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STRESSES IN THE VICINITY OF A PENNY-SHAPED CRACK IN A TRANSVERSELY ISOTROPIC THICK PLATE

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A steady state elastic mixed boundary value problem for a thick plate is considered in the paper. The faces of a penny-shaped crack situated in the mid-plane of the plate are opened up by the application of pressure. Solution valid for large values of the ratio between the thickness of the plate and the diameter of the crack has been obtained. The normal component of displacement on the crack faces has been calculated and the shape of the deformed crack has been shown graphically for a chosen value of that ratio.

Key Words: Penny-Shaped Crack; Transversely Isotropic Material; Shape of Deformed Crack

Introduction

The determination of the distribution of stress in an elastic solid in the vicinity of a crack plays a central part in the theories of fracture and for that reason is of some technical importance. Much of this work is based upon an analysis of the stress near a "penny-shaped" crack first discussed by Sneddon[1]. In that paper, Sneddon reduced the problem of finding the stresses in an infinite elastic solid occupying the region $|r|\geqslant 0$, $|z|\geqslant 0$ containing a penny-shaped crack described by $0\leqslant r\leqslant 1$, $z=0$, to the mixed boundary value problem of determining the stresses in a semi-infinite solid $|r|\geqslant 0$, $z\geqslant 0$. Using Sneddon's method Lowengrub[2] solved the problem of an isotropic plate of finite thickness and infinite radius containing a penny-shaped crack in the mid-plane. The present paper considers the problem of an infinite transversely isotropic plate whose thickness is assumed to be δ-times the diameter of the penny-shaped crack lying in the mid-plane. As expected, mixed conditions on the boundary give rise to dual integral equations which have been reduced to a Fredholm integral equation. Analytical expressions are derived for the stresses and displacements in the case of large values of δ. Numerical calculations have been carried out for a particular value of δ and the shape of the crack is illustrated by a figure.

Boundary Conditions

We take the crack-radius to be unity and hence the crack-faces occupy the regions $0\leqslant r\leqslant 1$, $z=0\pm$ where (r, θ, z) are cylindrical polar coordinates referred to the centre of the crack as origin. For symmetrical deformation of the elastic medium, the displacement of such a point may be taken to have the components (u_r, o, u_z) in this coordinate system and the non-vanishing components of the stress tensor will be σ_{rr}, $\sigma_{\theta\theta}$, σ_{xx} and σ_{zz}.

We have the following boundary conditions:

on $z=0$; $\sigma_{zz}(r, 0) = -p(r), 0<r<1$; ...(1)

$u_z = (r, 0) = 0, 1<r<\infty$; ...(2)

$\sigma_{rz}(r,0) = 0, 0<r<\infty$; ...(3)

on $z=\delta$: $\sigma_{rz}(r, \delta) = 0, r \geqslant 0$; ...(4)

$u_z(r, \delta) = 0, r \geqslant 0$; ...(5)

Basic Equations

The stress-strain relations for the transversely isotropic material under consideration are:

$$\sigma_{rr} = C_{11}\, e_{rr} + C_{12} e_{\theta\theta} + C_{13}\, e_{zz},$$

$$\sigma_{\theta\theta} = C_{12}\, e_{rr} + C_{11}\, e_{\theta\theta} + C_{13}\, e_{zz}$$

$$\sigma_{zz} = C_{13}(e_{rr} + e_{\theta\theta}) + C_{33} e_{zz}$$

and

$$\sigma_{rz} = C_{44} \gamma_{rz} \qquad \ldots(6)$$

where C_{11}, C_{12}, ..., C_{44} are elastic constants.

It is known that the solutions of the equations of the equilibrium can be obtained in terms of two stress functions ϕ_1 and ϕ_2 and the expressions for the relevant components of stress and displacement can be written as:

$$\sigma_{zz} = C_{13}\left(\frac{\delta^2}{\delta r^2} + \frac{1}{r}\frac{\delta}{\delta r}\right)(\phi_1 + \phi_2) + C_{33}\frac{\delta^2}{\delta z^2}(\lambda_1 \phi_1 + \lambda_2' \phi_2)\,; \qquad \ldots(7)$$

$$\sigma_{rz} = C_{44}\frac{\delta^2}{\delta r\, \delta z}[(1+\lambda_1)\,\phi_1 + (1+\lambda_2)\,\phi_2]\,; \qquad \ldots(8)$$

$$u_r = \frac{\delta}{\delta r}(\phi_1 + \phi_2) \qquad \ldots(9)$$

and

$$u_2 = \frac{\delta}{\delta z}(\lambda_1 \phi_1 + \lambda_2 \phi_2) \qquad \ldots(10)$$

where ϕ_1 and ϕ_2 are solutions of

$$\left(\frac{\delta^2}{\delta z^2} + \frac{1}{r}\frac{\delta}{\delta r} + \nu_i^2 \frac{\delta^2}{\delta z^2}\right)\phi_i = 0 \quad (i = 1,2)\,, \qquad \ldots(11)$$

in which ν_1^2, ν_2^2 are the roots of the quadratic equation:

$$C_{11}C_{44}\nu^4+(2C_{13}C_{44}-C_{11}(C_{33}+C_{13}^2)\nu^2+C_{33}C_{44}=0 \qquad \ldots(12)$$

The values of λ_1 and λ_2 correspond to the values ν_1^2 and ν_2^2 respectively of ν and

$$\frac{\lambda(C_{13}+C_{44})+C_{44}}{C_{11}}=\frac{\lambda C_{33}}{\lambda C_{44}+C_{13}+C_{44}}=\nu^2 \qquad \ldots(13)$$

Derivation of Dual Integral Equations

As solutions of eq. (11), let us assume

$$\phi_1=\int_0^\infty \xi^{-1}A_1(\xi)\cosh(\xi\nu_1)(\delta-z)\operatorname{cosech}(\xi\nu_1\delta)J_0(\xi r)\mathrm{d}\xi \qquad \ldots(14)$$

and

$$\phi_2=\int_0^\infty \xi^{-1}A_2(\xi)\cosh(\xi\nu_2)(\delta-z)\operatorname{cosech}(\xi\nu_2\delta)J_0(\xi r)\mathrm{d}\xi \qquad \ldots(15)$$

Using (14) and (15), we get from (8), (7) and (9) the relations

$$\sigma_{rz}=C_{44}\Big[\int_0^\infty \xi\{\nu_1(1+\lambda_1)A_1(\xi)\sinh(\xi\nu_1)(\delta-z)\operatorname{cosech}(\xi\nu_1\delta)$$
$$+\nu_2(1+\lambda_2)A_2(\xi)\sinh(\xi\nu_2)(\delta-z)\operatorname{cosech}(\xi\nu_2\delta)\}J_1(\xi r)\mathrm{d}\xi\Big] \qquad \ldots(16)$$

$$\sigma_{zz}=-C_{13}\Big[\int_0^\infty \xi\{A_1(\xi)\cosh(\xi\nu_1)(\delta-z)\operatorname{cosech}(\xi\nu_1\delta)+A_2(\xi)\cosh(\xi\nu_2)(\delta-z)$$
$$\times\operatorname{cosech}(\xi\nu_2\delta)\}J_0(\xi r)\mathrm{d}\xi\Big]+C_{33}\Big[\int_0^\infty \xi\{\lambda_1\nu_1^2A_1(\xi)\cosh(\xi\nu_1)(\delta-z)$$
$$\times\operatorname{cosech}(\xi\nu_1\delta)+\lambda_2\nu_2^2A_2(\xi)\cosh(\xi\nu_2)(\delta-z)\operatorname{cosech}(\xi\nu_2\delta)\}J_0(\xi r)\mathrm{d}\xi\Big] \qquad \ldots(17)$$

and

$$u_z=-\int_0^\infty\{\lambda_1\nu_1A_1(\xi)\sinh(\xi\nu_1)(\delta-z)\operatorname{cosech}(\xi\nu_1\delta)+\lambda_2\nu_2A_2(\xi)\sinh(\xi\nu_2)(\delta-z)$$
$$\times\operatorname{cosech}(\xi\nu_2\delta)\}J_0(\xi r)\mathrm{d}\xi\,. \qquad \ldots(18)$$

The authors observe that the boundary conditions (4) and (5) satisfy the relations (16) and (18).

Imposing the condition (3) we get from (16) the relation

$$\nu_1(1+\lambda_1)A_1(\xi)+\nu_2(1+\lambda_2)A_2(\xi)=0 \qquad \ldots (19)$$

which gives

$$A_2(\xi)=-\frac{\nu_1(1+\lambda_1)}{\nu_2(1+\lambda_2)}A_1(\xi) \qquad \ldots (20)$$

With the help of relation (20) we get from (17) and (18) the normal components of stress and displacement on the crack-plane as

$$\sigma_{zz}(r,0)=k_1\int_0^\infty \xi A_1(\xi)\coth(\xi\nu_1\delta)J_0(\xi r)\mathrm{d}\xi+k_2\int_0^\infty \xi A_1(\xi)\coth(\xi\nu_2\delta)J_0(\xi r)\mathrm{d}\xi \qquad \ldots (21)$$

and

$$u_z(r,0)=-\int_0^\infty\left\{\lambda_1\nu_1+\frac{\lambda_2\nu_1(1+\lambda_1)}{(1+\lambda_2)}\right\}A_1(\xi)J_0(\xi r)\mathrm{d}\xi \qquad \ldots (22)$$

where

$$k_1=C_{33}\lambda_1\nu_1^2-C_{13}$$

and

$$k_2=C_{13}\frac{\nu_1(1+\lambda_1)}{\nu_2(1+\lambda_2)}-C_{33}\frac{\lambda_2\nu_1\nu_2(1+\lambda_1)}{(1+\lambda_2)} \qquad \ldots (23)$$

If we substitute the conditions (1) and (2) into equations (21) and (22) we find that the function $A_1(\xi)$ must satisfy the pair of dual integral equations:

$$\int_0^\infty \xi A_1(\xi)[1+H(\xi\delta)]J_0(\xi r)\mathrm{d}\xi=-p_0,\ 0<r<1 \qquad \ldots (24)$$

and

$$\int_0^\infty A_1(\xi)J_0(\xi r)\mathrm{d}\xi=0,\ 1<r<\infty \qquad \ldots (25)$$

where

$$1+H(\xi\delta)=\frac{k_1\coth(\xi\nu_1\delta)+k_2\coth(\xi\nu_2\delta)}{k_1+k_2} \qquad \ldots (26)$$

Solution of the Dual Integral Equations

To solve the pair of dual integral equations (24), (25) we use Lowengrub's[2] method which consists in reducing the equations (24) and (25) to a single integral equation by assuming that the weight function $H(\xi\delta)$ can be expanded in the form:

$$H(\xi\delta)=\frac{2}{k_1+k_2}\left[k_1\sum_{m=1}^{\infty}(1+m\xi\nu_1\delta)e^{-2m\xi\nu_1\delta}+k_2\sum_{m=1}^{\infty}(1+m\xi\nu_2\delta)e^{-2m\xi\nu_2\delta}\right] \quad \ldots(27)$$

for δ much greater than 1.

Using the substitution

$$A_1(\xi)=\int_0^{\infty}\phi(t)\sin(\xi t)\mathrm{d}t, \quad \phi(0)=0 \quad \ldots(28)$$

and

$$\phi(t)=\frac{2}{\pi}\int_0^{\infty}\frac{vf(v)\,\mathrm{d}v}{\sqrt{(t^2-v^2)}}. \quad \ldots(29)$$

Eqs (24) and (25) can be reduced to the Fredholm integral equation of the second kind:

$$f(r)+4\pi^{-2}\delta^{-3}\int_0^{\infty}f(v)k(r,v)\,\mathrm{d}v=-p_0 \quad \ldots(30)$$

where the kernel $k(r, v)$ is defined by the set of three equations (Lowengrub[2]):

$$H^*(w)=\delta\int_0^{\infty}\xi H(\xi\delta)\sin(\xi w)\mathrm{d}\xi\,; \quad \ldots(31)$$

$$N(u,t)=\frac{1}{2}[H^*(u+t)-H^*(u-t)]\,; \quad \ldots(32)$$

and

$$k(r,v)=\delta^2 v\int_v^1\frac{\mathrm{d}t}{\sqrt{(t^2-v^2)}}\int_0^r\frac{N(u,t)\,\mathrm{d}u}{\sqrt{(r^2-u^2)}}. \quad \ldots(33)$$

Now, using the expansion (27) for H we find from eq. (33) that:

$$H^*(u)=\frac{1}{2(k_1+k_2)}\delta^{-2}\left[k_1\sum_{m=1}^{\infty}m^{-3}\left\{u(1+(u/2m\nu_1\delta)^2)^{-2}\right.\right.$$
$$\left.+\frac{1}{2}u(1+(u/2m\nu_1\delta)^2)^{-3}(3-(u/2m\nu_1\delta)^2\right\}$$
$$+k_2\sum_{m=1}^{\infty}m^{-3}\left\{u(1+(u/2m\nu_2\delta)^2)^{-2}\right.$$
$$\left.\left.+\frac{1}{2}u(1+(u/2m\nu_2\delta)^2)^{-3}(3-(u/2m\nu_2\delta)^2)\right\}\right] \qquad \ldots (34)$$

so that

$$\int_0^1 t\left[\delta^2 k_1(w,t)\mathrm{d}t=\sum_{m=1}^{\infty}m^{-3}(I_{1'}-I_{2'}+I_{3'}-I_4)\right.$$
$$+\sum_{m=1}^{\infty}(m^{-3}(I_{1''}-I_{2''}+I_{3''}-I_{4''})(\equiv E(w)\text{ (say)}, \qquad \ldots (35)$$

where

$$I_{1'}=\frac{k_1}{4(k_1+k_2)}\int_0^1\frac{t(w+t)\mathrm{d}t}{[1+\{(w+t)/2m\nu_1\delta\}^2]^2}$$
$$=\frac{k_1}{k_1+k_2}\left[m^3\nu_1^3\delta^3\left\{\tan^{-1}\left(\frac{w+1}{2m\nu_1\delta}\right)-\tan^{-1}(w/2m\nu_1\delta)\right\}\right.$$
$$\left.-\frac{1}{2}m^2\nu_1^2\delta^2\left\{1+\left(\frac{w+1}{2m\nu_1\delta}\right)^2\right\}^{-1}\right]$$
$$I_{2'}=\frac{k_1}{4(k_1+k_2)}\int_0^1\frac{t(w-t)\mathrm{d}t}{[1+\{(w-t)/2m\nu_1\delta\}^2]^2}$$
$$=\frac{k_1}{k_1+k_2}\left[m^3\nu_1^3\delta^3\left\{\tan^{-1}\left(\frac{w-1}{2m\nu_1\delta}\right)-\tan^{-1}(w/2m\nu_1\delta)\right\}\right.$$
$$\left.-\frac{1}{2}m^2\nu_1^2\delta^2\left\{1+\left(\frac{w-1}{2m\nu_1\delta}\right)^2\right\}^{-1}\right]$$
$$I_{3'}=\frac{k_1}{8(k_1+k_2)}\int_0^1\frac{t(w+t)[3-\{(w+t)/2m\nu_1\delta\}^2]\mathrm{d}t}{[1+\{w+t)/2m\nu_1\delta\}^2]^3}$$
$$=\frac{k_1}{16(k_1+k_2)}\left[\{(w+2)(w+1)^2+4\mathrm{w}m^2\nu_1^2\delta^2\}\left\{1+\left(\frac{(w+1)}{2m\nu_1\delta}\right)^2\right\}^{-2}\right.$$
$$\left.-(w^3+4wm^2\nu_1^2\delta^2)(1+w^2/4\nu_1^2\delta^2m^2)^{-1}\right].$$

$$I_{4'}=\frac{k_1}{8(k_1+k_2)}\int_0^1\frac{t(w-t)[3-\{(w-t)/2mv_1\delta\}^2]\mathrm{d}t}{[1+\{w-t)/2mv_1\delta\}^2]^3}$$

$$=\frac{k_1}{16(k_1+k_2)}\left[\{(w-2)(w-1)^2+4wm^2v_1^2\delta^2\}\left\{1+\left(\frac{(w-1)}{2mv_1\delta}\right)^2\right\}^{-2}\right.$$

$$\left.-(w^3+4wm^2v_1^2\delta^2)(1+w^2/4v_1^2\delta^2m^2)^{-1}\right]$$

$$I_{1''}=\frac{k_2}{4(k_1+k_2)}\int_0^1\frac{t(w+t)\mathrm{d}t}{[1+\{(w+t)/2mv_2\delta\}^2]^2}$$

$$=\frac{k_2}{k_1+k_2}\left[m^3v_2^3\delta^3\left\{\tan^{-1}\left(\frac{w+1}{2mv_2\delta}\right)-\tan^{-1}(w/2mv_2\delta)\right\}\right.$$

$$\left.-\frac{1}{2}m^2v_2^2\delta^2\left\{1+\left(\frac{w+1}{2mv_2\delta}\right)^2\right\}^{-1}\right],$$

$$I_{2''}=\frac{k_2}{4(k_1+k_2)}\int_0^1\frac{t(w-t)\mathrm{d}t}{[1+\{(w-t)/2mv_2\delta\}^2]^2}$$

$$=\frac{k_2}{k_1+k_2}\left[m^3v_2^3\delta^3\left\{\tan^{-1}\left(\frac{w-1}{2mv_2\delta}\right)-\tan^{-1}(w/2mv_2\delta)\right\}\right.$$

$$\left.+\frac{1}{2}m^2v_2^2\delta^3\left\{1+\left(\frac{w-1}{2mv_2\delta}\right)^2\right\}^{-1}\right]$$

$$I_{3''}=\frac{k_2}{8(k_1+k_2)}\int_0^1\frac{t(w+t)[3-\{(w+t)/2mv_2\delta\}^2]\mathrm{d}t}{[1+\{(w+t)/2mv_2\delta\}^2]^3}$$

$$=\frac{k_2}{16(k_1+k_2)}\left[\left\{(w+2)(w+1)^2+4wm^2v_2^2\delta^2\right\}\left\{1+\left(\frac{(w+1)}{2mv_2\delta}\right)^2\right\}^{-2}\right.$$

$$\left.-(w^3+4wm^2v_2^2\delta^2)(1+w^2/4v_2^2\delta^2m^2)^{-1}\right]$$

and

$$I_{4''}=\frac{k_2}{8(k_1+k_2)}\int_0^1\frac{t(w-t)[3-\{(w-t)/2mv_2\delta\}^2]\mathrm{d}t}{[1+\{(w-t)/2mv_2\delta\}^2]^3}$$

$$=\frac{k_2}{16(k_1+k_2)}\left[\left\{(w-2)(w-1)^2+4wm^2v_2^2\delta^2\right\}\left\{1+\left(\frac{(w-1)}{2mv_2\delta}\right)^2\right\}^{-2}\right.$$

$$\left.-(w^3+4wm^2v_2^2\delta^2)(1+w^2/4v_2^2\delta^2m^2)^{-1}\right]$$

Since the solution of (30) is taken up the order δ^{-4} [Lowengrub[2]], we replace $I_{1'}$, $I_{2'}$, $I_{3'}$, $I_{4'}$, $I_{1''}$, $I_{2''}$, $I_{3''}$ and $I_{4''}$ by their expansions in powers of δ upto the term containing δ^{-2}. We find that,

$$E(w)=\frac{k_1}{k_1+k_2}\left[\frac{5}{12}\zeta(3)-\frac{7}{40\nu_1^2}\delta^{-2}\zeta(5)(5w^2+1)\right]$$
$$+\frac{k_2}{k_1+k_2}\left[\frac{5}{12}\zeta(3)-\frac{7}{40\nu_2^2}\delta^{-2}\zeta(5)(5w^2+1)\right]+0(\delta^{-4}) \qquad \ldots(36)$$

where $\zeta(n)$ denotes the Reimann-Zeta function.

Using (34) and (35) we get the expression

$$f(r)=-p_0\left[1-\frac{5}{6\pi\delta^3}\zeta(3)+\frac{7\zeta(5)(5r^2+2)}{40\pi\delta^5(k_1+k_2)}\left(\frac{k_1}{\nu_1^2}+\frac{k_2}{\nu_2^2}\right)\right]+0(\delta^{-7}). \qquad \ldots(37)$$

We also find from equations (28) and (29) that

$$A_1(\xi)=-\frac{2}{\pi}\left[f_0 D-\frac{2}{3}f_1 D^3\right]\xi^{-1}\sin\xi+0(\delta^{-7})$$

where

$$f_0=-p_0\left[1-\frac{5\zeta(3)}{6\pi\delta^3}+\frac{7\zeta(5)}{20\pi\delta^5}\left(\frac{k_1}{\nu_1^2}+\frac{k_2}{\nu_2^2}\right)\Big/(k_1+k_2)\right]$$

and

$$f_1=\frac{7\zeta(5)}{8\pi\delta^5(k_1+k_2)}\left(\frac{k_1}{\nu_1^2}+\frac{k^2}{\nu_2^2}\right)$$

and

$D=d/d\xi$.

Shape of the Deformed Crack

We find an approximate expression-upto the order δ^{-6}—for the displacement $u_z(r,0)$ on $z=0$, $0\leqslant r\leqslant 1$ using Eq. (18) as follows:

$$u_0(r)=u_z(r,0)=Mp_0\sqrt{1-r^2}\left[1-\frac{5\zeta(3)}{6\pi\delta^3}+\frac{7\zeta(5)}{90\pi\delta^5}\cdot\frac{k_1/\nu_1^2+k_2/\nu_2^2}{k_1+k_2}\right.$$
$$\left.\times(7+5r^2)+0(\delta^{-7})\right] \qquad \ldots(38)$$

where

$$M=\frac{2\{\lambda_1\nu_1(1+\lambda_2)+\lambda_2\nu_1(1+\lambda_1)\}}{\pi(1+\lambda_2)} \qquad \ldots(39)$$

Let us consider the case of Magnesium for which the roots of the Eq. (12) are real and unequal.

The elastic constants are (Hearmon[3]):

$C_{13} = 0.181 \times 10^{12}$ dynes/cm^2

$C_{33} = 0.587 \times 10^{12}$ dynes/cm^2

The values of the roots are:

$\nu_1^2 = 1.985,$

$\nu_2^2 = 0.523$

and the corresponding values of λ_1 and λ_2 are:

$\lambda_1 = 2.732, \quad \lambda_2 = 0.365.$

The values of the Zeta-function are:

$$\zeta(3) \equiv \sum_{m=1}^{\infty} m^{-3} = 1.202, \qquad \zeta(5) \equiv \sum_{m=1}^{\infty} m^{-5} = 1.0369$$

Using these numerical values we obtain from (38) that:

$$u_0(r) = Mp_0\sqrt{(1-r^2)}\,[1 - 0.319(\delta^{-3}) + 0.013(\delta^{-5})(7 + 5r^2) + 0(\delta^{-7})].$$

Choosing $\delta = 3$ the normal component of displacement $u_0(r)$ is shown graphically for $0 \leqslant r \leqslant 1$ in Fig. 1.

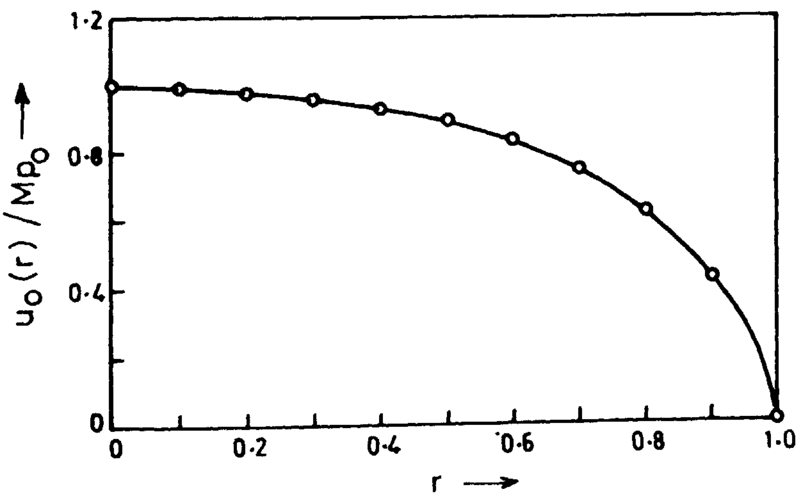

Fig 1

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