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ON THE STABILITY OF QUASI TWO DIMENSIONAL HYDRODYNAMIC FLOWS

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In two- and three-dimensional turbulent media, the cascading characteristics of its invariants are well known. A variational principle is set up here for the three-dimensional flows by invoking the differential decay rates of its invariants, the total energy and the mean square helicity density. An exact solution of the equation for the quasi-two dimensional case is presented. The stability of such a flow is also discussed.

Key Words: Invariants; Stability; Quasi-2-Dimensional Flows; Variational Principle

Introduction

There is a considerable experimental and observational evidence that self-organisation of flow into large structures is not precluded by turbulence and strong nonlinearities. Self-organisation is a process by which nonlinear interactions between small fluid elements result in the formation of large ordered coherent structures.

Hasegawa[1] has described the general features of a system capable of exhibiting self-organisation. The system is described by a nonlinear partial differential equation with dissipation. The system has two or more quadratic or higher order conserved quantities in the absence of dissipation. If the spectral behaviour of these invariants in the inertial range is such that one of them transfers towards large spatial scales and the other to small spatial scales, they would have differential dissipation rates when the dissipation is introduced.

The formation of large scale structures is a consequence of an inverse cascade of energy. Kraichnan[2,3] suggested that the formation of large-scale structures in two-dimensional incompressible turbulence is due to an inverse cascade driven by negative viscosity instabilities.

The self-organisation processes in two-dimensional hydrodynamic turbulence has been studied by Rhines[4], Bretherton and Haidvogel[5], Hasegawa[6] and by Woltjer[7], Taylor[8], Montgomery *et al.*[9], Mathaeus and Montgomery[10] in magnetohydrodynamic turbulence. An excellent review on self-organisation processes in continuous media has been presented by Hasegawa[1].

The importance of mean square helicity as an invariant and its role in inverse cascade in three dimensional turbulent fluids has been brought out by Levich and Tzvetkov[11] in the context of the earth's atmosphere. Recently, Frisch *et al.*[12] and Sulem *et al.*[13] discussed the generation of large scale structures in three dimensional flows lacking parity-invariance, in terms of kinetic α-effect.

A model of solar granulation through inverse cascade has been presented by Krishan[14]. Granules are cellular velocity patterns observed on the solar surface, believed to be manifestations of convective phenomenon occurring in the sub-photospheric layers. The formation of these cellular patterns on all scales has been interpreted to be the result of self-organisation processes occurring in the turbulent medium of the solar atmosphere.

The presence of very large structures in the Universe like the Great Wall indicates the existence of a hierarchy of well-ordered coherent formations up to the very large scales of a few hundred Mpc (Huchra and Geller[15]). Krishan and Sivaram[16] have explained the clustering of galaxies on several scales by inverse cascade in a turbulent medium. Satya Narayanan[17] studied self-organisation processes in Quasi-2 dimensional hydrodynamic flows.

In this paper, a variational principle connecting the invariants E (energy) and I (mean square helicity density) for a 3-dimensional incompressible fluid is set up following the arguments given in Hasegawa[1]. The resulting variational equation is highly nonlinear and hence closed form solutions are hard to obtain. However, the author has presented a closed form solution of the equation under some simplifying but physically realizable situations.

Variational Equation

Large helicity fluctuations present in a turbulent medium play an important role in the inverse cascading processes. The helicity density γ, a measure of the knottedness of the vorticity field, is defined as $\gamma = \nabla \cdot \omega$, $\omega = \nabla \times V$, where V and ω are the velocity and vorticity, respectively.

The quantity I defined as

$$I = \int \langle \gamma(x)\, \gamma(x+r) \rangle \mathrm{d}^3 x \qquad \ldots (1)$$

where $\langle \quad \rangle$ denotes an average over an ensemble, is an invariant of an ideal 3D hydrodynamic system in addition to the total energy E. By assuming a quasi normal distribution of helicities, the invariant I can be expressed as

$$I = C_1 \int [E(k)]^2 \, \mathrm{d}k, \qquad \ldots (2)$$

where C_1 is a constant and $E = \int E(k)\, \mathrm{d}k$ is the total energy density.

It can be shown that the inertial range for the energy invariant to be

$$E(k)\, a\, k^{-5/3} \text{ and } E\, a L^{2/3}, \qquad \ldots (3)$$

and for the I invariant to be

$$E(k)\, a\, k^{-1} \text{ and } E\, a \log L(t), \qquad \ldots (4)$$

where $L(t)$ is the largest length scale excited at time t (Levich and Tzvetkov[11]). Hasegawa[1] formulated a variational principle using the two invariants, the energy and enstrophy of a 2D system. Along the same lines, we set up a variational equation using the invariants 'I' and 'E' of a 3D system to get:

$$\delta \int (V \cdot \omega)^2 \, \mathrm{d}^3 x - \lambda \delta \int V^2 \mathrm{d}^3 x = 0. \qquad \ldots (5)$$

$$\int (V\cdot \omega)[\delta V\cdot \omega + V\cdot \delta\omega]\, d^3x - \lambda \int V\cdot \delta V d^3x = 0. \qquad \ldots (6)$$

Here λ is the Lagrange multiplier.

It can be shown that

$$V\cdot \delta\omega = \delta V\cdot \omega + \nabla\cdot[\delta V\times V].$$

Eq. (6) then becomes:

$$\int (V\cdot \omega)\, 2[\delta V\cdot \omega - \lambda V/2(V\cdot \omega)]\, d^3x \pm \int (V\cdot \omega)[\nabla\cdot(\delta V\times V)]\, d^3x = 0. \qquad \ldots (7)$$

Manipulating the second integral, eq. (7) can be written as

$$\int 2(\mathbf{V}\cdot \omega)\, \delta\mathbf{V}\cdot \left[\omega - \frac{\lambda \mathbf{V}}{2(\mathbf{V}\cdot\omega)} - \frac{\mathbf{V}\times\nabla(\mathbf{V}\cdot\omega)}{2(\mathbf{V}\cdot\omega)}\right] d^3x + \int_S (V\cdot \omega)(\delta V\times V)\cdot ds = 0. \qquad \ldots (8)$$

Applying the boundary condition that $\omega\cdot n$ vanishes on the boundary S, eq. (8) reduces to

$$2\,\omega(V\cdot \omega) - \lambda V - V\times \nabla(V\cdot\omega) = 0. \qquad \ldots (9)$$

The corresponding equation for a two-dimensional system with enstrophy and energy as its invariants is

$$\nabla\times\nabla\times V - \alpha V = 0. \qquad \ldots (10)$$

This is a linear equation whose solution can be written down immediately. However, eq. (9) is highly nonlinear and more difficult to solve.

The self-organised state described by eq. (9) should be a stationary solution of the Navier-Stokes equation (without dissipation and gravity)

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{\nabla p}{\rho} \qquad \ldots (11)$$

and

$$\nabla\cdot V = 0. \qquad \ldots (12)$$

Solution of the Variational Equation for a Special Case

Quasi 2-dimensional Case:

The largest dimension of fully 3D structures is given by the ratio $1/E^2 = L = L_z$, where L_z is the characteristic vertical scale. When the correlation length of helicity fluctuations reaches the limit L_z, it can grow only in the horizontal plane. Another consequence of the growth of the correlation length is that the velocity and vorticity become aligned, which reduces the nonlinear term $(V\cdot\nabla)V = \nabla\times(V\times\omega)$ of the Navier-Stokes equation and thus retards the flow of energy to small spatial scales. With the growth of correlation length only in the horizontal plane, the system becomes more and more anisotropic. In these circumstances, the vertical component of velocity V_z becomes independent of (x, y, z) and the horizontal components V_x and V_y independent of z, leading to $\omega_{x,y} = (\nabla \times V)_{x,y} = 0$.

The invariant I becomes

$$I = \int \langle (V_z \omega_z)^2 \rangle \, dx\, dy\, dz$$

$$L_z \langle V_z^2 \rangle k^2 V_k^2 k^{-2} \, \alpha \, V_k^2 = kE(k) \, \alpha \, L^{2/3}, \qquad \ldots (13)$$

and from $I = \int I(k)\, dk$, one finds

$$I(k) \, \alpha \, k^{-5/3} \qquad \ldots (14)$$

Thus $I(k)$ spectrum for the quasi 2D case, coincides with the energy spectrum of 2D turbulence $E(k) \, \alpha \, k^{-5/3}$, corresponding to the inverse cascade.

We assume the velocity field to be

$$V = V[V_x(x,y), V_y(x,y), V_z]$$

where V_z is a constant. The variational equation can be recast

$$\left(\nabla^2 + \frac{\lambda}{V_z^2} \right) \mathbf{V}_H = 0, \qquad \ldots (16)$$

where $V_H = (V_x, V_y)$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Stability

There is a very extensive literature concerning the field of hydrodynamic stability[18]. Bayly[19] studied the stability of quasi-2-dimensional steady flows *via* an analysis of a Floquet system of ODE. The stability of certain very special flows which are exact solutions of the Navier-Stokes equations have been considered by Craik and Criminale[20]. Friedlander and Vashik[21] have discussed the instability criteria of the flow of an inviscid incompressible fluid. They have obtained a geometric estimate from below on the growth rate of a small perturbation of a three dimensional flow.

Consider the dynamical system of ODE's:

$$\dot{X} = -U(X) \qquad \ldots (17)$$

$$\dot{\xi} = \left[\frac{\partial U}{\partial X} \right]^T \xi \qquad \ldots (18)$$

$$\dot{b} = -\left[\frac{\partial U}{\partial X} \right]^T b - [(\nabla \times U) \times b \cdot \xi] \frac{\xi}{|\xi|^2} \qquad \ldots (19)$$

The dot denotes differentiation with respect to time t. The initial conditions at $t = 0$ are

$$X = X_0, \quad \xi = \xi_0, \quad b = b_0,$$

with $\xi_0 \cdot b_0 = 0$.

The quantity $\xi_0 / | \xi_0 |$ is the direction of the spatial wave vector. The matrix $\partial U / \partial X$ has components $\partial u_i / \partial x_j$, $i,j = 1,2,3$.

The vector $\mathbf{b}(X_0, \xi_0, t)$ is the amplitude of a high-frequency wavelet localized at x_0. If σ is the growth rate of the perturbation of the equilibrium solution, then it was shown using WKB methods by Friedlander and Vashik[21] that

$$\overline{\lim_{t \to \infty}} \, (1/t) \ln \sup_{\{x_0, \xi_0, b_0} \| b(X, \xi_\delta t)_0 \| \le \sigma \qquad \ldots (20)$$

$$| \xi_0 | = 1, \; \xi_0 \cdot b_0 = 0$$

They also showed that at a stagnation point of the flow where $\nabla \times U = 0$, the nature of the stability would depend on the eigenvalues of the matrix given by

$$\left[\frac{\partial U}{\partial X} \right]^T \qquad \ldots (21)$$

If one of the eigenvalues of the above matrix is positive, then such a flow admits exponentially growing solution.

The authors apply the stability results obtained by Friedlander and Vashik[21] to see if the solution of the variational equation for quasi two dimensional flow admits stable and unstable solutions.

For a quasi-two dimensional steady flow the stream function satisfying the differential equation (16) can be written as

$$\psi(x, y, V_z) = \frac{\hat{\psi}}{V_z} \sin x \sin y \qquad \ldots (22)$$

where $\hat{\psi}$ is a constant. It is easy to see that the stagnation point is at the origin (0,0). Moreover, $\nabla \times U = 0$ at (0,0).

By simple algebra we can show that

$$-\left[\frac{\partial U}{\partial X} \right]^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \ldots (23)$$

The eigenvalues of the above matrix are ∓ 1. Thus the self-organised flow satisfied by the differential equation (16) can admit both stable and unstable solutions to small perturbations of the ordered coherent structure.

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