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AXISYMMETRIC THERMOELASTIC PROBLEM OF AN EXTERNALLY CRACKED INFINITE TRANSVERSELY ISOTROPIC MEDIUM CONTAINING A PENNY-SHAPED CRACK

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A steady-state thermoelastic problem for an infinite transversely isotropic solid containing a penny-shaped crack and concentric external circular crack has been considered. The crack faces are assumed to be free of stress and are subjected to prescribed axisymmetric temperature distributions. This paper develops power series representations for the stress intensity factors at the boundary of the penny-shaped crack and at the perimeter of the externally cracked region. These series representations are in terms of a non-dimensional parameter which is the ratio of the radius of the penny-shaped crack to the radius of the external circular crack.

Key Words: **Thermal Stress; Penny-shaped Crack; External Circular Crack; Transverse Isotropy**

Introduction

The calculation of thermal stresses in an infinite isotropic elastic medium containing a penny-shaped crack was first carried out by Olesiak and Sneddon[7]. The thermoelastic problems for anisotropic bodies containing cracks have not been treated exhaustively. The present paper is concerned with the determination of the components of stress and displacement in a transversely isotropic elastic material containing coplanar internal and external circular cracks. The corresponding isotropic problem has been solved by Bhomick and Das[6]. It has been assumed that the crack-faces are stress-free and are subjected to prescribed axisymmetric temperature. Expressions for the normal components of stress and displacement in the crack-plane have been presented for a constant temperature distributed over the faces of the internal crack. The stress intensity factors have been found. The corresponding problem for a homogeneous isotropic body has been considered by Selvadurai and Singh[1]. Results presented in this paper are comparable with those obtained by Mehta[6] and Selvadurai and Singh[1]. The variation of the stress intensity factors is shown graphically for different values of the ratio between the radii of the internal and external cracks.

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Basic Equations of Thermoelasticity

Let us consider the temperature and displacement fields in an anisotropic elastic medium which is conducting heat. In the problem it will be considered that the axis of z is the axis of elastic symmetry and that the position of a typical point of the medium may be expressed in terms of cylindrical polar coordinates (r, θ, z). The displacement vector at such a point may be taken to have the components (u_r, θ, u_z) for symmetrical deformation of the elastic medium in the present co-ordinate system and the non-vanishing components of the stress tensor will be σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz}.

In absence of any heat source in the medium. the temperature $T(r, z)$ at any point satisfies the steady-state heat conduction equation:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial T}{\partial r}+L^2\frac{\partial^2 T}{\partial z^2}=0\,, \qquad \ldots(1)$$

where L^2 is the ratio of coefficients of conductivity along z-axis and in z-plane.

The stress-stain relations in a transversely isotropic material are:

$$\sigma_{rr}=C_{11}\,e_{rr}+C_{12}\,e_{\theta\theta}+C_{13}\,e_{zz}-b_1\,T;$$

$$\sigma_{\theta\theta}=C_{12}\,e_{rr}+C_{11}\,e_{\theta\theta}+C_{13}\,e_{zz}-b_1\,T;$$

$$\sigma_{zz}=C_{13}(e_{rr}+e_{\theta\theta})+C_{33}\,e_{zz}-b_1\,T$$

and

$$\sigma_{rz}=C_{44}e_{rz}\,,\quad \text{where}\quad b_1=(C_{11}+C_{12})\alpha_1+C_{13}\,\alpha_2$$
$$b_2=2C_{13}\,\alpha_1+C_{33}\,\alpha_2\,. \qquad \ldots(2)$$

Here α_1, α_2 are the co-efficients of linear thermal expansion in the z-plane and along the z-axis respectively; C_{11}, C_{12}, ..., C_{44} being elastic constants. The strain-displacements relations are:

$$e_{rr}=\frac{\partial u}{\partial r}\,,\; e_{yy}=\frac{u}{r}\,,\; e_{zz}=\frac{\partial w}{\partial z}\,,\; \text{and}\; e_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\,. \qquad \ldots(3)$$

The equations of equilibrium reduce to:

$$\frac{\partial}{\partial r}(\sigma_{rr})+\frac{\partial}{\partial z}(\sigma_{rz})+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=0$$

and

$$\frac{\partial}{\partial r}(\sigma_{rz})+\frac{1}{r}\cdot\sigma_{rz}+\frac{\partial}{\partial z}(\sigma_{zz})=0\,. \qquad \ldots(4)$$

Following Bhattacharya[2] we substitute from (2) the values of stress components in the two equations of (3) and get,

$$C_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+C_{44}\frac{\partial^2 u}{\partial z^2}+(C_{13}+C_{44})\frac{\partial^2 w}{\partial r\,\partial z}=\alpha_1\frac{\partial T}{\partial r},$$

and

$$C_{44}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial w}{\partial r}\right)+C_{33}\frac{\partial^2 w}{\partial z^2}+(C_{13}+C_{44})\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)=\alpha_2\frac{\partial T}{\partial z}. \quad \ldots(5)$$

Let us assume

$$u=\frac{\partial}{\partial r}(\phi+\overline{T}) \quad \text{and} \quad w=\frac{\partial}{\partial z}(\lambda\phi+\mu\overline{T}), \quad \ldots(6)$$

where ϕ and $\overline{T}$ are functions of r and z; λ, μ are some constants. Eq. (5) are satisfied if:

$$C_{11}\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \phi}{\partial r}\right)+[C_{44}+\lambda(C_{13}+C_{44})]\frac{\partial^2 \phi}{\partial z^2}=0 \quad \ldots(7)$$

and

$$C_{11}\left(\frac{\partial^2 \overline{T}}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \overline{T}}{\partial r}\right)+[C_{44}+\mu(C_{13}+C_{44})]\frac{\partial^2 \overline{T}}{\partial z^2}=\alpha_1 T. \quad \ldots(8)$$

The first equation of (7) and (8) will give a non-zero solution only if they are identical and this occurs if

$$\frac{\lambda(C_{13}+C_{44})+C_{44}}{C_{11}}=\frac{\lambda C_{33}}{\lambda C_{44}+C_{13}+C_{44}}=\nu^2 \text{ (say)}. \quad \ldots(9)$$

Then ϕ_1 and ϕ_2 are the solutions of

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\nu_i^2\frac{\partial^2}{\partial z^2}\right)\phi_i=0, \qquad (i=1,2) \quad \ldots(9a)$$

and ν_1^2, ν_2^2 are the roots of the equation

$$C_{11}C_{44}\nu^4+(2C_{13}C_{44}+C_{13}^2-C_{11}C_{33})\nu^2+C_{33}C_{44}=0. \quad \ldots(9b)$$

The components of the displacements are then given by,

$$u_r=\frac{\partial}{\partial r}(\phi_1+\phi_2+\overline{T}), \quad \text{and} \quad \ldots(9c)$$

$$\omega = u_z = \frac{\partial}{\partial z}(\lambda_1\phi_1 + \lambda_2\phi_2) + \mu\bar{T}), \qquad \ldots (10)$$

where λ_1, λ_2 are the two values of λ corresponding to the values ν_1^2 and ν_2^2 respectively of ν^2 and

$$\mu = \frac{b_1 L^2(C_{13} + C_{14}) + b_2(C_{44} - L^2 C_{11})}{b_1(C_{33} - L^2 C_{44}) - b_2(C_{13} + C_{44})} \qquad \ldots (11)$$

The solution of the equations of equilibrium (in the absence of body forces) is obtained in terms of two stress functions ϕ_1, ϕ_2 and a function $\bar{T}$ of r, z and the expressions for the stresses and displacements can be written as:

$$\sigma_{rr} = \left(C_{11}\frac{\partial^2}{\partial r^2} + C_{12}\frac{1}{r}\cdot\frac{\partial}{\partial r}\right)(\phi_1 + \phi_2 + \bar{T}) + C_{13}\frac{\partial^2}{\partial z^2}(\lambda_1\phi_2 + \lambda_2\phi_2 + \mu\bar{T}) - b_1 T; \qquad \ldots (12)$$

$$\sigma_{\theta\theta} = \left(C_{12}\frac{\partial^2}{\partial r^2} + C_{11}\frac{1}{r}\cdot\frac{\partial}{\partial r}\right)(\phi_1 + \phi_2 + \bar{T}) + C_{13}\frac{\partial^2}{\partial z^2}(\lambda_1\phi_2 + \lambda_2\phi_2 + \mu\bar{T}) - b_1 T; \qquad \ldots (13)$$

$$\sigma_{zz} = C_{13}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r}\right)(\phi_1 + \phi_2 + \bar{T}) + C_{33}\frac{\partial^2}{\partial z^2}(\lambda_1\phi_1 + \lambda_2\phi_2 + \mu\bar{T}) - b_2 T; \qquad \ldots (14)$$

and

$$\sigma_{rz} = C_{44}\frac{\partial^2}{\partial r\,\partial z}[(1 + \lambda_1)\phi_1 + (1 + \lambda_2)\phi_2 + (1 + \mu)\bar{T}]. \qquad \ldots (15)$$

The above results have already been obtained by Bhattacharya[2].

Boundary Conditions

We assume that the faces of the crack are described by the relations $0 < r < a$, $r > b$, $z = 0\pm$. We suppose that there is no external force acting on the crack-faces and that the faces $z = 0+$, $0 < r < a$, $r > b$ are heated (or cooled) exactly in the same way as the faces $z = 0-$, $0 < r < a$, $r > b$. Then following Sneddon[7] we reduce the crack problem for the infinite medium $|z| \geq 0$ to the mixed boundary value problem for the semi-infinite medium $z \geq 0$ in which the thermal and elastic conditions on the boundary $z = 0$ are:

$$T(r, 0) = f(r); 0 \leq r < a, \qquad \ldots (16)$$

$$\frac{\partial}{\partial z} T(r, 0) = 0; \; a < r < b, \qquad \ldots (17)$$

$$T(r, 0) = 0; \; b < r < \infty, \qquad \ldots (18)$$

and

$$\sigma_{rz}(\mathrm{r}, 0) = 0; 0 \leq r < \infty \qquad \ldots (19)$$

$\sigma_{zz}(r, 0); 0 \leqslant r < a, b < r < r < \infty$, ... (20)

and

$u_z(r, 0) = 0; a < r < b$, ... (21)

where $f(r)$ is prescribed. Moreover, the temperature and the components of the stress tensor and the displacement vector tend to zero as $\sqrt{(r^2 + z^2)} \to \infty$. The boundary conditions (16) through (21) are depicted in Fig. 1.

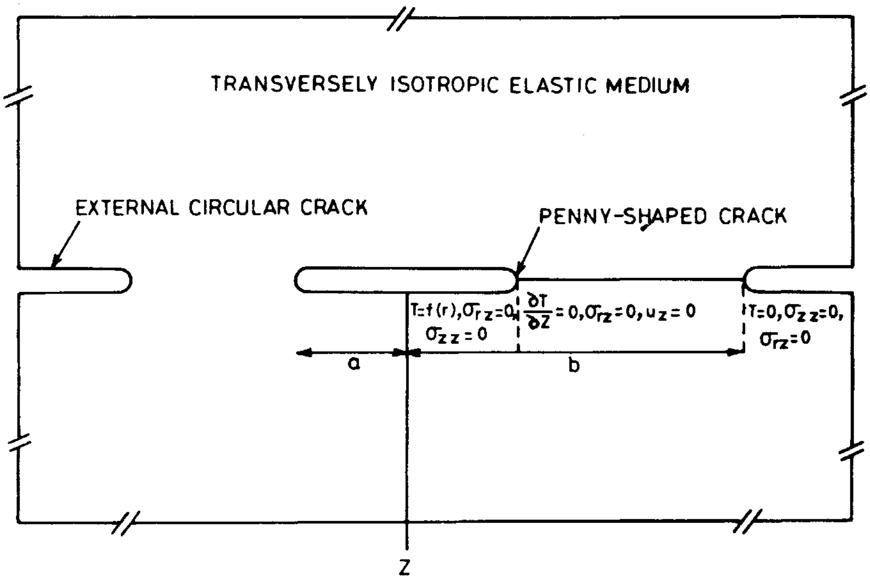


Fig 1 Mixed boundary conditions for the plane of crack $Z = 0$

Formulation of the Problem: Derivation of Triple Integral Equations

A suitable Hankel integral representation of the temperature field satisfying the Laplace's eq. (1) and vanishing at infinity is taken in the form:

$$T(r, z) = \int_0^\infty \theta(\xi) \cdot \exp\left(-\frac{\xi z}{L}\right) J_0(\xi r)\, d\xi . \qquad \ldots (22)$$

Imposing the conditions (16), (17) and (18), leads to the derivation of the triple integral equations:

$$\int_0^\infty \theta(\xi) J_0(\xi r)\, d\xi = f(r), \; 0 \leqslant r < a , \qquad \ldots (23)$$

$$\int_0^\infty \xi\, \theta(\xi) J_0(\xi r)\, d\xi = 0, \; a < r < b , \qquad \ldots (24)$$

and

$$\int_0^\infty \theta(\xi) J_0(\xi r)\, d\xi = 0, \; b < r < \infty . \qquad \ldots (25)$$

As solutions of equations (9a) let us assume:

$$\phi_1 = \int_0^\infty \xi^{-1} A_1(\xi)\, J_0(\xi r) \exp\left(-\frac{\xi z}{\nu_1}\right) d\xi \qquad \ldots (26)$$

and

$$\phi_2 = \int_0^\infty \xi^{-1} A_2(\xi)\, J_0(\xi r) \exp\left(-\frac{\xi z}{\nu_2}\right) d\xi . \qquad \ldots (27)$$

Let us suppose that:

$$\bar{T} = \int_0^\infty A(\xi)\, \theta(\xi)\, J_0(\xi r) \exp\left(-\frac{\xi z}{L}\right) d\xi \qquad \ldots (28)$$

It can be found that

$$\xi^2 A(\xi) = \frac{L^2[\alpha_1(C_{33} - L^2 C_{44}) - \alpha_2(C_{13} + C_{44})]}{(C_{44} - L^2 C_{11})(C_{33} - L^2 C_{44}) + L^2(C_{13} + C_{44})^2} = \text{constant} = P(\text{say}). \qquad \ldots (29)$$

Using the condition (19) and the result (29) we get

$$A_2(\xi) = \frac{\nu_2(1+\lambda_1)}{\nu_1(1+\lambda_2)} A_1(\xi) - \frac{\nu_2(1+\mu)}{L(1+\lambda_2)} P \xi^{-1} \theta(\xi) . \qquad \ldots (30)$$

Employing the relations (29) and (30) in (14) and (10), we get the normal components of stress and displacement on the crack plane as:

$$\sigma_{zz}(r, 0) = K_1 \int_0^\infty \xi A_1(\xi)\, J_0(\xi r)\, d\xi + K_2 \int_0^\infty \theta(\xi)\, J_0(\xi r)\, d\xi \qquad \ldots (31)$$

and

$$u_z(r, 0) = -\int_0^\infty \left[\frac{\lambda_1 - \lambda_2}{\nu_1(1+\lambda_2)} A_1(\xi) + \frac{\mu - \lambda_2}{L(1+\lambda_2)} P \xi^{-1} \theta(\xi)\right] J_0(\xi r)\, d\xi , \qquad \ldots (32)$$

where

$$K_1 = C_{13}\left\{\frac{\nu_2(1+\mu)}{L(1+\lambda_2)} - 1\right\} + C_{33}\left\{\frac{\lambda_1}{\nu_1^2} - \frac{\lambda_2(1+\lambda_2)}{\nu_1 \nu_2(1+\lambda_2)}\right\}$$

and ... (33)

$$K_2 = C_{13}P\left\{\frac{\nu_2(1+\mu)}{L(1+\lambda_2)} - 1\right\} - C_{33}P\left\{\frac{\lambda_2(1+\mu)}{\nu_2 L(1+\lambda_2)} - \frac{\mu}{L^2}\right\} - b_2$$

Conditions (20) and (21) are satisfied if $A_1(\xi)$ is solution of the system of triple integral equations:

$$\int_0^\infty A_1(\xi)\, J_0(\xi r)\, \mathrm{d}\xi = -\frac{K_2}{K_1}\int_0^\infty \theta(\xi)\, J_0(\xi r)\, \mathrm{d}\xi; \qquad 0 \leqslant r < a, \quad b < r < \infty \qquad \ldots (34)$$

and

$$\int_0^\infty A_1(\xi)\, J_0(\xi r)\, \mathrm{d}\xi = \frac{\nu_1(\lambda_2 - \mu)\, P}{L(\lambda_1 - \lambda_2)}\int_0^\infty \xi^{-1}\theta(\xi) J_0(\xi r)\, \mathrm{d}\xi; \qquad a < r < b. \qquad \ldots (35)$$

Let $\xi A_1(\xi) + \frac{\nu_1(\lambda_2 - \mu)\, P}{L(\lambda_1 - \lambda_2)}\theta(\xi) = B(\xi)$

so that,

$$A_1(\xi) = \xi^{-1}B(\xi) - \frac{\nu_1(\lambda_2 - \mu)\, P}{L(\lambda_1 - \lambda_2)}\xi^{-1}\theta(\xi) \qquad \ldots (36)$$

Then the triple integral eqs (34) and (35) become:

$$\int_0^\infty B(\xi)\, J_0(\xi r)\, \mathrm{d}\xi = -Qf(r); \qquad 0 \leqslant r < a,$$

$$\int_0^\infty \xi^{-1}B(\xi)\, J_0(\xi r)\, \mathrm{d}\xi = 0; \qquad a < r < b \qquad \ldots (37)$$

and

$$\int_0^\infty B(\xi)\, J_0(\xi r)\, \mathrm{d}\xi = 0; \qquad b < r < \infty;$$

where

$$Q = \frac{K_2}{K_1} - \frac{\nu_1(\lambda_2 - \mu)\, P}{L(\lambda_1 - \lambda_2)}.$$

Solution of the Heat Conduction Problem

The heat conduction problem defined by the system of triple integral eq. (23), (24) and (25) have been reduced to the solution of a pair of Fredholm integral equation by Bhowmick, Bandyopadhyay and Das[4]. Approximate solutions of these equations for small values of the ratio k between the radii of the internal and external circular cracks have been derived and finally the temperature distribution over the crack-plane $z=0$ has been found to be:

$$
\begin{aligned}
T(r,0)=\frac{f_0}{8\pi}\Bigg[\Bigg\{&16\gamma_0^{(1)}+4\gamma_2^{(1)}\left(2+\frac{r^2}{a^2}\right)+\gamma_4^{(1)}\left(6+\frac{2r^2}{a^2}+\frac{9r^4}{4a^4}\right)\\
&+\gamma_6^{(1)}\left(5+\frac{3r^3}{2a^2}+\frac{9r^4}{8a^4}+\frac{25r^6}{16a^6}\right)+\gamma_8^{(1)}\left(\frac{35}{8}+\frac{5r^2}{4a^2}+\frac{27r^4}{32a^4}+\frac{25r^6}{32a^6}\right.\\
&\left.+\frac{1225r^8}{1024a^8}\right)\Bigg\}\sin^{-1}\frac{a}{r}-\frac{\sqrt{(r^2-a^2)}}{a}\left\{4\gamma_2^{(1)}+\gamma_4^{(1)}\left(\frac{7}{2}+\frac{9r^2}{4a^2}\right)\right.\\
&\left.+\gamma_6^{(1)}\left(\frac{37}{12}+\frac{13r^2}{6a^2}+\frac{25r^4}{16a^4}\right)+\gamma_8^{(1)}\left(\frac{533}{192}+\frac{769r^2}{384a^2}+\frac{2425r^2}{1536a^4}+\frac{1225r^6}{1024a^6}\right)\right\}\\
&+\left\{16\gamma_0^{(2)}+4\gamma_2^{(2)}\left(2+\frac{b^2}{r^2}\right)+\gamma_4^{(2)}\left(6+\frac{2b^2}{r^2}+\frac{9b^4}{4r^4}\right)+\gamma_6^{(2)}\left(5+\frac{3b^2}{2r^2}\right.\right.\\
&\left.\left.+\frac{9b^4}{8r^4}+\frac{25b^6}{16r^6}\right)+\gamma_8^{(2)}\left(\frac{35}{8}+\frac{5b^2}{4r^2}+\frac{27b^4}{32r^4}+\frac{25b^6}{32r^6}+\frac{1225b^8}{1024r^8}\right)\right\}\frac{b}{r}\sin^{-1}\frac{r}{b}\\
&-\frac{b\sqrt{(b^2-r^2)}}{r^2}\left\{4\gamma_2^{(2)}+\gamma_4^{(2)}\left(\frac{7}{2}+\frac{9b^2}{4r^2}\right)+\gamma_6^{(2)}\left(\frac{37}{12}+\frac{13b^2}{6r^2}+\frac{25b^4}{16r^4}\right)\right.\\
&\left.+\gamma_8^{(2)}\left(\frac{533}{192}+\frac{769b^2}{384r^2}+\frac{2425b^4}{1536r^4}+\frac{1225b^6}{1024r^6}\right)\right\}\Bigg]; \qquad a<r<b, \qquad \ldots(38)
\end{aligned}
$$

where $f(r)=f_0$ (constant) and $\gamma_i^{(1)}$, $\gamma_i^{(2)}$, $i=0, 2, 4, 6, 8$ are constant terms containing different powers of $k\,(a/b)$ as follows:

$$\gamma_0^{(1)}=1+0.40528\,k+0.16425\,k^2-0.02352\,k^3+0.02697\,k^4-0.01211\,k^5$$
$$+0.01139\,k^6-0.00679\,k^7+0.0063\,k^8-0.0712\,k^9$$

$$\gamma_2^{(1)}=0.207019\,k^3+0.1095\,k^4-0.00966\,k^5+0.02038\,k^6-0.0076\,k^7$$
$$+0.00919\,k^8-0.0049\,k^9.$$

$$\gamma_4^{(1)}=0.21615\,k^5+0.0876\,k^6-0.00568\,k^7+0.01716\,k^8-0.00574\,k^9.$$

$$\gamma_6^{(1)}=0.18527\,k^7+0.07509\,k^8-0.00388\,k^9$$

$$\gamma_8^{(1)}=0.16469\,k^9$$

and

$$\gamma_0^{(2)}=-0.63662\,k-0.325801\,k^2+0.10765\,k^3-0.01369\,k^4+0.01365\,k^5$$

$$-0.00352\,k^6+0.00382\,k^7-0.00101\,k^8+0.00125\,k^9$$

$$\gamma_2^{(2)}=-0.42441\,k^3-0.17201\,k^4+0.10005\,k^5-0.01296\,k^6+0.01563\,k^7$$
$$-0.00443\,k^8+0.00543\,k^9$$

$$\gamma_4^{(2)}=-0.33953\,k^5-0.13760\,k^6+0.08974\,k^7-0.01168\,k^8+0.01521\,k^9$$

$$\gamma_6^{(2)}=-0.29103\,k^7-0.11795\,k^8+0.08154\,k^9$$

$$\gamma_8^{(1)}=-0.25869\,k^9$$

Considering $a=2$, $b=5$ so that $k=0.4$, we find the values of the constants $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ as follows:

$\gamma_0^{(1)}=1.1875$, $\gamma_2^{(1)}=0.0201$, $\gamma_4^{(1)}=0.0026$, $\gamma_6^{(1)}=0.0004$, $\gamma_8^{(1)}=0$,

$\gamma_6^{(2)}=-0.2892$, $\gamma_2^{(2)}=-0.0306$, $\gamma_4^{(2)}=-0.0039$, $\gamma_6^{(2)}=-0.0005$,

$\gamma_8^{(2)}=-0.0001$.

Solution of the Thermoelastic Problem

Following the method described by Selvadurai and Singh[1] we assume that

$$\int_0^\infty \xi^{-1}B(\xi)\,J_0(\xi r)\,\mathrm{d}\xi = f_1(r)\,;\ 0<r<a \qquad \ldots(39)$$

$$f_2(r);\ b<r<\infty \qquad \ldots(40)$$

which gives

$$f_1(r)=p_1(r)+\frac{2}{\pi}\int_b^\infty \frac{tf_2(t)\cdot(a^2-r^2)^{1/2}\,\mathrm{d}t}{(t^2-a^2)^{1/2}(t^2-r^2)}\,;0<r<a \qquad \ldots(41)$$

and

$$f_2(r)=\frac{2}{\pi}\int_0^a \frac{tf_1(t)\cdot(r^2-b^2)^{1/2}\,\mathrm{d}t}{(b^2-t^2)^{1/2}(r^2-t^2)}\,;\ b<r<\infty\,, \qquad \ldots(42)$$

where

$$P_1(r)=\frac{-2Q}{\pi}\int_r^a\left\{\int_0^s \frac{tf(t)\,\mathrm{d}t}{\sqrt{(s^2-t^2)}}\right\}\frac{\mathrm{d}s}{\sqrt{(s^2-r^2)}} \qquad \ldots(43)$$

We also assume that

$$F_1(r)=\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{tf_1(t)\,\mathrm{d}t}{\sqrt{(t^2-r^2)}}\,;\quad 0<r<a \qquad \ldots(44)$$

and

$$F_2(r)=\frac{d}{dr}\int_b^r \frac{tf_2(t)\,dt}{\sqrt{(r^2-t^2)}}\,;\ b<r<\infty\ . \qquad \ldots(45)$$

The triple integral eq. (37) are then reduced to a pair of simultaneous Fredholm Integeral equations:

$$F_1(s)+\frac{2s}{\pi}\int_b^\infty \frac{F_2(u)\,du}{(u^2-s^2)}=\frac{d}{ds}\int_s^a \frac{rp_1(r)\,dr}{\sqrt{(r^2-s^2)}}\,;\ 0<s<a \qquad \ldots(46)$$

and

$$F_2(s)+\frac{2}{\pi}\int_0^a \frac{uF_1(u)\,du}{(u^2-s^2)}=0;\ b<s<\infty\ . \qquad \ldots(47)$$

Solution for a Particular Type of Temperature Distribution

(A) *Let us assume,*

$$f(r)=f_0\ (\text{constant});\ 0<r\leqslant a\ . \qquad \ldots(48)$$

Then the eqs (46) and (47) reduce to:

$$F_1(as_1)+\frac{2ks_1}{\pi}\int_1^\infty \frac{F_2(bu_1)\,du_1}{(u_1^2-s_1^2k^2)}=Qf_0\,as_1\ ;\ 0<s_1<1 \qquad \ldots(49)$$

and

$$F_2(bs_2)+\frac{2k^2}{\pi}\int_0^1 \frac{u_1F_1(au_1)}{(s_2^2-k^2u_1^2)}du_1=0\ ;\ 1<s_2<\infty\ , \qquad \ldots(50)$$

where $k=\frac{a}{b}$; $s_1=\frac{s}{a}$ and $s_2=\frac{s}{b}$ and $u_1=\frac{u}{b}$ in (49); $u_1=\frac{u}{a}$ in (50). Assuming $k\ll 1$ so that terms containing powers of k higher than 8 can be neglected, we get from (49) and (50):

$$F_1(as_1)=Qf_0\,a\left[s_1+\frac{4s_1}{9\pi^2}k^3+\frac{4k^5}{5\pi^2}\left(\frac{s_1}{5}+\frac{s_1^3}{3}\right)+\frac{16k^6s_1}{81\pi^4}+\frac{4s_1k^7}{\pi^2}\left(\frac{1}{49}+\frac{s_1^2}{35}+\frac{s_1^4}{21}\right)\right.$$
$$\left.+\frac{16k^8}{\pi^4}\left(\frac{s_1}{75}+\frac{s_1^3}{135}\right)+0(k^9)\right];\ 0<s_1<1 \qquad \ldots(51)$$

$$F_2(bs_2)=-Qf_0\,a\left[\frac{2k^2}{3\pi s_2^2}+\frac{2k^4}{5\pi s_2^4}+\frac{8k^5}{27\pi^3s_2^2}+\frac{2k^6}{7\pi s_2^6}+\frac{16k^7}{\pi^3s_2^2}\left(\frac{1}{75}+\frac{1}{90s_2^2}\right)\right.$$

$$+\frac{2k^8}{9\pi s_2^2}\left(\frac{16}{27\pi^4}+\frac{1}{s_2^6}\right)+0(k^9)\Bigg] ; \; 1<s_2<a . \qquad \ldots(52)$$

The function $B(\xi)$ can be expressed in the form:

$$B(\xi)=\frac{2}{\pi}\,\xi^2\left[-\int_0^a F_1(s)\,\mathrm{d}s\int_0^s \frac{rJ_0(\xi r)\,\mathrm{d}r}{\sqrt{(s^2-r^2)}}+\int_b^\infty F_2(s)\,\mathrm{d}s\int_s^\infty \frac{rJ_0(\xi r)\,\mathrm{d}r}{(r^2-s^2)^{1/2}}\right]. \qquad \ldots(53)$$

we have from (31)

$$\sigma_{zz}(r,0)=-\int_0^\infty B(\xi)\,J_0(\xi r)\,\mathrm{d}\xi \qquad \ldots(54)$$

Using (53) and (54) it can be shown that:

$$\sigma_{zz}(r,0)=\frac{2}{\pi}\left[-\frac{F_1(a)}{\sqrt{(r^2-a^2)}}+\int_0^a\frac{F_1'(s)\,\mathrm{d}s}{\sqrt{(r^2-s^2)}}+\frac{F_2(b)}{(b^2-r^2)^{1/2}}+\int_b^\infty\frac{F_2'(s)\,\mathrm{d}s}{\sqrt{(s^2-r^2)}}\right];\; a<r<b \qquad \ldots(55)$$

where $F_1'(s)$ and $F_2'(s)$ denote derivatives of the respective functions.

The expression for the normal component of the displacement u_2 on the plane of crack can be written in the form:

$$u_z(r,0)=\frac{2(\lambda_1-\lambda_2)Qf_0}{\pi\nu_1(1+\lambda_2)(a^2-r^2)^{(-1/2)}}\left[\left(1+\frac{4k^3}{9\pi^2}+\frac{56k^5}{225\pi^2}+\frac{16k^6}{81\pi^4}+\frac{116k^7}{735\pi^2}+\frac{512k^8}{2025\pi^4}\right)\right.$$
$$\left.+\left(\frac{r}{a}\right)^2\left(\frac{8k^5}{45\pi^2}+\frac{8k^7}{63\pi^2}+\frac{32k^8}{405\pi^4}\right)+\frac{32k^7}{315\pi^2}\left(\frac{r}{a}\right)^4\right];\;0\leqslant r<a \qquad \ldots(56)$$

$$=0;\quad a<r<b$$

$$=\frac{4Qf_0k^3(\lambda_1-\lambda_2)}{3\pi\nu_1(1+\lambda_2)(r^2-b^2)^{1/2}}\left(\frac{b}{r}\right)^2\left[1+\frac{k^2}{5}+\frac{4k^3}{9\pi^2}+\frac{3k^4}{35}+\frac{184k^5}{450\pi^2}+\frac{k^6}{21}+\frac{16k^6}{81\pi^4}\right)$$
$$+\left(\frac{b}{r}\right)^2\left(\frac{2k^2}{5}+\frac{4k^4}{35}+\frac{8k^5}{45}+\frac{2k^6}{35}\right)+\left(\frac{b}{r}\right)^4\left(\frac{8k^4}{35}+\frac{8k^6}{105}\right)+\frac{16k^8}{105}\left(\frac{b}{r}\right)^6\Bigg];$$
$$b<r<\infty \qquad \ldots(57)$$

(B) The Stress Intensity Factors

The stress intensity factors at the crack-boundaries $r=a$ and $r=b$ are defined by:

$$k_a=\lim_{r\to a^+}[2(r-a)]^{1/2}\cdot\sigma_{zz}(r,0) \qquad \ldots(58)$$

and

$$k_b = \lim_{r \to b^-} [2(b-r)]^{1/2} \cdot \sigma_{zz}(r, 0) \qquad \ldots (59)$$

respectively. From (55), (58) and (59) it follows that:

$$k_a = -\frac{2}{\pi} Q f_0 \sqrt{a}\left[1 + \frac{4k^3}{9\pi^2} + \frac{32k^5}{75\pi^2} + \frac{16k^6}{81\pi^4} + 0(k^7)\right] \qquad \ldots (60)$$

and

$$k_b = -\frac{2}{\pi} Q f_0 \sqrt{(ak)}\left[\frac{2k^2}{3\pi} + \frac{2k^4}{5\pi} + \frac{8k^5}{27\pi^3} + \frac{2k^6}{7\pi} + 0(k^7)\right] \qquad \ldots (61)$$

(*C*) *Work Done*

The work done in opening the penny-shaped crack is given by

$$W = 2\pi Q f_0 \int_0^a r u_z(r, 0)\, dr. \qquad \ldots (62)$$

From (56) and (62) we obtain:

$$W = \frac{4Q^2 a^3 f_0(\lambda_1 - \lambda_2)}{3\nu_1(1+\lambda_2)}\left[1 + \frac{4k^3}{9\pi^2} + \frac{8k^5}{25\pi^2} + \frac{16k^6}{81\pi^4} + \frac{284k^7}{1225\pi^2} + \frac{64k^8}{225\pi^4} + 0(k^9)\right]. \qquad \ldots (63)$$

Numerical Results

Here we consider the case of magnesium (Mg) for which the roots of equation (10) are real. The elastic constants are (Hearmon[3]):

$C_{11} = 0.565 \times 10^{12}$ dynes/cm^2

$C_{12} = 0.232 \times 10^{12}$ dynes/cm^2

$C_{13} = 0.181 \times 10^{12}$ dynes/cm^2

$C_{33} = 0.587 \times 10^{12}$ dynes/cm^2

$C_{44} = 0.168 \times 10^{12}$ dynes/cm^2

The coefficients of linear thermal expansion along and perpendicular to the z-axis are:

$\alpha_1 = 27.7 \times 10^{-6}$ cm/°C

Fig 2

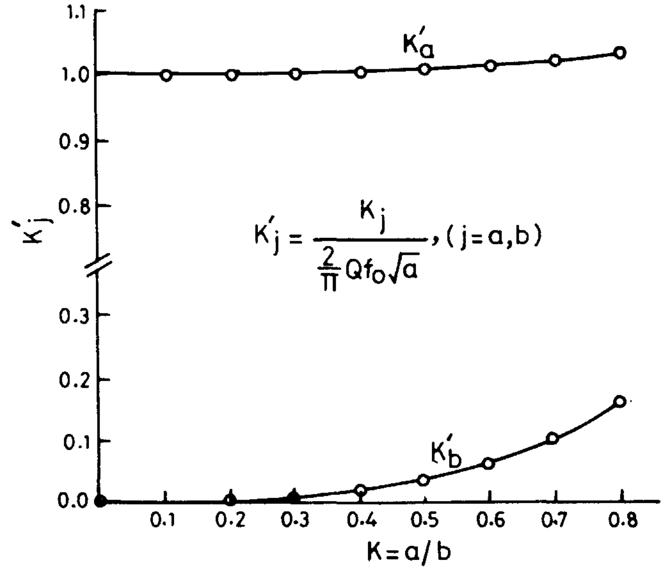
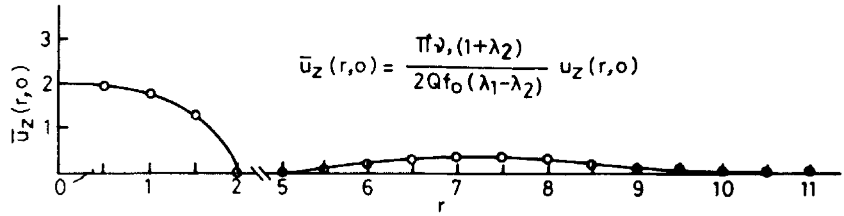


Fig 3



Figs 2&3 Variations of stress intensity factors and displacement for different values of *k*

and

$\alpha_2 = 26.6 \times 10^{-6}$ cm/°C.

$\nu_1^2 = 1.985$ and $\nu_2^2 = 0.523$

$b_1 = 26.89 \times 10^6$;

$b_2 = 25.64 \times 10^6$;

$\lambda_1 = 2.732$;

$\lambda_2 = 0.365$;

Taking $L = 1$ we get the following:

$\mu = -0.343$;

$P = -52.053 \times 10^{-6}$

$K_1 = 0.115 \times 10^{12}$

$K_2 = -1.595 \times 10^6$

$Q = 16.724 \times 10^{-6}$.

With these values of the constants we calculate the values of stress intensity factors k_a and k_b and displacement for different values of k.

The variations of k_a and k_b and of displacement $u_z(r, 0)$ are shown in Figs 2 and 3.

Conclusions

The stress intensity factors computed in (60) and (61) indicate that when the faces of the penny-shaped crack are exposed to a constant temperature f_0 and those of the external circular crack are subject to constant temperature zero the stress intensity factor k_a at the boundary of the penny-shaped crack ($r=a$) is greater than the stress intensity factor k_b at the boundary of the external crack ($r=b$). Also it is evident that as $b \to \infty$ (i.e., $k \to 0$) the result (61) yields the classical result for the stress intensity factor at the boundary of a heated or cooled penny-shaped crack in an infinite elastic solid[1].

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