independently. However, Wolf's original prescription for going from the two-point function to the quantities that arise in radiative transfer theory turned out to be not very convenient, and a much more elegant prescription was soon given by Sudarshan in 1979[9]. This was just a transcription of the entire WWM technology from quantum mechanics to optics! I will describe it now in the context of paraxial optics[10]—this is the situation where one has a beam of light propagating along a given spatial direction, say the z-axis; and one is interested in the properties across any transverse plane, and how they develop during propagation.

Let then $\boldsymbol{\Gamma}(x_\perp, x'_\perp)$ denote the two-point correlation function of a beam across a transverse plane, so $x_\perp$ and $x'_\perp$ are two-dimensional vectors, and the common coordinate z is suppressed. So except for the normalization condition it is just like the density matrix for a two-dimensional quantum system. In optics, one has the total beam intensity given by

$$I = \int \mathrm{d}^2 x_\perp \, \boldsymbol{\Gamma}(x_\perp; x_\perp) \qquad \ldots (18)$$

whereas in quantum mechanics one would have demanded that this integral be unity. Now the WWM transform of $\boldsymbol{\Gamma}(x_\perp; x'_\perp)$ is called the Wolf function or generalized ray distribution function $W(x_\perp; p_\perp)$:

$$W(\xi) = W(x_\perp; p_\perp) = (2\pi)^{-2} \int \mathrm{d}^2 x'_\perp \, \boldsymbol{\Gamma}\left(x_\perp - \frac{1}{2} x'_\perp; x_\perp + \frac{1}{2} x'_\perp\right) e^{ip_\perp \cdot x'_\perp},$$

$$\xi = \begin{pmatrix} x_\perp \\ p_\perp \end{pmatrix} \qquad \ldots (19)$$

One says that $W(x_\perp; p_\perp)$ is the "intensity" of generalized rays of light at transverse position $x_\perp$ and transverse direction $p_\perp$. We must remember of course that while $W(\xi)$ is definitely real, it could sometimes be negative. So in the terminology introduced by Sudarshan[9], when $W(\xi)$ is positive we say we have bright rays, and when it is negative we say we have dark rays. But the true justification for the use of the name "generalised rays" really comes from the connection to the symplectic groups. Let me explain.

In ray optics proper there is a class of optical systems called First Order Optical Systems—FOS for short-defined by the way they act on the position and direction coordinates of geometrical rays[11]. Remembering the paraxial limit, an incoming ray is specified by the four variables $\xi = (x_\perp; p_\perp)$ of transverse position and direction. FOS's stand in one-to-one correspondence with matrices $S \in Sp(4, R)$ and act on rays in this way:

FOS Action on Geometrical Rays

$$\begin{pmatrix} x_\perp \\ p_\perp \end{pmatrix}_{in} \rightarrow \begin{pmatrix} x'_\perp \\ p'_\perp \end{pmatrix}_{out} = S \begin{pmatrix} x_\perp \\ p_\perp \end{pmatrix}_{in}, S \in Sp(4, R). \quad \ldots (20)$$

One may ask why the symplectic nature comes in here at all. The reason is beautiful and deep: it is because ray optics is governed by Fermat's principle, and this is a variational or extremum principle just as in Lagrangian mechanics. So the entire canonical structure and machinery is automatically there and one cannot escape it! Now when one goes to wave optics (let us ignore polarization here), there is a wave amplitude $\Psi(x_\perp)$ over any transverse plane, and the passage of the wave through any *FOS* is given by the action of an integral kernel on the amplitude: it is just the action by the metaplectic unitary operator $\boldsymbol{U(S)}$ encountered earlier!

FOS Action on Waves

$\Psi' = \boldsymbol{U(S)}\,\Psi$:

$$\Psi'(x'_\perp) = \int d^2_{x_\perp} < x'_\perp |\boldsymbol{U(S)}| x_\perp > \Psi(x_\perp). \quad \ldots (21)$$

I am using here the suggestive and natural notation of quantum mechanics in two dimensions. This integral kernel $< x'_\perp |\boldsymbol{U(S)}| x_\perp >$ is called the generalized Huyghens kernel, and is the exponential of a quadratic expression in $x'_\perp$ and $x_\perp$[12]. What the definition of the generalized ray distribution function achieves is that this action of *FOS*'s on waves gets reduced to the kind of action we had on geometrical rays, but now in the context of a full wave theory. We go from Ψ to $\boldsymbol{\Gamma}$ and then on to W to find:

FOS Action on Generalized Rays

$$W'(\xi) = W(S^{-1}\xi) \quad \ldots (22)$$

With our improved appreciation of the significance of the symplectic groups, we can in fact regard this simple action of *FOS*'s as the prime motivation for introducing the Wolf function and creating the concept of generalized rays rather than as a bridge to the concepts of radiative transfer theory.

For axially symmetric optical systems we have a simplification from the group $Sp(4, R)$ to $Sp(2, R)$. I might indicate some examples of physical systems and give their corresponding matrix representatives in this case[11]:

Free propagation over distance D: $f(D) = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix}$

Lens of power g: $l(g) = \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix}$

Magnification: $m(\xi) = \begin{pmatrix} e^{\xi} & 0 \\ 0 & e^{-\xi} \end{pmatrix}$

$$\text{Fourier transformation: } \mathscr{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \ldots (23)$$

The key point is that the action of integral transformations on the wave amplitude gets reduced to simple point transformations on generalized rays. The rays are *generalized*, not geometrical, because we are presenting here a complete wave theory, not a short wave length approximation to it!

Now I would like to outline some further uses of the symplectic groups in quantum mechanics and optics, in conjunction with the WWM method. We shall be concerned with general n-dimensional quantum systems - paraxial optics corresponding to $n=2$. For any quantum state (or optical beam) the variances or spreads in the basic coordinates and momenta contain a certain amount of important information about the state of the system. If the state, i.e., the two-point function or density matrix or Wigner-Wolf function, is furthermore Gaussian, these variances essentially determine the state completely (the additional information being the means). In general, for any state, we define the real symmetric $2n$-dimensional variance matrix $\boldsymbol{V}$ as

$$\boldsymbol{V} = (V_{ab}),$$

$$V_{ab} = \frac{1}{2} Tr\hat{\boldsymbol{\Gamma}}\{\hat{\xi}_a - \xi^{(0)}_a, \hat{\xi}_b - \xi^{(0)}_b\} / Tr\hat{\boldsymbol{\Gamma}}, \qquad \ldots (24)$$

where $\xi^{(0)}_a$ are the mean values of the $\hat{\xi}_a$. In terms of the Wigner-Wolf function this has a neat form

$$V_{ab} = \int d^{2n}\xi\ \xi_a \xi_b\, W(\xi) / \int d^{2n}\xi\, W(\xi) - \xi^{(0)}_a\, \xi^{(0)}_b \qquad \ldots (25)$$

Since the means are easy to handle, I omit them hereafter. Under action by any symplectic transformation (FOS in optics) this matrix has a nice law of change:

$$\boldsymbol{\Gamma}' = \boldsymbol{U}(\boldsymbol{S})\hat{\boldsymbol{\Gamma}}\boldsymbol{U}(\boldsymbol{S})^{-1} \rightarrow \boldsymbol{V}' = \boldsymbol{S}\boldsymbol{V}\boldsymbol{S}^{T} \qquad \ldots (26)$$

If we assume that we have a Gaussian phase space distribution, the knowledge of $\boldsymbol{V}$ fixes $W(\xi)$ completely:

$$\text{Gaussian distribution: } W(\xi) \simeq \exp\left(-\frac{1}{2}\xi^{T}\boldsymbol{V}^{-1}\xi\right) \qquad \ldots (27)$$

Certain properties of the variance matrix are evident upon inspection—these are that it has to be real symmetric positive definite. But not every such matrix qualifies to be the variance matrix of some physical state! There are certain inequalities that $\boldsymbol{V}$ must obey in order to be physically allowed. It is these that ensure that when we go back from $W(\xi)$ to the coordinate space kernel of $\boldsymbol{\Gamma}$, then this operator is positive semidefinite. In quantum mechanics these are the usual Heisenberg uncertainty relations, or rather extensions of them taking into account all the elements of the variance matrix. In optics they are similar uncertainty relations for the positions and directions of hypothetical rays in wave optics. One can ask if there is

a way to state these conditions on V in an explicitly symplectic invariant form. It is interesting that this can be done[13]; the key is a very nice theorem due to Williamson which I would like to describe at this point[14].

As is well known, any real symmetric matrix can be brought to diagonal form by a suitable real orthogonal transformation, and then the diagonal elements are the eigenvalues of the original matrix. However, under $Sp(2n, R)$, the variance matrix transforms by a symmetric symplectic transformation, as in (26), and this is not a similarity transformation. Of course the reality, symmetry and positive definiteness of V are maintained. Now the theorem of Williamson says: if V is symmetric positive definite, we can definitely find a symplectic matrix S such that SVS^T is diagonal. Of course the resulting diagonal elements will in general not be eigenvalues of V. (Notice by the way that if V is not symmetric positive (or negative) definite, it may not be possible to "diagonalise" it with a symplectic S). The complete set of inequalities on V expressing the uncertainty principles in invariant form are just these: with S chosen such that SVS^T is diagonal, the elements of V must obey the ordinary Heisenberg relations for each degree of freedom:

$$\Delta q'_r \Delta p'_r \geq \frac{1}{2}, \quad r = 1, 2, ..., n. \qquad \ldots (28)$$

These then ensure that $\hat{\Gamma}$ as an operator is nonnegative. It is possible to express these inequalities directly in terms of V, without having to pass to its Williamson diagonal form at all[13]. It involves working with the traces $Tr(\beta V)^{2l}$ for $l = 1, 2, ..., n$, which are individually $Sp(2n, R)$ invariant. But I will omit the details.

The Gaussian pure states of n-dimensional quantum systems have a very interesting group theoretical structure which only the combination of phase space and symplectic methods can reveal in a reasonably straightforward way. Let me just indicate how it goes[15]. Such states are described by normalised complex Gaussian wave functions:

$$\Psi(q) \simeq \exp\left\{-\frac{1}{2} q^T \left(u + iv\right) q\right\} \qquad \ldots (29)$$

So apart from a phase each such state is specified by a pair of real symmetric $n \times n$ matrices (u, v), with the former being positive definite. It is now turns out that such states are in one-to-one correspondence with points of the coset space $Sp(2n, R)/U(n)$, where $U(n)$ in the maximal compact subgroup of the noncompact group $Sp(2n, R)$. When we act on this $\Psi(q)$ by the unitary operator $U(S)$ for some $S \in Sp(2n, R)$, we get another similar Gaussian pure state; and any two such states can be connected in this way. (We are ignoring here the niceties of the metaplectic group!). The change in Ψ is given as a point transformation on the coset space by a very pretty formula; the point (u, v) jumps to another point (u', v') specified as follows:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, R), \Lambda = (iu - v)^{-1}:$$

$$\Psi' = U(S)\Psi \Rightarrow \Lambda' = (A\Lambda + B)(C\Lambda + D)^{-1} \qquad \ldots (30)$$

It would take me too far to present any more details but I thought I should at least mention this result. By the way, for one degree of freedom, Λ becomes a complex number in the lower half plane, and we have here a Mobius transformation. So what we have above is the n-dimensional matrix generalisation of the complex half plane and these transformations, which is quite an intricate structure[16].

Returning to the properties of variance matrices for general states, the combined use of phase space and group theoretical ideas help us understand in a comprehensive way the notion of squeezing. Both theoretically and experimentally this is of considerable interest these days. While much work has been done, it turns out that the methods I have been describing can teach us something new even for one-dimensional systems. They also show how to define squeezing for systems with n degrees of freedom.

Recall that for one degree of freedom, in dimensionless form, such as for a single mode of the radiation field, the Heisenberg uncertainty principle says

$$\Delta q \Delta p \geqslant \frac{1}{2} \qquad \ldots (31)$$

Here there is no reference to the cross variance $\Delta(qp)$. Usually, a state is said to be squeezed if one of the two spreads, Δq or Δp, is less than $\frac{1}{\sqrt{2}}$. We have proposed and analyzed in detail a more satisfactory definition of squeezing, involving the complete 2×2 variance matrix, and possessing a reasonable degree of invariance[17]. Our definition is:

$$V = \begin{pmatrix} (\Delta q)^2 & \Delta(qp) \\ \Delta(qp) & (\Delta p)^2 \end{pmatrix}, \det V \geqslant \frac{1}{4}:$$

$$\text{Squeezed state} \Leftrightarrow \text{lesser eigenvalue of } V < \frac{1}{2} \qquad \ldots (32)$$

Remembering the general $Sp(2, R)$ transformation law of $\boldsymbol{V}$, namely as in (26),

$$\boldsymbol{S} \in Sp(2, R): \boldsymbol{V} \to \boldsymbol{V}' = \boldsymbol{S}\boldsymbol{V}\boldsymbol{S}^T, \qquad \ldots (33)$$

we see that our definition of squeezing is invariant under the $U(1)$ or $SO(2)$ subgroup of $Sp(2, R)$, but of course not under the entire group of symplectic transformations. And one can give good physical arguments why this should be so.

Now for states of n-dimensional systems, in a similar way, one can justify the search for a definition of squeezed states in terms of the variance matrix, such that the definition is $U(n)$ invariant, but not $Sp(2n, R)$ invariant. Here, for $n \geqslant 2$, something essentially new arises. We must remember that $\boldsymbol{V}$ is a $2n \times 2n$ matrix, while the subgroup of $Sp(2n, R)$ of interest here is $U(n)$. In one dimension, since $U(1)$ and $SO(2)$ are the same, any 2×2 variance matrix is diagonalisable by some $U(1)$ element, so we were able to state our squeezing definition in (32) in terms of the eigenvalues of $\boldsymbol{V}$. But for $n \geqslant 2$ dimensions, a corresponding statement does not

hold. Let us now propose our definition of squeezing in n-dimensions[16]. Look at the diagonal elements of V: they are the squares of the uncertainties $(\Delta q_1)^2, (\Delta q_2)^2, \ldots, (\Delta q_n)^2, (\Delta p_1)^2, \ldots, (\Delta p_n)^2$. We shall say a given state is squeezed if any diagonal element of V is less that $1/2$, or if this is so for any $U(n)$ transform of V. This is an explicitly $U(n)$-invariant definition, and it makes good physical sense. However, it is not expressed in terms of the eigenvalues of V. Now, in general, a $2n \times 2n$ variance matrix cannot be brought to diagonal form by action of elements in the $U(n)$ subgroup of $Sp(2n, R)$. What comes as a nice surprise is this: the definition of squeezing we have just given is equivalent to the following:

State with variance matrix V is squeezed $\Leftrightarrow$ the smallest eigenvalue of V is less that $1/2$. ... (34)

I hope I have communicated the sense in which we have something nontrivial here. It is worth stressing that it is experience with uses of phase space and symplectic group ideas that allows us to tackle this problem of squeezing in a kind of global way. It turns out that one can also give a broad and physically meaningful classification of states by classifying their variance matrices into distinct families, along these lines.

The final technical item I would like to present concerns the problem of going beyond the two-point correlation function in optics. We saw that generalized rays were introduced in terms of this function. Now there is an infinite hierarchy of correlation functions, all of which are needed for a complete statistical description[18]. The two-point function is essentially the lowest in this hierarchy. A general correlation function $\Gamma^{(m, n)}$ of order (m, n) involves the expectation value of m factors of the analytic signal A and n factors of the complex conjugate A^*. A careful analysis now shows that only for the diagonal functions $\Gamma^{(m, m)}$ it is possible to give an alternative account in terms of generalized rays[19]. We are unable to do it for the functions with $m \neq n$. So the phase space method in this sense goes only part of the way. All the information contained in the functions $\Gamma^{(m, m)}$ can be reexpressed via a sequence of phase space functions W_m in an invertible manner:

$$\Gamma^{(m, m)}(\ldots x_\perp \ldots; \ldots x'_\perp \ldots) \leftrightarrow W_m(\xi^{(1)} \ldots \xi^{(m)}) \quad \ldots (35)$$

This function W_m can be viewed as the m-generalized rays joint distribution function, and it has a simple behaviour under the action of symplectic *FOS*'s. Incidentally, the Bose nature of light shows up in a nonlocal correlation among generalized rays. One finds that for any pair of arguments $\xi^{(j)}$ and $\xi^{(k)}$ the following integral relation holds for W_m[19]:

$$W_m(\ldots \xi^{(j)} \ldots \xi^{(k)} \ldots) = \int d^4\xi \cdot \exp\{-i\xi^T \beta(\xi^{(j)} - \xi^{(k)})\}$$

$$W_m\left(\ldots \frac{1}{2}(\xi^{(j)} + \xi^{(k)} - \xi) \ldots \frac{1}{2}(\xi^{(j)} + \xi^{(k)} + \xi) \ldots\right) \quad \ldots (36)$$

This is explicitly covariant under $Sp(4, R)$ transformations. There is a kind of complementarity here. Either one works in configuration space where the Bose property is simple while action by symplectic *FOS*'s is nonlocal; or one works with

phase space and generalised rays, which makes *FOS* action simple but the Bose nature nonlocal. All this goes through with essentially no changes for the hierarchy of quantum higher order correlation functions as well.

Let me now conclude with a few remarks. I hope to have shown you how the use of phase space methods and symplectic group structure has led to new insights and points of view in optics. These are techniques learnt and perfected in the arena of quantum mechanics, and then successfully transferred to another field. From the epistemological point of view the following rather interesting comparison can be made. In quantum mechanics the meaning of the uncertainty principle, and the problem of describing localisable particles in terms of probability amplitudes as waves, lead to profound issues of observation and existence. In optics there are similar uncertainty principles for ray variables in the wave theory, but clearly the problems of interpretation are nowhere as deep. However a new criterion makes its appearance—that of convenience of description—and in this context the kind of complementarity I pointed out above comes up. This goes to show that ways of thinking acquired in the understanding and interpretation of quantum mechanics come up and become useful in other areas as well. And it even seems that the subjective criterion of convenience of description is relevant in discerning what is real and what exists. I refer here to the natural question: do generalised rays really exist or are they merely a mathematical artefact and a matter of convenience? I will leave these and related questions to the interested reader to ponder over, and only express the hope that she will find some satisfaction in these ways of looking at an old friend with new eyes.

Acknowledgement

It is a pleasure to acknowledge with gratitude the lessons I have learnt in all of the above from R. Simon, and in much of it from E.C.G. Sudarshan.

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