Proc. Indian natn. Sci. Acad., **60**, A, No. 3, 1994, pp. 535-542.

FLOW OF THIRD GRADE FLUID THROUGH STENOSED TUBES

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(*Received* 21 *May* 1993; *Revised* 29 *December* 1993; *Accepted* 4 *January* 1994)

The effects of the stenotic geometry and the non-Newtonian parameter of a third grade fluid on the resistive impedance and wall shearing stress are estimated for the flow in a stenosed tube. The results are compared with the similar studies corresponding to other non-Newtonian and Newtonian fluids. Possible biomedical implications related to blood flow of the study are discussed.

Key Words: Third Grade Fluid; Stenosis; Resistive Impedance; Wall Shear Stress; Blood Flow

Introduction

The rheological behaviour of suspensions in fluids cannot always be described by classical theory. The most favoured approach to modify classical continuum theory for analysing such flows has been the introduction of higher order kinematic variables. Several constitutive assumptions have been made to model the non-Newtonian nature of suspensions. One class that has gained support for the above study is the fluids of differential type which was first considered systematically by Rivlin and Ericksen[1]. Fluids of grade '*n*' form a sub-class of fluids of the differential type. In recent years, thermodynamics of fluids of grades 2 and 3 have attracted considerable attention[2,3]. However, only a few boundary value problems involving such fluids have been solved till today, some of them exactly[4].

Due to its application in various fields like physiological flows and polymer science, the flow of non-Newtonian fluids through converging and diverging channels has received great consideration during the last few decades. Blood being a suspension of cells in plasma behaves like a non-Newtonian fluid[5]. The non-linear relationship of shear stress and shear rate is an important feature of non-newtonian fluids. Majhi and Usha[6] have analysed the flow of blood in tubes by representing blood as a third grade fluid. With the help of the experimental data of Gaetghens[7], they have identified the range of values of the non-Newtonian parameter of the fluid for representing blood flow. The rheologic and hydrodynamic properties of blood flow through arteries play a vital role in the etiology and pathogenesis of arterial stenosis which occurs due to the deposition of fatty substances suspended in blood. Numerous studies, analytical as well as experimental, with different perspectives have been carried out to analyse the flow of both Newtonian and non-Newtonian fluids through stenosed tubes[8,9,10,11,12,13,14]. In the present study, an effort has been made to analyse the flow of a third grade fluid through a stenosed tube with cosine wall variation keeping in view the hemodynamics associated with arterial diseases.

Analysis

A steady, axially symmetric, laminar and fully developed flow of a third grade fluid through a rigid circular tube of finite length L and radius R_0 with mild stenosis (Fig. 1) has been considered. In the analysis, cylinderical polar coordinates (r, θ, z) have been used taking z-axis as the axis of the tube. In axi-symmetric flow problems with mild stenosis the radial velocity is negligible compared to the axial velocity[15]. Hence, we assume the velocity components in the respective coordinate directions as $(0,0, w(r,z))$. Following Young[8], the inner radius, $R(z)$, of the stenosed tube is defined as

$$R=\begin{cases} R_0-\dfrac{\delta}{2}\left[1+\cos\dfrac{2\pi}{L_0}\left(z-d-\dfrac{L_0}{2}\right)\right], & d\leqslant z\leqslant d+L_0 \\ R_0 & 0\leqslant z\leqslant d \text{ and } L_0\leqslant z\leqslant L \end{cases} \quad \ldots(1)$$

Here R_0 is the radius of the tube outside the stenotic region, L_0 is the length of the stenosis, d indicates its location and δ is the maximum perturbance of the stenosis growth which is assumed to be much smaller in comparison to R_0.

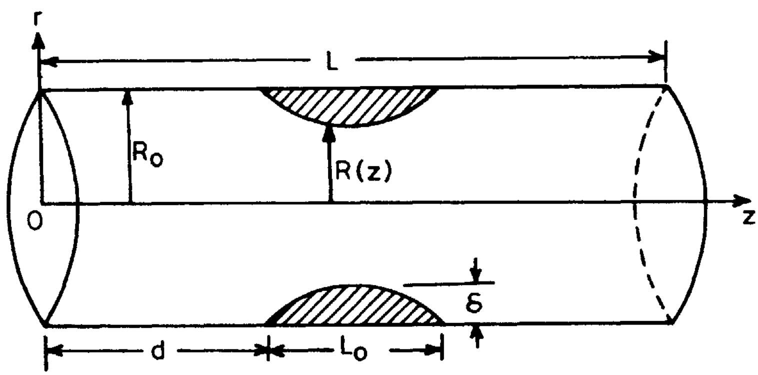

Fig 1 Geometry of a stenosed tube

The constitutive equation for a third grade fluid[16] is

$$\tau=-PI+\mu A_1+\alpha_1 A_2+\alpha_2 A_1^2+\beta_1 A_3+\beta_2[A_1 A_2+A_2 A_1]+\beta_3(TrA_1^2)A_1. \quad \ldots(2)$$

Here τ is the Cauchy stress tensor, PI is the spherical stress, μ is the coefficient of viscosity, α_i's and β_i's are material moduli. A_1, A_2, A_3 are the first three Rivlin-Ericksen tensors defined through

$$A_1=\text{grad } v+(\text{grad } v)^T$$

and

$$A_n=\dot{A}_{n-1}+A_{n-1}(\text{grad } v)+(\text{grad } v)^T A_{n-1}, \; n=2, 3,$$

where v denotes the velocity field, (T) stands for the matrix transposition and dot $(\cdot)$ denotes the material time differentiation. It has been shown by Fosdick and Rajagopal[17] that the fluid represented by the constitutive relation (2) is required to satisfy the Clausius-Duhem inequality along with the Helmholtz free energy assumption for the fluid to be compatible with thermodynamics. Then in the present case the material moduli must satisfy

$$\mu>0,\ \alpha_1>0,\ |\alpha_1+\alpha_2|<\sqrt{24\mu\beta_3},$$
$$\beta_1=\beta_2=0 \text{ and } \beta_3\geqslant 0$$

The general constitutive equation[8,11] for the non-Newtonian behaviour under the present flow situation can be written as,

$$\left(-\frac{\mathrm{d}w}{\mathrm{d}r}\right)=f(\tau), \qquad \dots(3)$$

where τ is the shear stress, w is the axial velocity and $f(\tau)$ is the general function prescribed for a given fluid.

Following Bird, *et al.*[18] and Govier and Aziz[19] the volumetric flow rate, Q, can be written in the form of a Robinowitsch equation as:

$$Q=\frac{\pi R^3}{\tau_R^3}\int_0^{\tau_R}\tau^2 f(\tau)\mathrm{d}\tau, \qquad \dots(4)$$

where

$$\tau=\frac{-r}{2}\frac{\mathrm{d}p}{\mathrm{d}z},\ \tau_R=\frac{-R}{2}\frac{\mathrm{d}p}{\mathrm{d}z}. \qquad \dots(5)$$

Here τ_R is the shear stress at the wall ($r=R$) and $\mathrm{d}p/\mathrm{d}z$ is the pressure gradient. The constitutive relation for a third grade fluid under the above assumptions becomes[6]

$$\tau=\left[\mu+B\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right]\frac{\mathrm{d}w}{\mathrm{d}r} \qquad \dots(6)$$

where μ is the viscosity and B is the material modulus characterising the non-Newtonian nature of the fluid. Assuming B to be small enough to neglect terms containing its second and higher powers, one gets from equations (2) and (5)

$$f(\tau)=\left(-\frac{\mathrm{d}w}{\mathrm{d}r}\right)=\frac{9\,B\tau^3}{4\,\mu^4}-\frac{\tau}{\mu}. \qquad \dots(7)$$

Then equations (4) and (5) along with (7) give

$$\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^3-\frac{8\mu^3}{3BR^2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)+\frac{64\mu^4 Q}{3\pi R^6 B}=0 \qquad \dots(8)$$

The real root of the above equation relevant for the present analysis is given by,

$$\frac{dp}{dz}=\left\{\frac{-32Q\mu^4}{3\pi R^6 B}+\sqrt{\left(\frac{-32Q\mu^4}{3\pi R^6 B}\right)^2-\left(\frac{8\mu^3}{9BR^2}\right)^3}\right\}^{(1/3)}$$
$$+\left\{\frac{-32Q\mu^4}{3\pi R^6 B}-\sqrt{\left(\frac{-32Q\mu^4}{3\pi R^6 B}\right)^2-\left(\frac{8\mu^3}{9BR^2}\right)^3}\right\}^{(1/3)} \qquad \ldots(9)$$

Neglecting the second and higher powers in B, one obtains

$$\frac{dp}{dz}=\frac{8Q\mu}{\pi R^4}+\left(\frac{3Q}{\pi}\right)^3\left(\frac{16B}{R^{10}}\right). \qquad \ldots(10)$$

Integrating equations (9) with respect to z over the length of the tube and using the conditions $p=p_0$ at $z=0$, $p=p_1$ at $z=L$, and eq. (1) one gets the corresponding pressure drop as

$$p_1-p_0=2d[k_1+k_2]+\int_d^{d+L_0}\left[\frac{k_1}{(R/R_0)^4}+\frac{k_2}{(R/R_0)^{10}}\right]dz, \qquad \ldots(11)$$

where

$$k_1=\frac{8Q\mu}{\pi R_0^4} \quad \text{and} \quad k_2=\left(\frac{3Q}{\pi}\right)^3\left(\frac{16B}{R_0^{10}}\right).$$

The resistive impedance to flow, λ_r, of a stenosed artery[21,22,12] is defined as

$$\lambda_r=\frac{p_0-p_1}{Q} \qquad \ldots(12)$$

In the case of no stenosis ($R=R_0$ all along the tube) the resistive impedance to flow, λ_n, is given as

$$\lambda_n=\frac{-L}{Q}[k_1+k_2]. \qquad \ldots(13)$$

Then the ratio $\lambda=\lambda_r/\lambda_n$ is written as

$$\lambda=1-\frac{L_0}{L}+\frac{L_0}{2\pi(k_1+k_2)L}\int_0^{2\pi}\left[\frac{k_1}{(a+b\cos\phi)^4}+\frac{k_2}{(a+b\cos\phi)^{10}}\right]d\phi, \qquad \ldots(14)$$

where

$$\phi=\frac{2\pi}{L_0}(z-d-L_0/2), \quad b=\delta/2R_0, \quad a=1-b..$$

Representing τ_s and τ_n as the wall shear stresses at the middle of the tube with stenosis and without stenosis respectively, one gets,

$$\tau_s = -\frac{R_0}{2}[k_1(1-\delta/R_0)^{-3} + k_2(1-\delta/R_0)^{-9}] \qquad \ldots(15)$$

and

$$\tau_n = (-R_0/2)(k_1 + k_2) \qquad \ldots(16)$$

Then the ratio of the wall shear stresses, $T = \tau_s/\tau_n$, can be calculated as:

$$T = \frac{1}{(k_1 + k_2)}[k_1(1-\delta/R_0)^{-3} + k_2(1-\delta/R_0)^{-9}] \qquad \ldots(17)$$

Discussion

The parameters λ and T are calculated numerically from eq. (14) and (17) for various values of the parameters δ/R_0, B and L_0/L. For computational purpose the numerical values of the tube (artery) radius and the corresponding flow rate (blood) are taken from the data given by Hogan and Henrikson[23] keeping in view the application aspects in blood rheology. The corresponding results are presented graphically in Figs 2,3 and 4.

From equation14, it is clear that λ remains always greater than 1 and increases with B. When $B=0$ (Newtonian case), the expression (14) for λ reduces to the corresponding expression reported by Young[8] and Shukla *et al.*[11].

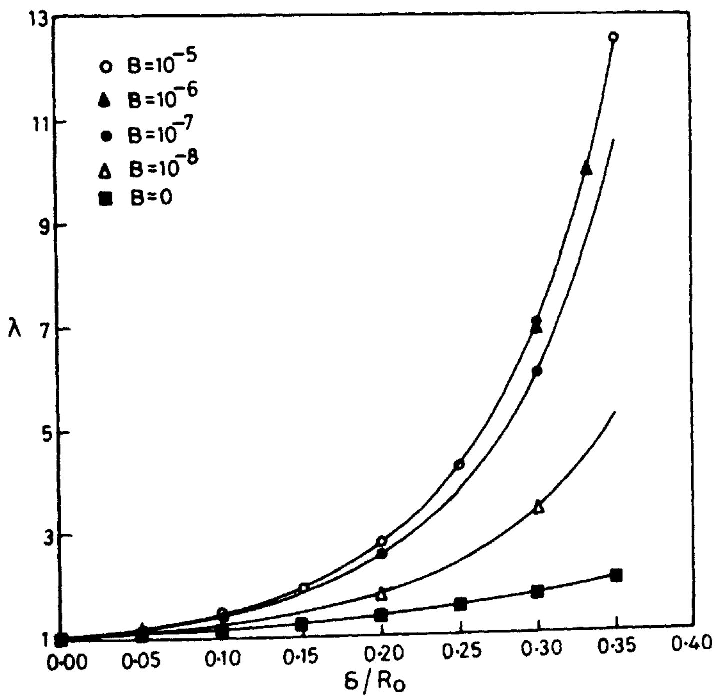

Fig 2 Variation of λ with δ/R_0 for various B

Fig. 2 gives the variation of λ as δ/R_0 varies for specific values of B ranging between 0 and 10^{-5} (cgs units) when the value of L_0/L is fixed as 0.6. For values of B in the range $0 \leqslant B \leqslant 10^{-5}$ the rate at which λ increases with δ/R_0 is comparatively higher over the range $0.25 < \delta/R_0 < 1$ than over $0 \leqslant \delta/R_0 \leqslant 0.25$ and this rate is increasing function of B. There shows no significant variation in λ for values of $B \geqslant 10^{-6}$.

The physical implications of the above observations are:

(i) the resistive impedance in stenosed tube is more than that of the uniform tube; and

(ii) the non-Newtonian nature of a fluid causes significant changes in the impedance which is an increasing function of the material modulus that characterises the non-Newtonian behaviour.

The former observation is associated with vessel geometry where as the later with the nonlinear features of the fluid.

Several investigators[23,24,25] have considered the effect of the length of the stenosis on flow parameters and arrived at the conclusion that the flow does not depend strongly on length. But the investigations of Young[8], Shukla *et al.*[11] and Chaturani *et al.*[12] show that the length of the stenosis has significant influence on the corresponding flow parameters. Therefore, it is of interest to study the effect of L_0/L on the flow variables λ and T in the present case. From eq. (17), it is clear that for a fixed flow rate T does not depend on L_0/L. Fig. 3

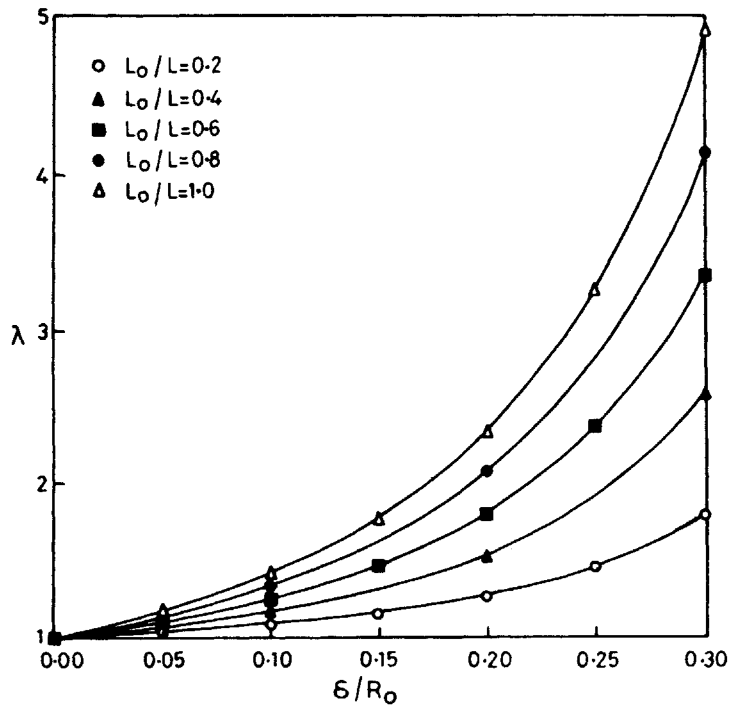

Fig 3 Variation of λ with δ/R_0 for various values of L_0/L

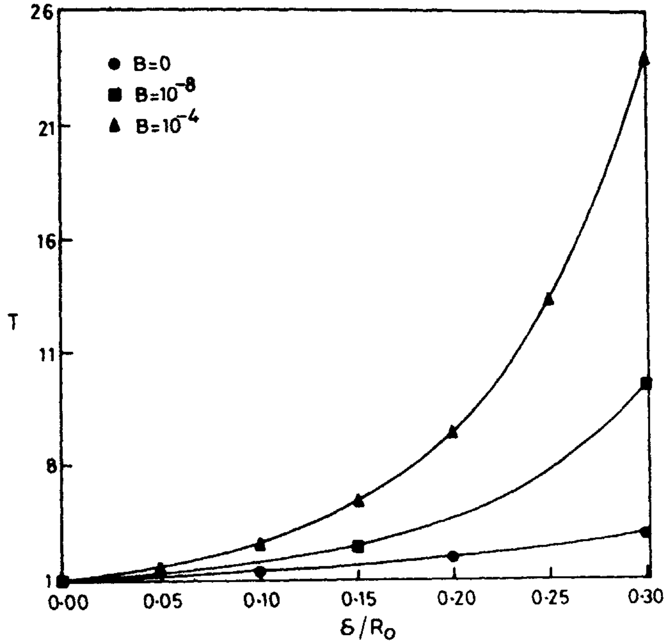

Fig 4 Variation of T with δ/R_0 for different B

gives the effect of L_0/L on λ. It is noted from the figure that the resistive impedance to flow in stenosed tubes increases with the length of the stenosis (for fixed values of B and δ/R_0) and *vice-versa.* This observation is in confirmation with the physics of the situation and supports the findings in the references[9,10,13].

It is clear from eq. (17) that the ratio of the wall shear stress T varies in the range $1 \leqslant T < \infty$ as δ/R_0 takes values between 0 and 1. This fact can be seen from Fig. 4, which graphs T against δ/R_0. The same observation has also been noted in cases of other non-Newtonian and Newtonian fluids[12,13,14]. In the present case the rate of increment in T increases with B for fixed values of δ/R_0.

The comparison of the present results with the corresponding results of Casson and power law models[11] indicates that in all the three cases the flow parameters λ and T increase with δ/R_0 when other parameters assume constant values. These parameters are also increasing functions of L_0/L for fixed δ/R_0 in all the three cases. However, a basic difference noted from the graphs of λ and T is that they fall above the corresponding Newtonian plot in the present case whereas in other two cases they fall below[11].

In conclusion one may remark that both the parameters, namely the resistance to flow and the wall shear stress increase as the size of the stenosis (either L_0/L or δ/R_0, or both) increases when the non-Newtonian parameter B assumes a fixed value. Such effects are also significant when compared with Newtonian case and increase with B. Therefore, in diseased conditions like myocardial infarction, strokes, hypertension etc., it is more appropriate to approximate blood as a non-Newtonian fluid, since Newtonian approximation may lead to inaccurate predictions. As one can see from the present analysis the difference between the two estimates is substantial. The impedance is an important physiological factor that gives the measure of the volume of blood received by different organs. From this point of view the present analysis may be useful for analysing the blood flow in diseased state. Therefore, the physiological parameters λ and T and the rheological parameter B may play a crucial role in comparing the normal subjects and the patients in the diagnosis and treatment of diseases associated with stenosis.

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