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ROLE OF TIME DELAY IN CLIMATE RESEARCH

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The two crucial roles played by time delay in theoretical climate research are discussed. One is the well-known use of delays in computations of attractor dimension by embedding methods. The other is the ability of delay parameter in a delay differential equation to bring about bifurcations when they are appropriately changed. The latter case is illustrated by results from an one dimensional energy balance model with delayed albedo feedback.

Key Words: Delay; Embedding; Bifurcations

Introduction

The concept of time delay is very important in non-linear dynamics in general and climate research in particular. Ruelle[1] and Packard *et al.*[2] first showed how to use delay coordinates in building up of an embedding space for estimation of attractor dimension. With a scalar time series $X_n(t)$ sampled $N+1$ times at interval τ, i.e.

$$t=0, \tau, 2\tau, ..., N\tau, \qquad \ldots (1)$$

the delay vector

$$X_n = (X(n\tau), X((n+1)\tau), ..., X((n+N)\tau)) \qquad \ldots (2)$$

can easily be constructed. Provided that the dimension (m) of such an embedding space is large enough (i.e., $m > 2d+1$, where d is the dimension of the attractor) the attractor set of a dynamical system can be safely embedded into it (Whitney[3]; Takens[4]; Fraedrich[5]). Recently, this concept has been utilized by Grassberger and Procaccia[6] very effectively to design an algorithm for attractor dimension estimation. Although we will present this procedure in some detail, we will first discuss the delays in the natural climate system and the sources thereof.

There are several components of the terrestrial climate system. The most important of these are the atmosphere, the hydrosphere, the cryosphere and the biosphere. Of these the behaviour of first three can be quantified and their characteristic time scales are known to be very different, i.e., these components are delayed with respect to one another. It was Lorenz[7] who first pointed out the possibility of non-linear interaction between these different time scales resulting in irregular fluctuations that resemble the time series of the terrestrial climate. Such climatic fluctuations have been referred to as the climatic change due to internal causes, or more simply, as the internal climatic change.

There are many physical sources of delay in the terrestrial climate system, some of which can be easily identified. Ice, for instance, can change on widely different time scales, starting from snow (~seasons) to continent sized ice sheets (~millions of years). Similarly, oceans exhibit different time scales of change, with the upper oceans changing on seasonal time scale and the deep oceans changing on the time scale of thousands of years. Bare land has the fastest time scale of response. It has even been postulated there is a phase lag in the development of cumulus clouds with interesting consequences (Davies[8]).

This paper describes an one dimensional energy balance climate model with delayed albedo feedback. It is shown how this model responds to changes on different time scales.

Delays in Attractor Dimension Calculation

The basic concept of a delay vector is quite old and had been developed by statisticians. However, Whitney's[3] embedding theorem is new in that it gives some quantitative theresholds so that attractor reconstruction using the delay coordinates is a valid procedure. This reconstruction has, however, turned out to be hard to realize in practice, because measured time series are very often noisy and of short duration (Vautard *et al.*[9]). So much so that the very first task of deciding whether a given time series is indeed chaotic or just noisy is often a hard one (Provenzale *et al.*[10]). When both are present the problem is of course magnified many times. Another outcome of the noise is that a great deal of effort has been spent towards noise reduction. One method of reducing noise is through appropriate choice of basis vectors, where the vectors are determined by the time series itself. Usually the basis is chosen to be the eigenvectors of two-point correlation matrix calculated from data. This method, originally called Principal Component Analysis by the statisticians, is given different names: Singular Value Decomposition, Singular Spectrum Analysis Proper Orthogonal Vectors. The eigenvectors themselves are often called Karhunen-Loeve vectors or Empirical Orthogonal Functions.

Another important area of uncertainty is determination of optimal m or τ. Although Whitney's[3] theorem guarantees validity of attractor reconstruction as long as $m > 2d+1$, the attractor dimension d is not known in advance. Several methods exist for optimal computation of embedding dimension (m) or the delay (τ).

Although in principle any delay τ can be used, in practice the delay cannot be too small because in that case the delay vectors are hardly distinguishable from one another and it also cannot be too large because the delay vectors become statistically independent. A good (and usual) choice is to plot the autocorrelation as a function of τ, and find first point at which the autocorrelation is zero. Although this method suffices as a "thumb rule", it is intellectually more satisfying if such a criterion can be obtained from information theory which leads to a nonlinear notion of independence (Abarbanel *et al.*[11]). Fraser and Swinney[12] suggested that we compute average mutual information $I(t)$ defined as

$$I(t)=\sum_{n=1}^{N} P(X(n), X(n+t)) \log_2 \left[\frac{P(X(n), X(n+t))}{P(X(n)\, P(X(n+t))}\right], \qquad \ldots (3)$$

where

$P(a, b)$ is the joint probability of measurement of a and b within their respective ensembles.

Then the first minimum $I(t)$ is a good choice for lag time τ. However, like the linear case this is also only a prescription.

Attractor Reconstruction

Starting from the original time series $X_n(t)$ we construct (Ruelle[1]; Packard *et al.*[2]) $N+1$ new time series by applying to each term of the series shifts of

$0, \tau, 2\tau, \ldots\ldots, (N-1)\tau, N\tau$

Grassberger and Procaccia[6] gave the following widely used algorithm: compute values of the (linear) correlation function $C(r)$ defined as

$$C(r)=\frac{1}{N^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} H(r-|\mathbf{X}_i-\mathbf{X}_j|) \qquad \ldots (4)$$

where $H(x)$ is the Heaviside function.

A log-log plot of $C(r)$ against r yields a straight line. Starting from a small values of n these straight lines are successively plotted, and the slopes reach a plateau after an initial increase. This limiting slope is the attractor dimension (d), and the value of n for which this limiting slope is attained gives the minimum number of variables needed to fully describe the system.

One Dimensional Energy Balance Climate Model: Role of Delay

A delay differential equation is infinite dimensional in nature, and it is expected that a wide range of behaviours will be exhibited by such an equation. This was confirmed by an one dimensional energy balance climate model of the Sellers[13] type. This model was extended by Bhattacharya[14] *et al.* using some realistic assumptions that are valid over ice-age time scales. Here, the model is described briefly, details may be found in Bhattacharya *et al.*[14].

In this model latitude is the only space variable, and the energy balance at the earth's surface is described by the time-dependent equation

$$G(X)\frac{\partial T}{\partial t}(X, t)= R_i[X, T(X, t)] - R_0[T(X, t)] + F\left[X, t, \frac{\partial T}{\partial X}(X, t), \frac{\partial^2 T}{\partial X^2}(X, t)\right] \qquad \ldots (5)$$

where $X=(2\Phi)/\pi$, and Φ is the colatitude. $T(X,t)$ is the surface temperature and $G(X)$ is the heat capacity of earth's surface. On the R.H.S the successive energy terms are the absorbed part of the incoming solar radiation at the surface, the outgoing longwave radiation at the surface, and the meridional heat transport. They are represented as

$$R_i\ \mu Q(X)\{1-\alpha[X,\ T(X,\ t)]\} \qquad \ldots(6)$$

$$R_0=C[T(X,\ t)]\cdot\sigma\cdot[T(X,\ t)]^4 \qquad \ldots(7)$$

$$F=\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}[\sin\phi\cdot K(\phi)]\frac{\partial T}{\partial\phi} \qquad \ldots(8)$$

Two new features were added to the basic Sellers'[13] model, and they affect the results greatly. Hence they are discussed here in some detail.

The first is the inclusion of delay in the albedo formulation. As indicated earlier, this is the single most important reason for the interesting behaviour of the model to be described later. Whenever, snow or ice is present at the surface the albedo is determined not only the by the current temperature but also by weighted past temperatures. This weight is a maximum at $t=\tau$, and has a cutoff at $t=2\tau$. In absence of any guidance from the past records, the weighing function has been taken to be of Gaussian shape, with the peak of the curve at the current time.

The second is the addition of a kink in the temperature-albedo curve. This is located at that temperature where baroclinic activity normally takes place, and this is thought to coincide with the edge of the large icecaps. This hypothesis is partially supported by observations of cloudiness at the icecaps (Lamb[15], Schwerdtfeger and Kachelhoffer[16]). However, although qualitatively plausible, this relation is again difficult to formulate; so in an ad hoc manner a triangular shaped jump was used.

Two other minor features were a new computation of global surface heat capacity and restriction of heat transport to a linear formulation that is determined by the present temperatures only.

Results

Time independent (steady-state) version of the model yielded five solutions for the present climate, as against three for the original Sellers[13] model. We well denote them by $T_1(X)$, ..., $T_5(X)$. $T_1(X)$, $T_3(X)$ and $T_5(X)$ are linearly stable, while $T_2(X)$ and $T_4(X)$ are linearly unstable. Moreover, $T_1(X)$ represents the present (interglacial) climate, $T_3(X)$ represents the glacial climate, and $T_5(X)$ represents the "deep freeze"—a hypothetical climate where the earth was entirely covered with ice. All these have been elaborated upon in Bhattacharya *et al.*[14].

Results obtained from the time-dependent version of this model emphasize the dominant role played by the delay parameter in a delay differential equation. For all the integrations performed the starting temperature and albedo were consistent with each other. From start to $t=2\tau$ the time lag tail was built

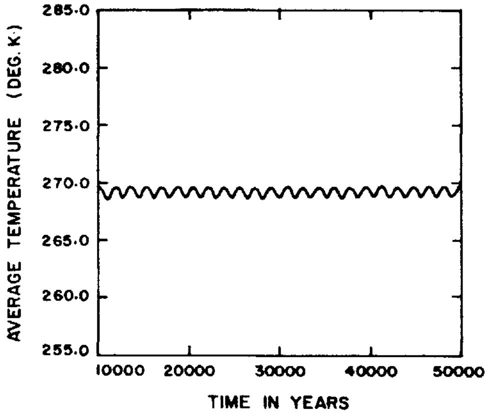

Fig. 1a

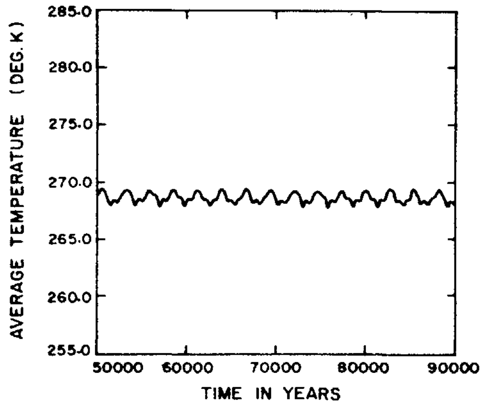

Fig. 1b

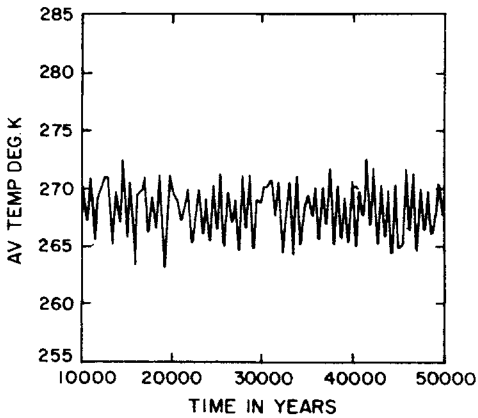

Fig. 1c

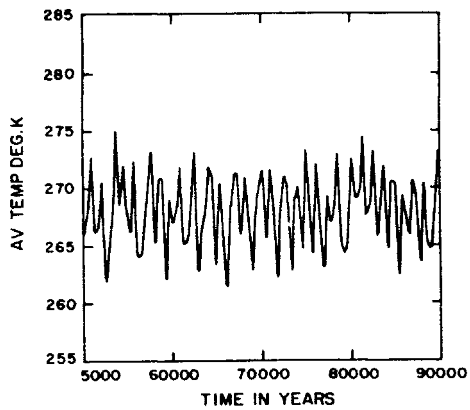

Fig. 1d

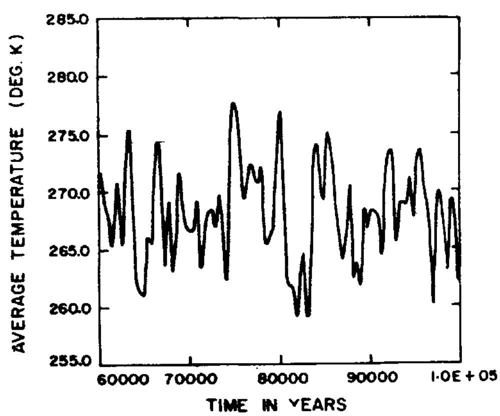

Fig. 1e

Figs 1 *a*–*e* Sustained oscillations of the hemispherically averaged temperature $\overline{T}(t)$ around climate 1 are shown. The values of time lag τ are marked in the figures. Time units are in 1000 years. For plotting purposes a running mean over 100 years are shown. The change in the character of oscillations at about $\tau = 1500$ years is evident.

up by keeping the albedo fixed at the starting value. Thereafter, the albedo was allowed to be determined by past temperatures as described before. No oscillations were obtained when the starting temperature were equal to or higher than for present climate.

Sustained oscillations were obtained for negative perturbations that were large but not too large, i.e.,

$$T_1(X) - 10\text{K} < T(X,0) < T_1(X) - 7.5\text{K} \quad \ldots (9)$$

In these cases the shape and amplitude of the oscillations were determined by τ and σ, where σ is the half width of the Gaussian distribution that determines the past albedo from the past times. The actual oscillations for $\tau = 500$, 750, 1000, 1500, and 5000 years are shown in Figs 1*a-e* where the hemispherically averaged temperature $\bar{T}(t)$ are plotted against the time t. It is seen that between $\tau = 1000$ and $\tau = 1500$ years a qualitative change in the nature of the oscillations take place from periodic to aperiodic. For $\tau = 500$ years, the oscillations are simply periodic, while for $\tau = 750$ and 1000 years the oscillations are quasiperiodic. Clearly the change in values of the time-lag parameter τ has brought about the bifurcations of the system from simply periodic to quasi periodic, and then to chaotic.

Discussions

We have seen two important roles of time delay in climate research. One pertains to the usefulness of time delay in dimension calculations of the attractors, the main motivation here being the detection of strange attractors which is a non-trivial task for a short and noisy time series. The other is the role of the delay parameter in a delay differential equation. We have seen that by properly changing τ we can bring about bifurcations in such an equation which is an infinite dimensional system. However the exact manner in which this happens is yet to be explained and that is subject of future research.

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