Proc. Indian natn. Sci. Acad., **60**, A, No. 4, 1994, pp. 657-674.

THERMOELASTIC STRESSES IN AN EXTERNALLY CRACKED INFINITE ELASTIC MEDIUM CONTAINING A COPLANAR PENNY-SHAPED CRACK

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(*Received* 30 *March* 1993; *Accepted* 16 *March* 1994)

An axisymmetric steady-state thermoelastic problem of an infiite isotropic meidum containing concentric internal and external circular cracks lying in the same plane is considered. The faces of the cracks are expôsed to prescribed temperature distributions. The normal stress, the normal displacement on the crack plane and the stress intensity factors at the boundaries of the cracks are expressed in power series in terms of the ratio between the radii of the cracks. These are illustrated graphically.

Key Words: Thermoelastic Stress; Penny-Shaped & External Circular Crack

Introduction

We consider the problem of an infinite elastic medium containing a crack covering the interior of a circle of radius 'a' and another crack covering the exterior of a coplanar concentric circle of radius 'b' ($a<b$) situated in exactly the same way as considered in the corresponding elastic problem by Selvadurai and Singh[5]. It is assumed that the crack faces are free of stress and are subjected to prescribed temperature distributions. The three part mixed conditions on the crack-plane give rise to triple integral equations which are reduced to simultaneous Fredholm integral equations. These are solved iteratively for small values of the ratio of the radius of the penny-shaped crack to the radius of the external crack. Expressions for the normal stress, the normal displacement on the crack plane and the stress intensity factors at the boundaries of the cracked regions are presented. Variations of these quantities are shown by graphs.

Basic Equations

We consider the temperature and displacement fields in an isotropic elastic medium which is conducting heat. If we assume that there is symmetry about an axis, which we take to be the z-axis, then the position of a typical point of the medium may conveniently be expressed by the cylindrical polar coordinates (r, θ, z) and the displacement vector will have the components (U_r, 0, u_z). The non-vanishing components of the stress tensor will be σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz}.

In the absence of any heat source in the medium, the steady state equations of thermoelasticity with symmetry about z-axis are (Nowacki[1] pg. 11).

$$\left.\begin{aligned} &\nabla^2 u_r - \frac{1}{r^2} u_r + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} - \frac{2(1+\nu)}{1-2\nu}\,\alpha\,\frac{\partial T}{\partial r} = 0\,, \\ &\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} - \frac{2(1+\nu)}{1-2\nu}\,\alpha\,\frac{\partial T}{\partial z} = 0 \end{aligned}\right\} \quad \ldots (1)$$

and $\nabla^2 T = 0$,

where $T = T(r, z)$ denotes the deviation in the temperature from a reference temperature T_0, say, the temperature of the medium in a state of zero stress and strain, α is the co-efficient of linear expansion of the medium, ν is the Poisson's ratio and

$$e = \frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad \ldots (2)$$

The solution set of equations (1) can be obtained in the form

$$u = \bar{\mathbf{u}} + \bar{\mathbf{u}}$$

where $\bar{\boldsymbol{u}}$ is the particular solution of the non-homogeneous system of equations (1) and $\bar{\boldsymbol{u}}$ is the general solution of the homogeneous system

$$\left.\begin{aligned} &\nabla^2 \bar{\boldsymbol{u}}_r - \frac{1}{r^2} \bar{\boldsymbol{u}}_r + \frac{1}{1-2\nu}\frac{\partial \bar{e}}{\partial r} = 0 \\ &\text{and} \\ &\nabla^2 \bar{\boldsymbol{u}}_z + \frac{1}{1-2\nu}\frac{\partial \bar{e}}{\partial z} = 0 , \end{aligned}\right\} \quad \ldots (3)$$

where

$$\bar{e} = \frac{\partial \bar{u}_r}{\partial r} + \frac{1}{r} \bar{u}_r + \frac{\partial \bar{u}_z}{\partial z}.$$

The particular solution of (1) can be derived by following the method of Goodier[2] who introduced the thermoelastic displacement potential Φ related to the displacement $\bar{\boldsymbol{u}}$ by means of

$$\bar{u}_r = \frac{\partial \Phi}{\partial r}, \quad \bar{u}_z = \frac{\partial \Phi}{\partial z}. \quad \ldots (4)$$

Substituting (4) in (1) and integrating we have,

$$\nabla^2 \Phi = mT, \quad \ldots (5)$$

where $m = \frac{1+\nu}{1-\nu}\alpha$.

The problem has thus been reduced to the solution of the Poisson eq. (5) and the system of displacement equations of theory of elasticity (3).

The solution of eq. (5) gives the function Φ and the relevant stress components corresponding to the displacements (4) are expressed by the relations

$$\left.\begin{aligned}\bar{\sigma}_{rz}&=2\mu\frac{\partial^2\Phi}{\partial r\,\partial z}\\ \text{and}\\ \bar{\sigma}_{zz}&=2\mu\left(\frac{\partial^2\Phi}{\partial z^2}-\nabla^2\Phi\right).\end{aligned}\right\}\qquad\ldots(6)$$

For an infinite medium (4) and (6) constitute the final solution whereas in a bounded body the function Φ can satisfy atmost a part of the conditions on the boundary and therefore an additional solution $\bar{u}$ satisfying (3) is always necessary. This can be found conveniently in terms of Love function Ψ according to the relations (Nowacki[1], p. 16)

$$\left.\begin{aligned}\bar{u}_r&=-\frac{1}{1-2\nu}\frac{\partial^2\Psi}{\partial r\,\partial z}\\ \text{and}\\ \bar{u}_z&=\frac{1}{1-2\nu}\left[2(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Psi.\end{aligned}\right\}\qquad\ldots(7)$$

The complementary stresses derivable from Ψ are given by

$$\left.\begin{aligned}\bar{\sigma}_{rz}&=\frac{2\mu}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Psi\\ \text{and}\\ \bar{\sigma}_{zz}&=\frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Psi\end{aligned}\right\}\qquad\ldots(8)$$

The function Ψ satisfies the biharmonic equation

$$\nabla^2\nabla^2\Psi=0.$$

Boundary Conditions

With a suitable choice of our unit of length we can assume that, the faces of the internal and external circular cracks can be described by $r\leq a$, $z=0\pm$ and $b\leq r<\infty$, $z=0\pm$. The faces of the internal crack are exposed to a known temperature distribution $f(r)$ while those of the external crack are kept at zero temperature. The problem of determining the state of stress in the infinite medium $r\geq 0$, $|z|\geq 0$ is equivalent to that for the semi infinite medium $r\geq 0$, $z\geq 0$ of which the plane boundary $z=0$ is subjected to the thermal and elastic conditions:

$$T(r,0)=f(r),\qquad 0\leq r<a,\qquad\ldots(8a)$$

$$\frac{\partial T}{\partial z}(r,0)=0,\qquad a<r<b,\qquad\ldots(9)$$

$$T(r, 0)=0\ , b<r<\infty\ , \qquad \ldots (10)$$

$$\sigma_{rz}(r, 0)=0\ , \qquad 0\leq r<\infty \qquad \ldots (11)$$

$$\sigma_{zz}(\mathrm{r}, 0)=0\ , \qquad 0\leq r<a,\ b<r<\infty \qquad \ldots (12)$$

and

$$u_z(r, 0)=0\ , \qquad a<r<b\ , \qquad \ldots (13)$$

where $f(r)$ is prescribed. Furthermore, the temperature and the components of the stress tensor and the displacement vector tend to zero as $\sqrt{(r^2+z^2)}\rightarrow\infty$. The boundary conditions (8) through (13) are depicted in Fig. 1.

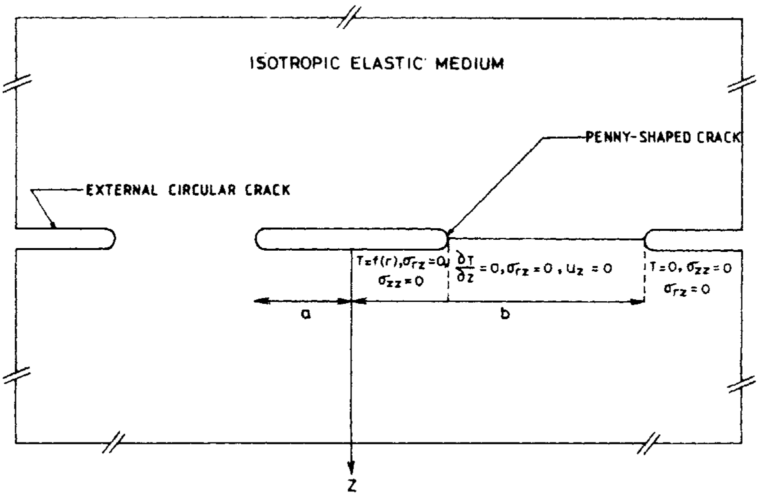

Fig 1 Mixed boundary conditions for the plane of crack $z=0$

Formulation of the Problem. Derivation of Triple Integral Equations

A suitable Hankel integral representation of the temperature field $T(r,z)$ satisfying the Laplace's equation

$$\nabla^2 \mathrm{T}=0$$

and vanishing at infinity is taken in the form

$$T(r, z)=\int_0^\infty \theta(\xi)\exp(-\xi z)\,J_0(\xi\gamma)\,\mathrm{d}\xi\ . \qquad \ldots (14)$$

Imposition of conditions (8), (9) and (10) leads to the derivation of the triple integral equations:

$$\int_0^\infty \theta(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=f(r)\ , \qquad 0\leq r<a\ ,$$

(*eqn. contd on next page*)

$$\left.\begin{aligned}&\int_0^\infty \xi\,\theta(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=0, \qquad a<r<b\\ &\text{and}\\ &\int_0^\infty \theta(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=0, \qquad r>b.\end{aligned}\right\} \qquad \ldots(15)$$

The potential $\Phi(r, z)$ of the thermoelastic displacements satisfying eqs (4) and (5) can be represented by the Hankel integral

$$\Phi(r, z)=-\frac{1}{2}\,m\int_0^\infty \xi^{-2}\,\theta(\xi)\exp(-\xi z)\,J_0(\xi r)\,\mathrm{d}\xi. \qquad \ldots(16)$$

To solve the isothermal elastic problem defined above, we make use of the stress function

$$\Psi(r, z)=-\int_0^\infty \xi^{-3}\,\psi(\xi)(2\nu+\xi z)\exp(-\xi z)\,J_0(\xi r)\,\mathrm{d}\xi. \qquad \ldots(17)$$

Using boundary conditions, eq. (11) is automatically satisfied and from eqs (12) and (13) we get,

$$\int_0^\infty \psi(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=-\frac{(1-2\nu)m}{2}\int_0^\infty \theta(\xi)\,J_0(\xi r)\,\mathrm{d}\xi, \quad 0\leqslant r<a,\ b<r<\infty \qquad \ldots(18)$$

$$\int_0^\infty \xi^{-1}\,\psi(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=0, \qquad a<r<b. \qquad \ldots(19)$$

$$\text{Let } -\frac{1}{2}(1-2\nu)\,mf(r)=F(r). \qquad \ldots(20)$$

Using boundary conditions (8) and (10), we get from equations (18) and (19)

$$\int_0^\infty \psi(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=F(r), \qquad 0\leq r<a, \qquad \ldots(21)$$

$$\int_0^\infty \xi^{-1}\,\psi(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=0, \qquad a<r<b \qquad \ldots(22)$$

and

$$\int_0^\infty \psi(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=0, \qquad b<r<\infty. \qquad \ldots(23)$$

Solution of the Heat Conduction Problem

The associated heat conduction problem defined by the system of triple integral eq. (15) has been reduced to the solution of a pair of Fredholm integral equations by Bhowmick, Bandyopadhyay and Das[4]. Approximate solutions of these equations for small values of the ratio k between the radii of the internal and external circular cracks have been derived and finally the temperature distribution over the crack-plane $z=0$ has been found to be

$$\begin{aligned}
T(r,0)=\frac{f_0}{8\pi}\Bigg[\Bigg\{&16\,\gamma_0^{(1)}+4\,\gamma_2^{(1)}\left(2+\frac{r^2}{a^2}\right)+\gamma_4^{(1)}\left(6+\frac{2r^2}{a^2}+\frac{9r^4}{4a^4}\right)\\
&+\gamma_6^{(1)}\left(5+\frac{3r^2}{2a^2}+\frac{9r^4}{8a^4}+\frac{25r^6}{16a^6}\right)+\gamma_8^{(1)}\left(\frac{35}{8}+\frac{5r^2}{4a^2}+\frac{27r^4}{32a^4}\right.\\
&\left.+\frac{25r^6}{32a^6}+\frac{1225r^8}{1024a^8}\right)\Bigg\}\sin^{-1}\frac{a}{r}+\Bigg\{16\,\gamma_0^{(2)}+4\,\gamma_2^{(2)}\left(2+\frac{b^2}{r^2}\right)\\
&+\gamma_4^{(2)}\left(6+\frac{2b^2}{r^2}+\frac{9b^4}{4r^4}\right)+\gamma_6^{(2)}\left(5+\frac{3b^2}{2r^2}+\frac{9b^4}{8r^4}+\frac{25b^6}{16r^6}\right)\\
&+\gamma_8^{(2)}\left(\frac{35}{8}+\frac{5b^2}{4r^2}+\frac{27b^4}{32r^4}+\frac{25b^6}{32r^6}+\frac{1225b^8}{1024r^8}\right)\Bigg\}\frac{b}{r}\sin^{-1}\frac{r}{b}\\
&-\frac{\sqrt{(r^2-a^2)}}{a}\Bigg\{4\,\gamma_2^{(1)}+\gamma_4^{(1)}\left(\frac{7}{2}+\frac{9r^2}{4a^2}\right)+\gamma_6^{(1)}\left(\frac{37}{12}+\frac{13r^2}{6a^2}+\frac{25r^4}{16a^4}\right)\\
&+\gamma_8^{(1)}\left(\frac{533}{192}+\frac{769r^2}{384a^2}+\frac{2425r^4}{1536a^4}+\frac{1225r^6}{1024a^6}\right)\Bigg\}\\
&-\frac{b\sqrt{(b^2-r^2)}}{r^2}\Bigg\{4\,\gamma_2^{(2)}+\gamma_4^{(2)}\left(\frac{7}{2}+\frac{9b^2}{4r^2}\right)+\gamma_6^{(2)}\left(\frac{37}{12}+\frac{13b^2}{6r^2}+\frac{25b^4}{16r^4}\right)\\
&+\gamma_8^{(2)}\left(\frac{533}{192}+\frac{769b^2}{384r^2}+\frac{2425b^4}{1536r^4}+\frac{1225b^6}{1024r^4}\right)\Bigg\}\Bigg],\qquad a<r<b, \qquad \ldots(24)
\end{aligned}$$

where $f(r)=f_0$ (constant) and $\gamma_i^{(1)}$, $\gamma_i^{(2)}$, $i=0, 2, 4, 6, 8$ are constant terms containing different powers of k. Considering $a=2$, $b=5$ so that $k=0.4$, we find the values of the constants $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ as follows:

$\gamma_0^{(1)}=1.1875$, $\gamma_2^{(1)}=0.0201$, $\gamma_4^{(1)}=0.0026$, $\gamma_6^{(1)}=0.0004$, $\gamma_8^{(1)}=0$,

$\gamma_0^{(2)}=-0.2892$, $\gamma_2^{(2)}=-0.0306$, $\gamma_4^{(2)}=-0.0039$, $\gamma_6^{(2)}=-0.0005$, $\gamma_8^{(2)}=-0.0001$.

Solution of the Thermoelastic Problem

To solve the system of eq. (21), (22) and (23), we assume that

$$\int_0^\infty \xi^{-1}\,\psi(\xi)\,J_0(\xi r)\,\mathrm{d}\xi=\begin{cases} g_1(r), & 0\le r<a \\ g_2(r), & b<r<\infty . \end{cases} \qquad \ldots(25)$$

Employing the method of Cooke[3] from the above system of eqs (21), (22), (23) and (25), we arrive at the pair of simultaneous integral equations

$$g_1(r)=\frac{2}{\pi}(a^2-r^2)^{1/2}\int_b^\infty \frac{t g_2(t)\,\mathrm{d}t}{(t^2-r^2)(t^2-a^2)^{1/2}}+h_1(r), \qquad 0\le r<a \qquad \ldots(26)$$

and

$$g_2(r)=\frac{2}{\pi}(r^2-b^2)^{1/2}\int_0^a \frac{t g_1(t)\,\mathrm{d}t}{(r^2-t^2)(b^2-t^2)^{1/2}}, \qquad b<r<\infty \qquad \ldots(27)$$

where

$$h_1(r)=\frac{2}{\pi}\int_r^a \frac{\mathrm{d}u}{(u^2-r^2)^{1/2}}\int_0^u \frac{sF(s)\,\mathrm{d}s}{(u^2-s^2)^{1/2}}. \qquad \ldots(28)$$

Introducing

$$g_1(r)=\frac{a}{r}(a^2-r^2)^{1/2}\left(\frac{b}{a}\right)^{3/2} p_1\left(\frac{r}{a}\right) \qquad \ldots(29)$$

and

$$g_2(r)=\frac{b}{r}(r^2-b^2)^{1/2}\,p_2\left(\frac{b}{r}\right), \qquad \ldots(30)$$

the above equations are further reduced to the pair of Fredholm integral equations:

$$p_1(v)=\int_0^1 M(u,v)\,p_2(u)\,\mathrm{d}u+q_1(v), \qquad 0\le v<1 \qquad \ldots(31)$$

and

$$p_2(v)=\int_0^1 M(u,v)\,p_1(u)\,\mathrm{d}u, \qquad 0\le v<1, \qquad \ldots(32)$$

where

$$M(u,v)=\frac{2}{\pi}vk^{3/2}\frac{(1-y^2)^{1/2}}{(1-k^2u^2v^2)(1-k^2u^2)^{1/2}}$$

(*eqn contd. on next page*)

and

$$q_1(v)=\frac{v\,h_1(av)\,k^{3/2}}{a(1-v^2)^{1/2}}, \qquad 0\leq v<1 \text{ and } k=\frac{a}{b}. \qquad \ldots(33)$$

Writing $p_\pm(v)=p_1(v)\pm p_2(v)$, we get

$$p_\pm(v)=\pm\int_0^1 M(u,v)\,p_\pm(u)\,\mathrm{d}u+q_1(v), \qquad 0\leq v<1. \qquad \ldots(34)$$

At this stage, we introduce the functions $G_1(r)$ and $G_2(r)$ defined by

$$\left.\begin{aligned} G_1(r)&=\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{t g_1(t)\,\mathrm{d}t}{(t^2-r^2)^{1/2}}, \qquad 0\leq r<a \\ &\text{and} \\ G_2(r)&=\frac{\mathrm{d}}{\mathrm{d}r}\int_b^r \frac{t g_2(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}}, \qquad b<r<\infty. \end{aligned}\right\} \qquad \ldots(35)$$

Solutions of the above Abel type integral equations are

$$\left.\begin{aligned} g_1(t)&=-\frac{2}{\pi}\int_t^a \frac{G_1(s)\,\mathrm{d}s}{(s^2-t^2)^{1/2}}, \qquad 0\leq t<a \\ &\text{and} \\ g_2(t)&=\frac{2}{\pi}\int_b^\infty \frac{G_2(s)\,\mathrm{d}s}{(t^2-s^2)^{1/2}}, \qquad b<t<\infty. \end{aligned}\right\} \qquad \ldots(36)$$

Now we calculate the quantities of physical interest. Inverse Hankel transform of (25) gives

$$\xi^{-2}\,\psi(\xi)=\int_0^a \lambda g_1(\lambda)\,J_0(\xi\lambda)\,\mathrm{d}\lambda+\int_b^\infty \lambda g_2(\lambda)\,J_0(\xi\lambda)\,\mathrm{d}\lambda. \qquad \ldots(37)$$

Inserting g_1 and g_2 from eq. (36) in (37) we get,

$$\xi^{-2}\,\psi(\xi)=-\frac{2}{\pi}\left[\int_0^a G_1(s)\,\mathrm{d}s\int_0^s \frac{\lambda J_0\ (\xi\lambda)\,\mathrm{d}\lambda}{(s^2-\lambda^2)^{1/2}}-\int_b^\infty G_2(s)\,\mathrm{d}s\int_s^\infty \frac{\lambda J_0(\xi\lambda)}{(\lambda^2-s^2)^{1/2}}\,\mathrm{d}\lambda\right]. \qquad \ldots(38)$$

The normal component of stress acting on the crack plane $z=0$ is

$$\sigma_{zz}(r,0)=-m\mu\,T(r,0)+\frac{2\mu}{\pi}\left[\int_0^a \frac{G_1'(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}}-\frac{G_1(a)}{(r^2-a^2)^{1/2}}+\int_b^\infty \frac{G_2'(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}} +\frac{G_2(b)}{(b^2-r^2)^{1/2}}\right], \qquad \ldots(39)$$

where the prime (′) denotes differentiation.

Similarly, the normal component of displacement on $z=0$ can be found in the following form:

$$\left.\begin{aligned} u_z(r,0) &= -\frac{2}{\pi}(1-\nu)\int_r^a \frac{G_1(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}}, && 0\le r<a \\ &= \frac{2(1-\nu)}{\pi}\int_b^r \frac{G_2(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}}, && b<r<\infty . \end{aligned}\right\} \quad \ldots (40)$$

Iterative Solutions of the Fredholm Integral Equations

In the particular case, when the faces of the crack are maintained at a constant temperature, then

$$F(r) = -\frac{1}{2}(1-2\nu)\,mf_0 = f_1 \text{ (say)}, \qquad 0\le r<a,$$

so that

$$\left.\begin{aligned} h_1(v) &= \frac{2f_1}{\pi}(a^2-r^2)^{1/2}, && 0\le r<a \\ &\text{and} \\ q_1(v) &= \frac{2f_1}{\pi}k^{3/2}\,v, && 0\le v<1 . \end{aligned}\right\} \quad \ldots (41)$$

The Fredholm integral eqs (34), then become

$$p_+(v) = \int_0^1 M(u,v)\,p_+(u)\,\mathrm{d}u + \frac{2f_1}{\pi}k^{3/2}v, \qquad 0\le v<1, \qquad \ldots (42)$$

and

$$p_-(v) = -\int_0^1 M(u,v)\,p_-(u)\,\mathrm{d}u + \frac{2f_1}{\pi}k^{3/2}v, \qquad 0\le v<1, \qquad \ldots (43)$$

where the kernel $M(u, v)$ is defined by eq. (33).

To solve the eq. (42) iteratively, let

$$p_+(v) = \frac{2f_1}{\pi}k^{3/2}\sum_{r=0}^{\infty} p_+^{(r)}(v), \qquad 0\le v<1 . \qquad \ldots (44)$$

Then we have the iterative formulae

$$p_+^{(0)}(v) = v$$

and

$$p_+^{(r+1)}(v) = \int_0^1 M(u, v)\, p_+^{(r)}(u)\, \mathrm{d}u . \qquad \ldots (45)$$

The kernel $M(u, v)$ can be expanded as

$$M(u, v) = \sum_{r=0}^{\infty} \beta_{2r}(u)\, v^{2r+1} , \qquad \ldots (46)$$

where the co-efficients β_{2r} are calculated as

$$\beta_0(u) = \frac{2k^{3/2}}{\pi}(1-u^2)^{1/2}\left(1 + \frac{1}{2}k^2u^2 + \frac{3}{8}k^4u^4 + \frac{5}{16}k^6u^6 + \frac{35}{128}k^8u^8 + \ldots\right)$$

$$\beta_2(u) = \frac{2k^{3/2}}{\pi}(1-u^2)^{1/2}\left(k^2u^2 + \frac{1}{2}k^4u^4 + \frac{3}{8}k^6u^6 + \frac{5}{16}k^8u^8 + \ldots\right),$$

$$\beta_4(u) = \frac{2k^{3/2}}{\pi}(1-u^2)^{1/2}\left(k^4u^4 + \frac{1}{2}k^6u^6 + \frac{3}{8}k^8u^8 + \ldots\right), \qquad \ldots (47)$$

$$\beta_6(u) = \frac{2k^{3/2}}{\pi}(1-u^2)^{1/2}\left(k^6u^6 + \frac{1}{2}k^8u^8 + \ldots\right)$$

and

$$\beta_8(u) = \frac{2k^{3/2}}{\pi}(1-u^2)^{1/2}\left(k^8u^8 + \ldots\right).$$

From the form, taken by $M(u, v)$, it is clear that each of the functions $p_+^{(r)}(v)$ may be expressed as

$$p_+^{(r)}(v) = \sum_{s=0}^{\infty} p_{2s}^{(r)}\, v^{2s+1} \qquad \ldots (48)$$

We assume that k is sufficiently small to enable us to omit powers of k higher than the ninth. The next iteration can be obtained from relations (45) where the new co-efficients $p_{2s}^{(r)}$ will be determined from the equations:

$$p_{2s}^{(r+1)} = \int_0^1 [p_0^{(r)}u + p_2^{(r)}u^3 + p_4^{(r)}u^5 + p_6^{(r)}u^7 + p_8^{(r)}u^9]\,\beta_{2s}(u)\, \mathrm{d}u , \qquad s = 0,1,2,3,4 . \qquad \ldots (49)$$

In case of $p_+^{(0)}(v)$, the initial value, we take

$$p_0^{(0)}=1\,,\qquad p_{2s}^{(0)}=0\,,\qquad s=1,2,3\,. \qquad \dots(50)$$

To find the co-efficients of the successive iteration, we calculate the twenty five quantities:

$$a_{2s+1,2r}=\int_0^1 u^{2i+1}\beta_{2s}(u)\,\mathrm{d}u\,,\qquad (i,s=0,1,2,3,4)\,. \qquad \dots(51)$$

The general term $p_{+}^{(r)}(v)$ will be given by (48) with the co-efficients satisfying the recurrence relations

$$p_{2s}^{(r+1)}=a_{1,2s}p_0^{(r)}+a_{3,2s}p_2^{(r)}+a_{5,2s}p_4^{(r)}+a_{7,2s}p_6^{(r)}+a_{3,2s}p_8^{(r)} \qquad \dots(52)$$

The quantities $a_{2s+1,2r}$ in eq. (51) are tabulated below:

$$a_{1,0}=\frac{2k^{3/2}}{\pi}\left[\frac{1}{3}+\frac{k^2}{15}+\frac{k^4}{35}+\frac{k^6}{63}+\frac{k^8}{99}\right],$$

$$a_{3,0}=\frac{2k^{3/2}}{\pi}\left[\frac{2}{15}+\frac{4k^2}{105}+\frac{2k^4}{105}+\frac{8k^6}{693}+\frac{10k^8}{1287}\right],$$

$$a_{5,0}=\frac{2k^{3/2}}{\pi}\left[\frac{8}{105}+\frac{8k^2}{315}+\frac{16k^4}{1155}+\frac{80k^6}{9009}+\frac{8k^8}{1287}\right],$$

$$a_{7,0}=\frac{2k^{3/2}}{\pi}\left[\frac{16}{315}+\frac{64k^2}{3465}+\frac{32k^4}{3003}+\frac{64k^6}{9009}+\frac{112k^8}{21879}\right],$$

$$a_{9,0}=\frac{2k^{3/2}}{\pi}\left[\frac{128}{3465}+\frac{128k^2}{9009}+\frac{128k^4}{15015}+\frac{128k^6}{21879}+\frac{1792k^8}{415701}\right],$$

$$a_{1,2}=\frac{2k^{3/2}}{\pi}\left[\frac{2k^2}{15}+\frac{4k^4}{105}+\frac{2k^6}{105}+\frac{8k^8}{693}\right],$$

$$a_{3,2}=\frac{2k^{3/2}}{\pi}\left[\frac{8k^2}{105}+\frac{8k^4}{315}+\frac{16k^6}{1155}+\frac{80k^8}{9009}\right],$$

$$a_{5,2}=\frac{2k^{3/2}}{\pi}\left[\frac{16k^2}{315}+\frac{64k^4}{3465}+\frac{32k^6}{3003}+\frac{64k^8}{9009}\right],$$

$$a_{7,2}=\frac{2k^{3/2}}{\pi}\left[\frac{128k^2}{3465}+\frac{128k^4}{9009}+\frac{128k^6}{15015}+\frac{128k^8}{21879}\right],$$

$$a_{9,2}=\frac{2k^{3/2}}{\pi}\left[\frac{256k^2}{9009}+\frac{512k^4}{45045}+\frac{256k^6}{36465}+\frac{2048k^8}{415701}\right],$$

$$a_{1,4}=\frac{2k^{3/2}}{\pi}\left[\frac{8k^4}{105}+\frac{8k^6}{315}+\frac{16k^8}{1155}\right],$$

$$a_{3,4}=\frac{2k^{3/2}}{\pi}\left[\frac{16k^4}{315}+\frac{65k^6}{3465}+\frac{32k^8}{3003}\right],$$

$$a_{5,4}=\frac{2k^{3/2}}{\pi}\left[\frac{128k^4}{3465}+\frac{128k^6}{9009}+\frac{128k^8}{15015}\right],$$

$$a_{7,4}=\frac{2k^{3/2}}{\pi}\left[\frac{256k^4}{9009}+\frac{512k^6}{45045}+\frac{256k^8}{36465}\right],$$

$$a_{9,4}=\frac{2k^{3/2}}{\pi}\left[\frac{1024k^4}{45045}+\frac{1024k^6}{109395}+\frac{4096k^8}{692835}\right],$$

$$a_{1,6}=\frac{2k^{3/2}}{\pi}\left[\frac{16k^6}{315}+\frac{64k^8}{3465}\right],$$

$$a_{3,6}=\frac{2k^{3/2}}{\pi}\left[\frac{128k^6}{3465}+\frac{128k^8}{9009}\right],$$

$$a_{5,6}=\frac{2k^{3/2}}{\pi}\left[\frac{256k^6}{9009}+\frac{512k^8}{45045}\right],$$

$$a_{7,6}=\frac{2k^{3/2}}{\pi}\left[\frac{1024k^6}{45045}+\frac{1024k^8}{109395}\right],$$

$$a_{9,6}=\frac{2k^{3/2}}{\pi}\left[\frac{2048k^6}{109395}+\frac{16384k^8}{2078505}\right],$$

$$a_{1,8}=\frac{2k^{3/2}}{\pi}\left[\frac{128k^8}{3465}\right],$$

$$a_{3,8}=\frac{2k^{3/2}}{\pi}\left[\frac{256k^8}{9009}\right],$$

$$a_{5,8}=\frac{2k^{3/2}}{\pi}\left[\frac{1024k^8}{45045}\right],$$

$$a_{7,8}=\frac{2k^{3/2}}{\pi}\left[\frac{2048k^8}{109395}\right],$$

and

$$a_{9,8}=\frac{2k^{3/2}}{\pi}\left[\frac{32768k^8}{2078505}\right].$$

It is found that upto terms of order less than ten:

$p_0^{(0)}=1,\ p_2^{(0)}+p_4^{(0)}=p_6^{(0)}=p_8^{(0)}=0\ ,$

$p_0^{(1)}=0.2122\,k^{3/2}+0.0424\,k^{7/2}+0.0182\,k^{11/2}+0.0101\,k^{15/2}+0.0064\,k^{19/2}\ ,$

$p_2^{(1)}=0.0849\,k^{7/2}+0.0242\,k^{11/2}+0.0121\,k^{15/2}+0.0073\,k^{19/2}\ ,$

$p_4^{(1)}=0.0485\,k^{11/2}+0.0162\,k^{15/2}+0.0088\,k^{19/2}\ ,$

$p_6^{(1)}=0.0323\,k^{15/2}+0.0118\,k^{19/2}\ ,$

$p_8^{(1)}=0.0235\,k^{19/2}\ ,$

$p_0^{(2)}=0.0450\,k^{3}+0.0252\,k^{5}+0.0161\,k^{7}+0.0110\,k^{9},$

$p_2^{(2)}=0.0180\,k^{5}+0.0128\,k^{7}+0.0093\,k^{9}\ ,$

$p_4^{(2)}=0.0103\,k^{7}+0.0082\,k^{9}\ ,$

$p_6^{(2)}=0.0069\,k^{9}\ ,$

$p_0^{(3)}=0.0095\,k^{9/2}+0.0087\,k^{13/2}+0.0073\,k^{17/2}\ ,$

$p_2^{(3)}=0.0038\,k^{13/2}+0.0041\,k^{17/2}\ ,$

$p_4^{(3)}=0.0022\,k^{17/2}\ ,$

$p_0^{(4)}=0.0021\,k^{6}+0.0025\,k^{8}\ ,$

$p_2^{(4)}=0.0008\,k^{8}\ ,$

$p_0^{(5)}=0.0004\,k^{15/2}+0.0007\,k^{19/2}\ ,$

$p_2^{(5)}=0.0002\,k^{19/2}$

and

$p_0^{(6)}=0.0001\,k^{9}\ .$

With these values of the co-efficients $p_{2s}^{(r)}$ we get an iterative solution of the equation (42) in the form

$$p_+(v)=\frac{2f_1}{\pi}\,k^{3/2}\sum_{r,s}p_{2s}^{(r)}\,v^{2s+1}\ ,\qquad 0\leq v<1\ .$$

The integral equation (42) differs from the equation (43) by a change in the sign of the kernel function $M(u, v)$ only. An iterative solution of the eq. (43) can be found in the form:

$$p_-(v) = \frac{2f_1}{\pi} k^{3/2} \sum_{r,s} (-)^r p_{2s}^{(r)} v^{2s+1}, \qquad 0 \le v < 1 .$$

Thus solutions of (31) and (32) are obtained as

$$p_1(v) = \frac{1}{2}[p_+(v) + p_-(v)] = \frac{2f_1}{\pi} k^{3/2} \sum_{s=0}^{3} \delta_{2s}^{(1)} v^{2s+1}, \qquad \ldots (53)$$

and

$$p_2(v) = \frac{1}{2}[p_+(v) - p_-(v)] = \frac{2f_1}{\pi} \sum_{s=0}^{4} \delta_{2s}^{(2)} v^{2s+1}, \qquad \ldots (54)$$

where

$\delta_0^{(1)} = 1 + 0.0450\,k^3 + 0.0252\,k^5 + 0.0021\,k^6 + 0.0161\,k^7 + 0.0025\,k^8 + 0.0111\,k^9$,

$\delta_2^{(1)} = 0.0180\,k^5 + 0.0128\,k^7 + 0.0008\,k^8 + 0.0093\,k^9$,

$\delta_4^{(1)} = 0.0103\,k^7 + 0.0082\,k^9$,

$\delta_6^{(1)} = 0.0069\,k^9$,

$\delta_0^{(2)} = 0.2122\,k^3 + 0.0424\,k^5 + 0.0095\,k^6 + 0.0182\,k^7 + 0.0087\,k^8 + 0.0105\,k^9$

$\qquad + 0.0073\,k^{10} + 0.0071\,k^{11}$,

$\delta_2^{(2)} = 0.0849\,k^5 + 0.0242\,k^7 + 0.0038\,k^8 + 0.0121\,k^9 + 0.0041\,k^{10} + 0.0075\,k^{11}$,

$\delta_4^{(2)} = 0.0485\,k^7 + 0.0162\,k^9 + 0.0022\,k^{10} + 0.0088\,k^{11}$,

$\delta_6^{(2)} = 0.0323\,k^9 + 0.0118\,k^{11}$

and

$\delta_8^{(2)} = 0.0235\,k^{11}$.

Inserting (53) and (54) in (29) and (30) we get,

$$g_1(r) = \frac{2f_1}{\pi}\left[\delta_0^{(1)} + \delta_2^{(1)}\left(\frac{r}{a}\right)^2 + \delta_4^{(1)}\left(\frac{r}{a}\right)^4 + \delta_6^{(1)}\left(\frac{r}{a}\right)^6\right]\sqrt{(a^2 - r^2)}$$

$$g_2(r)=\frac{2f_1}{\pi}\left(\frac{b}{r}\right)^2\left[\delta_0^{(2)}+\delta_2^{(2)}\left(\frac{b}{r}\right)^2+\delta_4^{(2)}\left(\frac{b}{r}\right)^4+\delta_6^{(2)}\left(\frac{b}{r}\right)^6+\delta_8^{(2)}\left(\frac{b}{r}\right)^8\right]\sqrt{(r^2-b^2)}\,. \qquad \ldots(55)$$

Substituting from (55) in (35), we get

$$G_1(r)=f_1\left[-r\delta_0^{(1)}-\left(\frac{3r^3}{2a^2}-\frac{r}{2}\right)\delta_2^{(1)}-\left(\frac{15r^5}{8a^4}-\frac{3r^3}{4a^2}\frac{r}{8}\right)\delta_4^{(1)}\right.$$
$$\left.-\left(\frac{35r^7}{16a^6}-\frac{15r^5}{16a^2}-\frac{3r^3}{16a^2}-\frac{r}{16}\right)\delta_6^{(1)}\right]$$

$$G_2(r)=bf_1\left[\frac{b^2}{r^2}\delta_0^{(2)}+\left(-\frac{b^2}{2r^2}+\frac{3b^4}{2r^4}\right)\delta_2^{(2)}+\left(-\frac{b^2}{8r^2}-\frac{3b^4}{4r^4}+\frac{15b^6}{8r^6}\right)\delta_4^{(2)}\right.$$
$$+\left(-\frac{5b^2}{16r^2}+\frac{9b^4}{16r^4}+\frac{5b^6}{16r^6}+\frac{7b^8}{16r^8}\right)\delta_6^{(2)}+\left(-\frac{5b^2}{128r^2}-\frac{3b^4}{32r^4}-\frac{15b^6}{64r^6}\right.$$
$$\left.\left.-\frac{35b^8}{32r^8}+\frac{315b^{10}}{128r^{10}}\right)\delta_8^{(2)}\right].$$

Now, from equation (39), we get,

$$\sigma_{zz}(r,0)=\frac{4\mu f_1}{\pi(1-2\nu)}\left[-\left\{\delta_0^{(1)}+\delta_2^{(1)}\left(\frac{9r^2}{4a^2}-\frac{1}{2}\right)+\delta_4^{(1)}\left(\frac{225r^4}{64a^4}-\frac{9r^2}{8a^2}-\frac{1}{8}\right)\right.\right.$$
$$\left.+\delta_6^{(1)}\left(\frac{1225r^6}{256a^6}-\frac{225r^4}{128a^4}-\frac{9r^2}{32a^2}-\frac{1}{16}\right)\right\}\sin^{-1}\frac{a}{r}+\frac{\sqrt{(r^2-a^2)}}{a}\left\{\frac{9}{4}\delta_2^{(1)}\right.$$
$$\left.+\left(\frac{225r^2}{64a^2}+\frac{111}{32}\right)\delta_4^{(1)}+\left(\frac{245r^4}{64a^4}+\frac{475r^2}{96a^2}+\frac{769}{192}\right)\delta_6^{(1)}\right\}$$
$$+\frac{a}{\sqrt{(r^2-a^2)}}\{\delta_0^{(1)}+\delta_2^{(1)}+\delta_4^{(1)}+\delta_6^{(1)}\}+\frac{b}{\sqrt{(b^2-r^2)}}\{\delta_0^{(2)}+\delta_2^{(2)}+\delta_4^{(2)}+\delta_6^{(2)}+\delta_8^{(2)}\}$$
$$+\frac{b}{r}\left\{\frac{b^2}{r^2}\left(\delta_0^{(2)}+\left(\frac{1}{2}-\frac{9b^2}{4r^2}\right)\delta_2^{(2)}+\left(\frac{1}{8}+\frac{9b^2}{8r^2}-\frac{225b^4}{64r^4}\right)\delta_4^{(2)}+\left(\frac{5}{16}-\frac{27b^2}{32r^2}\right.\right.\right.$$
$$\left.\left.-\frac{75b^4}{128r^4}-\frac{245b^6}{256r^6}\right)\delta_6^{(2)}\quad\sin^{-1}\frac{r}{b}-\frac{\sqrt{(b^2-r^2)}}{r}\left(\delta_0^{(2)}-\left(1+\frac{9b^2}{4r^2}\right)\delta_2^{(2)}\right.$$
$$\left.\left.\left.-\left(1+\frac{39b^2}{32r^2}+\frac{225b^4}{64r^4}\right)\delta_4^{(2)}-\left(1+\frac{335b^2}{192r^2}+\frac{235b^4}{192r^4}+\frac{245b^6}{256r^6}\right)\delta_6^{(2)}\right)\right\}\right]$$

$$-m\mu T(r,0),\qquad a<r<b\,. \qquad \ldots(56)$$

From eq. (40), the normal component of displacement vector on $z=0$ can be found in the following form:

$$u_z(r,0)=\begin{cases}\dfrac{2(1-\nu)}{1-2\nu}\,g_1(r), & 0\le r<a\\[2mm] \dfrac{2(1-\nu)}{1-2\nu}\,g_2(r), & b<r<\infty\,.\end{cases} \qquad \ldots(57)$$

The stress intensity factors at the boundaries $r=a$ and $r=b$ are defined by

$$K_a=\lim_{r\to a+}[2(r-a)]^{1/2}\,\sigma_{zz}(r,0)$$

and

$$K_b=\lim_{r\to b-}[2(b-r)]^{1/2}\,\sigma_{zz}(r,0)$$

respectively.

After some mathematical manipulation we obtain the following expressions for the stress intensity factors:

$$K_a=-\frac{4}{\pi}\cdot\frac{\mu}{1-2\nu}\,\frac{G_1(a)}{\sqrt{a}.}$$

$$=\frac{4f_1}{\pi}\cdot\frac{\mu}{1-2\nu}\cdot\sqrt{a}\,[1+0.0450\,k^3+0.0432\,k^5+0.0021\,k^6+0.0392\,k^7$$

$$+0.0033\,k^8+0.0355\,k^9] \qquad \ldots(58)$$

$$K_b=\frac{4}{\pi}\cdot\frac{\mu}{1-2\nu}\cdot\frac{G^2(b)}{\sqrt{b}}$$

$$=\frac{4f_1}{\pi}\cdot\frac{\mu}{1-2\nu}\cdot\sqrt{b}\,[0.2122\,k^3+0.1273\,k^5+0.0095\,k^6+0.0909\,k^7$$

$$+0.0125\,k^8+0.0711\,k^9+0.0136\,k^{10}+0.0587\,k^{11}]\,. \qquad \ldots(59)$$

The variation of K_a and K_b with $k(=a/b)$ is shown in Fig. 2 and the variation of $\sigma_{zz}(r,0)$ and $u_z(r,0)$ with r are shown in Figs 3 & 4 respectively.

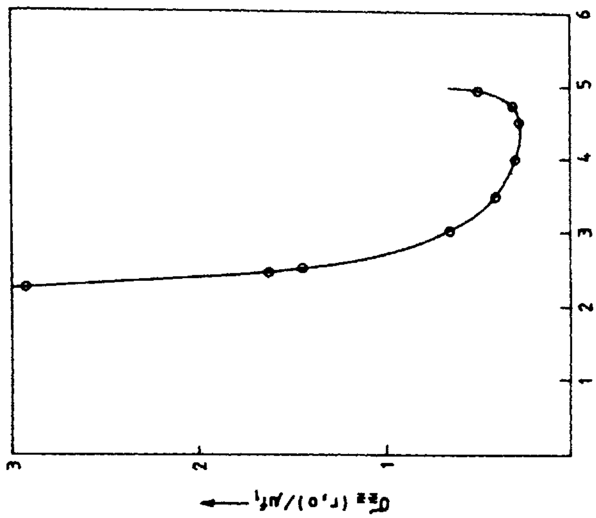

Fig 3 Variation of $\sigma_{zz}(r, 0)$ with r

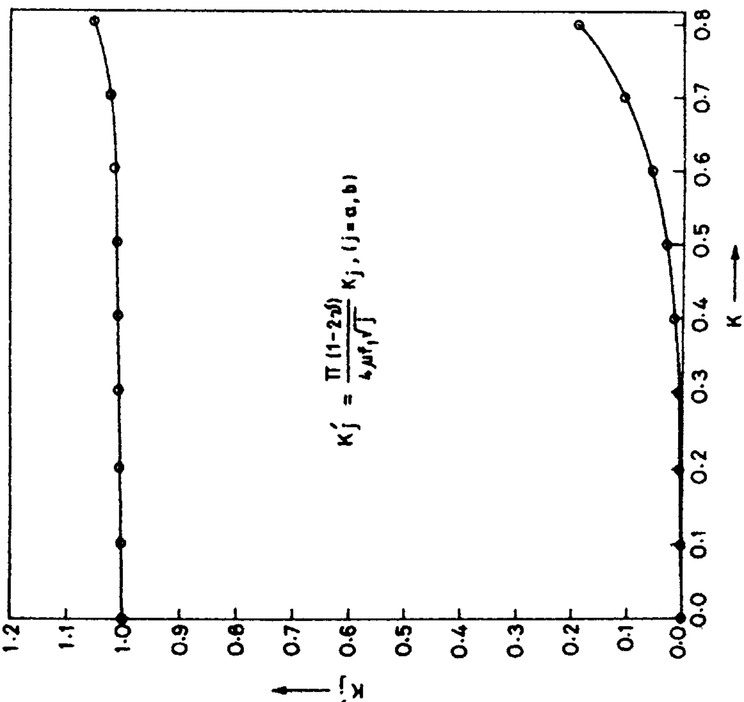

Fig 2 Variation of K_a, K_b with K

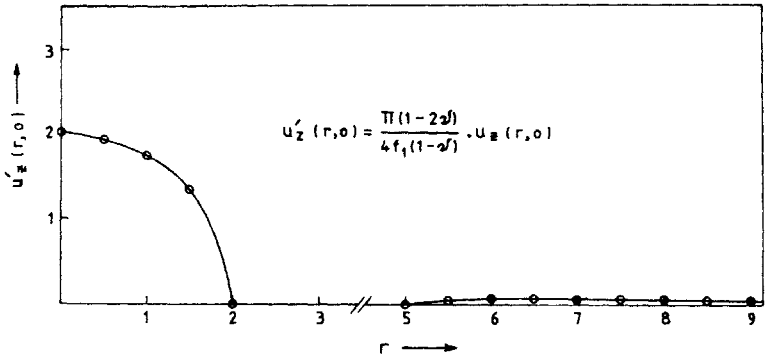

Fig 4 Variation of $u_z(r, 0)$ with r

Numerical Results

Let us consider the case of steel, for which the values of the relevant constants (Love[6]) are

$\mu = 8.19 \times 10^{11}$, $\nu = 0.310$, $\alpha_t = 10^{-5}/°C$.

With these values of the constants we calculate the normal component of stress $\sigma_{zz}(r, 0)$ for different values of r. The variation of $\sigma_{zz}(r, 0)$ is shown in Fig. 3.

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