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AIRFOIL DESIGN METHOD BASED ON THE QUASISOLUTIONS OF INVERSE BOUNDARY-VALUE PROBLEM

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A mathematically justified method of solving the inverse boundary-value problems of aerohydrodynamics is presented. In these problems an airfoil contour is sought by velocity distribution given on it as a function of the arc length. The method is based on the idea quasisolution from the theory of incorrect problems and consists in optimal modification of the initial data in order to obtain a unique stable solution from the given class of airfoil contours. The method is applied to the design of isolated airfoils in a plane steady potential flow of an ideal incompressible fluid. Further, it is generalized to the cases of subsonic gas flow, viscous incompressible fluid and viscous gas. Results of numerical calculations demonstrating the method abilities are presented.

Key Words : Inverse Boundary-Value Problems of Aerohydrodynamics; Quasisolution; Viscous Incompressible Fluid; Subsonic Gas Flow; Optimal Shape Design

Problem Statement for the Model of An Ideal Incompressible Fluid

A distinctive feature of the inverse boundary-value problems[1,27] is their constructive character, because they deal not with study of properties of a known object but with its design by given characteristics. The origin of the general mathematical theory of inverse boundary-value problems was stimulated by the problem of constructing an airfoil contour flowed by a steady potential stream of an ideal incompressible fluid with a velocity distribution given as a function of the arc length of the contour[17]. This problem will be called the principal inverse boundary-value problem and we shall present its complete statement.

In the complex z-plane (Fig 1*a*) a steady potential stream of ideal incompressible fluid flows past an unknown contour; the interior angle of the flow region at the trailing edge B is equal to $\varepsilon\pi$, $1 \leq \varepsilon \leq 2$. The velocity vector at infinity is parallel to the x-axis and its value v_∞ is given. The arc length s, $0 \leq s \leq L$ of L_z varies from $s = 0$ at B to $s = L$ at the same point as shown in Fig 1*a*. The velocity distribution is given along L_z (Fig 1*b*)

$$v = v(s), \qquad 0 \leq s \leq L, \qquad \ldots(1)$$

where a piecewise smooth function $v(s)$ vanishes at the stagnation point A ($s = s_*$) and is continuously differentiable in the vicinity of s_* (the last condition implies the smoothness of the contour at A). The sign of $v(s)$ depends on the direction of tracing of L_z and therefore $v(s) < 0$ for $0 < s < s_*$ $v(s) > 0$

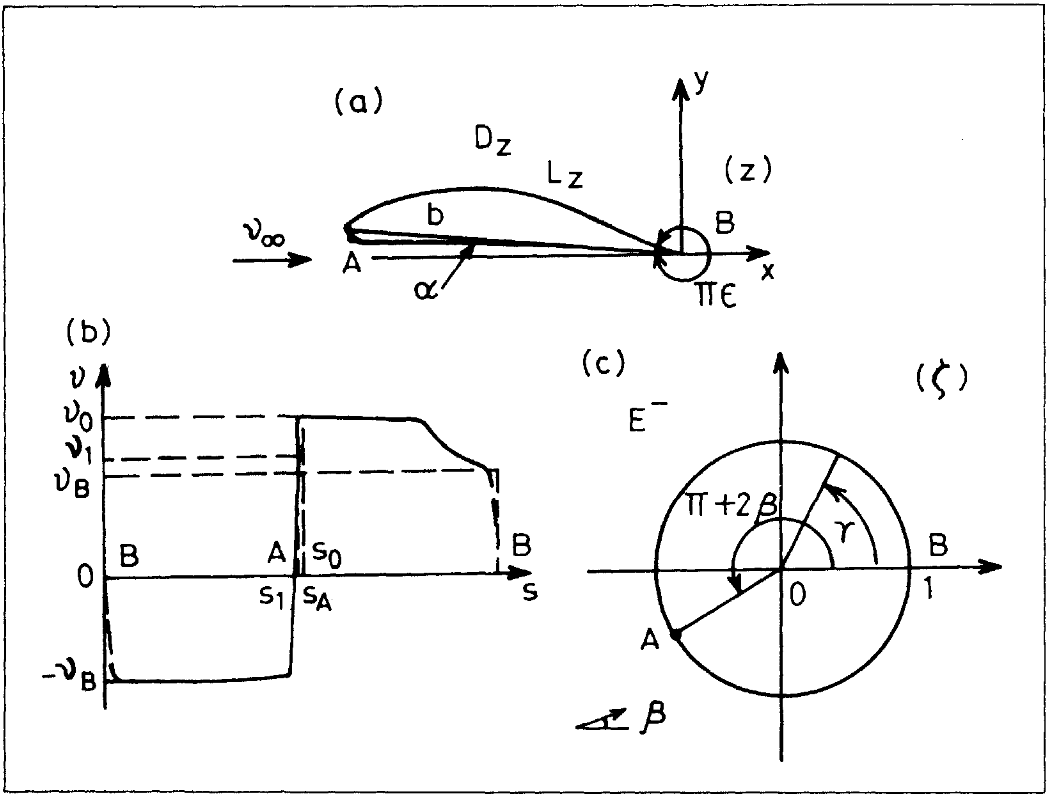

Fig 1a, b, c

for $s_* < s < L$, and $v(L) = - v(0) = v_* > 0$ for $\varepsilon = 2$ (infinitely thin trailing edge) or $v(0) = v(L) = 0$ for $1 \le \varepsilon < 2$ (the broken lines in neighbourhoods of $s = 0$ and $s = L$ in Fig 1*b*). It is necessary to determine the shape of the airfoil, the angle of attack α and the lift coefficient C_L.

The lift of the airfoil is *a priori* known when v is given as function of s

$$R = \rho v_\infty \Gamma, \quad \Gamma = \int_0^L v(s) ds, \qquad \ldots(2)$$

where Γ is the circulation and ρ is the density. Taking into account that as a rule the chord length b is somewhat smaller than half perimeter ($b/L \approx 0.48 \ldots 0.49$), one can estimate the lift coefficient as follows

$$C_L = 2\Gamma/(bv_\infty) \approx 4.2\Gamma/(Lv_\infty)$$

Further, knowing $v(s)$ one can specify the structure of L_z in a neighbourhood of stagnation points what is important for a numerical solution. For example, the radius of curvature of the contour a neighbourhood of A can be found by the following formula

$$r \approx 2v_\infty / v(s_*)$$

The value of ε can also be determined from the behaviour of $v(s)$ in neighbourhood of B, if we take into account that $\varepsilon = 2$ for $v_* \neq 0$ and $v(s) \sim -C|\bar{s}|^{2/\varepsilon}/\bar{s}$ for $v_* = 0$ (where $\bar{s} = s$ in neighbourhood of $s = 0$ and $\bar{s} = s - L$ in the neighbourhood of $s = L$, C = const > 0).

Assuming that the boundary layer on the airfoil flowed by a viscous fluid is sufficiently thin, we may prescribe a distribution $v(s)$ allowing us to calculate boundary layer characteristics before the solution of the inverse problem and to evaluate the drag coefficient C_D by using the Squire-Young formula

$$C_D \approx 2(v_* / v_\infty)^{3.2} \delta_0^{**} / b$$

where δ_0^{**} is a boundary layer momentum thickness at the trailing edge. Consequently, the aerodynamic ration can be approximately calculated *a priori*

$$K = C_L / C_D \approx \Gamma (v_\infty / v_*)^{3.2} / v_\infty \delta_0^{**})$$

Finally, parameterization by s allows to prescribe the velocity distribution which guarantees the flow without stalling, because the well-known criteria for separation of the boundary layer are expressed in terms of $v(s)$ We shall dwell on this question in detail.

Choice of Hydrodynamically Expedient Velocity Distributions

In a solution of the principal inverse boundary-value problem of aerohydrodynamics, the initial distribution of $v(s)$ must be chosen with regard for requirements of the employed model of flow and for restrictions imposed in the aerohydrodynamic design of airfoils. Therefore, the initial distribution $v(s)$ must be chosen so that there are neither stalling nor supersonic zones on the whole contour, and shape of L_z is physically real.

For the flow to be subsonic it is sufficient to impose a restriction on the maximum value of velocity. The same restriction appears also in the design of contours flowed by a fluid. There it follows from the absence of cavitation.

To prescribe the velocity distribution which guarantees the flow without stalling we may use the well-known criteria for separation of the boundary layer[22]. In the special case of an isolated profile in an incompressible fluid, assuming that the boundary layer is purely turbulent, from the condition of separation a simple approximate formula can be obtained for a velocity distribution on the diffusor section:

$$v(s)=v_0[1+K_0(s-s_0)]^{-m},\quad s_0\leq s\leq L, \qquad \ldots(3)$$

$$K_0=0.025(\delta_0^{**})^{-1},\ \delta_0^{**}=\delta^{**}(s_0),\ m=0.20\ldots0.25,$$

where $\delta^{**}(s)$ is a boundary layer momentum thickness, s_0 is the initial coordinate of the diffusor section on the airfoil. This formula was originally indicated by Stepanov. As a reasonable condition for the choice of $v(s)$ on the diffusor section of the airfoil we can assume that a specially constructed form parameter[16] for turbulent boundary layer calculation is constant. This condition allows to find the velocity distribution on the deceleration zone of airfoil as an exact solution of integro-differential equation.

Suppose that there are the acceleration zone $[s_*, s_0]$ and the deceleration zone $[s_0, L]$ on the upper surface $[s_*, L]$ of the airfoil (Fig 1*b*). We set

$$f(s)=av'(s)[v(s)]^{-b}\left[\int_{s_0}^{s}[v(\tau)]^{b-1}d\tau+C_0\right],\quad s_0\leq s\leq L, \qquad \ldots(4)$$

where $v(s)$ is supposed to be smooth in the interval $[s_0, L]$, a, b and C_0 are fixed constants. According to Loizansky[16], the criteria for unseparation of the boundary layer is the inequality $f(s)\geq f_0$, with $a=a_1=0.45$, $b=b_1=5.35$, $f_0=f_{01}=-0.0681$ for a laminar flow and with $a=a_2=1.17$, $b=b_2=4.75$, $f_0=f_{02}=-6$ for a turbulent flow. In the limit case of $f(s)=f_0$ from eq. (4) we obtain an integro-differential equation whose solution is the function eq. (3), where

$$m=f_0[a-f_0(b-1)]^{-1},\ K_0=v_0^{b-1}[a-f_0(b-1)](aC_0)^{-1},\ v_0=v(s_0).$$

The constant C_0 is determined from the continuity condition for the function $\delta^{**}(s)$ at s_0

$$\delta^{**}(s_0-0)=\delta^{**}(s_0+0)=\delta_0^{**} \qquad \ldots(5)$$

We note the form of C_0 under various assumptions concerning the character of the boundary layer.

If on the whole section $s_* \leq s \leq L$ the entire boundary layer is either laminar or turbulent, then from (5) it follows that

$$C_0 = \int_{s_*}^{s_0} [v(\tau)]^{b-1}\, d\gamma, \qquad \ldots(6)$$

where $b = b_1$ for the laminar flow and $b = b_2$ for the turbulent one.

If the laminar boundary layer becomes turbulent at some point s_1, then continuity condition for δ^{**} at s_1 yields

$$C_0 = 82.18 \nu^{0.41} v_1^{0.8} \left[\int_{s_*}^{s_1} [v(\tau)]^{4.35}\, d\tau \right]^{0.58} + \int_{s_1}^{s_0} [v(\tau)]^{3.75}\, d\tau, \qquad \ldots(7)$$

where $v_1 = v(s_1)$, ν is the kinematic viscosity. Thus, the values of K_0 and m in (3) can be explicitly found by using formulae (6) or (7) depending on the supposed structure of the boundary layer. We note that the velocity distribution in the form (3) was also obtained by Wortmann[31] from the constancy condition of $H=\delta^*/\delta^{**}$ in the deceleration zone of the airfoil, δ^* being the thickness of the boundary layer displacement. Narramore[18] used mini-computers to realize the method for determining the separation zone of the distribution (3) with arbitrary K_0 and m.

Stratford[25] constructed more complicated form of velocity distribution in an unseparated flow. It consists of two parts, the first being obtained by splitting the boundary layer onto sublayers and by analyzing them. The second part has the form (3), where the values of K_0 and m correspond to the case of $H=2$ for Wortmann's velocity distribution. This distribution was used by Liebeck[14] in the aerodynamic design of high-lift airfoil.

A Scheme of the Solution of the Principal Inverse Boundary-Value Problem of Aerohydrodynamics

The most complete early results on the solution of the inverse boundary-value problems of aerohydrodynamics were obtained by Eppler[9], Lighthill[15], Mangler[17], Stepanov[22], Tumashev[26]. Results obtained before 1965 were reviewed by Tumashev & Nuzhin[27], whereas the review of works up to 1980 is presented by Liebeck[14] and Aksent'ev *et al.*[1] Elizarov, Il'inskiy & Potashev[8] offered a classification of studies of inverse boundary-value problems of aerohydrodynamics according to the methods employed.

We show briefly a scheme of the solution of the inverse boundary-value problem of aerohydrodynamics by a conformal mapping. Under assumptions from the statement of the problem there exists a complex flow potential $w(z) = \varphi(x, y) + i\psi(x, y)$, where $z = x + iy$, φ is velocity potential, ψ is stream function and on L_z we have

$$\varphi(s)=\int_{S_*}^{s} v(\tau)d\tau, \quad 0\le s \le L \qquad \ldots(8)$$

Eq. (8) and the condition $\psi = 0$ determine the equation of projection of the boundary of the region, corresponding to the flow region and lying on an infinitely valent Riemann surface over the w-plane. We map it conformally on the exterior of a unit disk in the auxiliary plane (Fig. 1 (*c*)) by the function $w = w(\zeta)$, $w(\infty) = \infty$, $w(1) = \varphi(L) = \varphi_1$

$$w(\zeta)=U_0\,[\zeta e^{-i\alpha_0}+\zeta^{-1}e^{i\alpha_0}-2\cos\alpha_0+2i\sin\alpha_0\ln\zeta]+\varphi_1 \qquad \ldots(9)$$

where $U_0 = |dw/d\zeta|_\infty = \Gamma/(4\pi\sin\alpha_0)$, $\alpha_0 = -\arg(dw/d\zeta)\,|_{z=\infty}$ $\Gamma = \varphi(L) - \varphi(0)$ is the velocity circulation. From eq. (9) at $\zeta = e^{i\gamma}$ we have

$$\varphi(\gamma)=2U_0\,[\cos(\gamma-\alpha_0)-\cos\alpha_0]-\Gamma_\gamma/(2\pi)+\varphi_1, \qquad \ldots(10)$$

where α_0 is determined from the equation $2\Gamma(\alpha_0 + \cot\alpha_0) = \pi(2\varphi_1 - \Gamma)$.

Consider the function

$$\tilde{\chi}(\zeta)=\chi(\zeta)-\chi_0(\zeta) \qquad \ldots(11)$$

analytic in the region $E^- = \{\zeta : |\zeta| > 1\}$, where

$$\chi(\zeta) = \ln(dw/dz) = \ln v - i\theta \qquad \ldots(12)$$

and

$$\chi_0(\zeta) = \ln(1 - \zeta^{-1}e^{i\gamma^*}) + (2 - \varepsilon)\ln(1 - \zeta^{-1}) \qquad \ldots(13)$$

The function $\chi_0(\zeta)$ singles out the logarithmic singularities of $\zeta(\chi)$ at $\zeta = 1$ and $\zeta = \zeta_* = e^{i\gamma_*}$ (ζ_* is the preimage of A, $\gamma_* = \pi + 2\alpha_0$). Comparing $\varphi(s)$ and $\varphi(\gamma)$ in (8) and (10), we find the function $s = s(\gamma)$ on the circle $\zeta = e^{i\gamma}$, $0 \le \gamma \le 2\pi$, and in view of (1) we find Holder function

$$P(\gamma)\equiv \mathcal{R}\tilde{\chi}(e^{i\gamma})=\ln|v\,[s(\gamma)]\,|-\ln\left[2\left|\sin\frac{\gamma-\gamma_*}{2}\right|\right]-(2-\varepsilon)\ln\left(2\sin\frac{\gamma}{2}\right),$$

where $\mathcal{R}$ denotes real part of the complex function. Now we use the Schwartz's operator to determine the function $\tilde{\chi}(\zeta)$:

$$\tilde{\chi}(\zeta)=\frac{1}{2\pi}\int_0^{2\pi} P(\tau)\frac{\zeta+e^{i\tau}}{\zeta-e^{i\tau}}d\tau$$

Taking into account eqs (9), (11)-(13) we can write the integral representation of $z = z(\zeta)$ which maps E^- conformally onto the flow region with the correspondence $z(\infty) = \infty$, $z(1) = 0$

$$z(\zeta)=U_0 e^{i\alpha_0}\int_1^{\zeta}(1-1/\zeta)^{\varepsilon-1}\exp[-\tilde{\chi}(\zeta)]d\zeta \qquad \ldots(14)$$

Setting $\zeta = e^{i\gamma}$, we obtain a parametrization of the desired contour L_z. Knowing L_z one can easily find the chord length b, the angle of attack α, the lift coefficient C_L and other characteristics.

However, the constructed solution does not guarantee that the desired contour is closed and that the calculated free-stream velocity equals to the given one. Conditions ensuring these properties are called conditions for solvability. They have the following form

$$\int_0^{2\pi} P(\tau)e^{i\tau}d\tau= B_1 + iB_2, \qquad \ldots(15)$$

and

$$\int_0^{2\pi} P(\tau)d\tau = B_3, \qquad \ldots(16)$$

where $B_1 = \pi(1 - \varepsilon)$, $B_2 = 0$, $B_3 = 2\pi\ln v_\infty$. We note that if v_∞ is not given, the condition can be used to find it.

The relations (15) and (16) mean three Fourier coefficients for function $P(\gamma)$ are fixed. Mangler[17] and others[2] (See for example 23) proposed that if these conditions are not satisfied, then the values of corresponding coefficients should be replaced by the necessary ones, thus making the problem "correct". However, this approach does not require that the variation of the initial velocity distribution should be minimal. Strand[23] numerically realized a method to satisfy eqs. (15), (16) by modification of finite number of parameters, introduced into expression of $v(s)$ in order to keep closer to the initial data. In the special case and without proof of uniqueness and existence one of the possible general solutions was constructed by Strand[24] for the problem in question.

The solvability conditions are still not sufficient for the constructed solution to be physically realizable, since, generally speaking, the designed contour may be self-intersected. Well-known statements of inverse boundary-value problems of aerohydrodynamics, as rule, do not require univalence of the solution and their design methods may result in contours which are, generally speaking, non-intersected only on a multivalent Riemann surface. At the same time, the requirements of univalence of function $z(\zeta)$ represented by eq. (14) were studied by many (see, for example[3]). We present some of them, which can be used to construct solution of inverse boundary-value problems aerohydrodynamics

Let C_λ, $0 < \lambda \leq 1$ be the space of Hölder functions, and $H(A, \lambda)$ be a subset of functions from C_λ, satisfing Hölder condition with the coefficient $0 < A < \infty$ and the index λ. Then for the univalence of $z(\zeta)$ with the representation (14) in E^- it is sufficient that either of the following conditions holds

$$1^\circ \cdot 1 \leq \varepsilon < 2, P(\gamma) \in H(A,\lambda), A = (3-\varepsilon)/[2M(\lambda)],$$

$$M(\lambda) = 2^l \pi^{-1} \int_0^{\pi/2} t^l \csc t dt;$$

$$2^\circ \cdot P(2\gamma)/(2\sin\gamma) + \cos\gamma \in H(A,1), A = 34/(\pi e A_0^2), A_0 \leq \pi^2 e/4$$

and

$$P'(0) + i\int_0^{2\pi} P'(\gamma) + \sin\gamma)]\cot\frac{\gamma}{2}\, d_\gamma = 0.$$

We remark that the condition 1° guarantees the convexity of an unknown contour and the constant A_0 in 2° characterizes a special geometric property of the contour obtained by mapping $z(\zeta) = \zeta e^{1/\zeta}$.

Avkhadiev[3] (1987) obtained necessary conditions for univalence in terms of the initial data of the problem. Thus, if $s = s(\varphi)$, $s_* \mathrel{\dot{\leq}} s \leq L$, $0 \leq \varphi \leq \varphi_1$ is the inverse function of $\varphi(s)$ (see eq. (8), then for the solution of the inverse boundary-value problem of aerohydrodynamics to be univalent it is necessary that the following inequality holds

$$|\mu(\varphi+h)| + \mu(\varphi-h) - 2\mu(\varphi)| < 12|h|, \qquad \ldots(17)$$

$$\mu(\varphi) = \int_0^{\varphi} \ln\{v[s(\varphi)]\}\, d\varphi, \; 0 \leq \varphi \leq \varphi_1$$

for any three points φ, $\varphi \pm h \in [0, \varphi_1]$. A similar inequality must be valid on the interval $[0, \varphi_0]$ which corresponds $[0, s_*]$ of variation of s. We note that if (17) does not hold for any three such points, then by no means the solution will be univalent.

A Quasisolution of the Principal Inverse Boundary-Value Problem of Aerohydrodynamics

Thus, for a given velocity distribution $v(s)$ which satisfies the conditions formulated in the statement of the problem we cannot be sure that the solution lies in the required class, in particular, in the class of airfoils bounded by closed and non-intersecting contour, i.e., the problem appears to have no solution. Hence, one of the requirements from the definition of correctness in the sense of Hadamard is not satisfied, namely, the requirement for existence of a solution. The conditions (15) and (16) may be satisfied only in a special case. Therefore, it is necessary to define a generalized solution of the problem and to show how to find it. Moreover, in the case when the solvability conditions are satisfied, the generalized condition must coincide with the regular solution. We note as a preliminary that the methods developed in the theory of inverse boundary-value problems[27] allow to obtain explicit integral representation of the solution e.g. *See* eq. (14). By this the inverse boundary-value problems essentially differ from other inverse problems in which we cannot write a solution explicitly. In many cases having integral representations we can pass from conditionally-correct problems to problems correct in Hadamar sense, since it allows to check whether the constructed solution belongs to the correctness class or to indicate an algorithm for verification of this property.

It is well known that a quasisolution is a generalized solution of an incorrect problem which unlike the true solution is correct in Hadamar sense sufficiently general conditions. Thus, we have to define the quasisolution of the inverse boundary-value problem of aerohydrodynamics and to verify all conditions for its correctness in Hadamar sense. The definition given below is based on the idea by Ivanov[13] for quasisolution construction.

Let U be a correctness class, i.e., the class of required solutions singled out. Clearly, the conditions for solvability (15) and (16) must be satisfied for all elements of U. In addition, the definition of U may include various constructive conditions (e.g, univalence, unseparation of the flow, restriction on the airfoil thickness etc.).

Using (14)-(16) we can show that small variations of the function $P(\gamma)$ in the norm $\|.\|C_\lambda$ of the Banach space C_λ correspond to small variations of the function $z(\zeta)$ in the topology of the uniform convergence what means the stability of the solution of the problem. Since the required solution under the conditions (15) and (16) is constructed uniquely by the Schwartz operator with subsequent integration (see eq. (14) and the conditions for solvability (15) and (16) as well as sufficient conditions for univalence are expressed in terms of $P(\gamma)$, the quasisolution is suitably constructed by using the function $P(\gamma)$. We note that other constructive restrictions on U can be expressed in terms of $P(\gamma)$.

So, let the correctness class U consists of functions $P(\gamma)$, giving solutions of the problem in the form (14) from the prescribed class of solutions. If the function $P(\gamma)$ found by the initial data lies in U, then the corresponding function $z(\zeta)$ will be the required solution. Otherwise, it is necessary to "adjust" $P(\gamma)$ so, that the new function (we shall denote it by $P_*(\gamma)$) lies in U since, generally speaking, there are infinitely many such functions, we have to impose additional restrictions which single out the "optimum" function $P_*(\gamma)$. One of the restrictions is that the distance (in a certain sense) between $P_*(\gamma)$ and $P(\gamma)$ must be minimum.

Definition. *Quasisolution is a metamorphic (with the pole at* ∞*) and local univalent in* E^- *function* $z^*(\zeta)$*, which is uniquely determined from (14) by* $P_*(\gamma)$ *belonging to the correctness class U and minimizing a given functional* $J(Q, P)$ *on it, i.e.,* $\min J(Q,P) = J(P_*, P)$.

We note that we may take for J a norm $\|Q - P\|_H$ where H is a normed space containing U as a subset.

Now it is necessary to ensure existence, uniqueness and stability of the quasisolution by constructing U in a special way. In particular, existence and stability are guaranteed by compactness and strict convexity of U and uniqueness follows from its compactness[5-7]. We present some of the results.

Let the correctness class $U \subset H(A,\lambda) \subset C_\lambda$ for a fixed A, λ consist of functions satisfying eqs. (15) and (16) and any of conditions 1° - 2°. The correctness of the problem of finding the quasisolution is implied by the following assertion[5].

Let $H = L_2$, $J(Q,P) = J_1(Q,P) \equiv \|Q - P\|_{L_2}$ and $U \neq \phi$. Then the quasisolution $z_*(\zeta)$ exists, is unique and stable on U, and is uniquely determined by $P_*(\gamma)$ which minimizes the functional J_1 on U.

We assume that in finding the quasisolution the function $P(\gamma)$ determined by the initial data is constant everywhere except a fixed interval $[a, b]$ (in particular, when $[a, b] \equiv [0, 2\pi]$, the function $P(\gamma)$ is not constant anywhere) and the set U defined only by conditions (15) and (16). We present a function $P_*(\gamma)$ in the following form

$$P_*(\gamma)=\begin{Bmatrix} P(\gamma) + T(\gamma), & \gamma\in[a,b] \\ P(\gamma) & \gamma\notin[a,b] \end{Bmatrix},$$

where $T(\gamma)$ is a Hölder function, vanishing at the ends of the interval ($T(a) = T(b) = 0$). The conditions for solvability (15) and (16) expressed in terms of $T(\gamma)$ take the form

$$\int_a^b T(\tau)e^{i\tau}d\tau = B_1 - A_1 + i(B_2 - A_2), \quad \int_a^b T(\tau)d\tau = B_3 - A_3 \qquad \ldots(18)$$

where

$$A_1 + iA_2 = \int_0^{2\pi} P(\tau)e^{i\tau}d\tau, \quad A_3 = \int_0^{2\pi} P(\tau)d\tau.$$

The above assertion guarantees existence and uniqueness of $T(\gamma)$ with the properties (18) and uniform convergence of the minimizing sequence what allows to use uniformly convergent Fourier expansion of $T(\gamma)$. In addition, the above variational problem is reduced to the problem of minimization of functional

$$J = \sum_{k=0}^{\infty} \left(c_k^2 + d_k^2\right)$$

(c_k, d_k are the Fourier coefficients of the function $T(\gamma)$ under linear restrictions and its solution is obtained explicitly in[6].

Suppose now that $J(Q, P) = J_2(Q, P) \equiv \|Q' - P'\|L_2$ and $T(\gamma)$ is continuously differentiable. The solvability conditions take the form

$$\int_a^b T(\tau)e^{i\tau}d\tau = -(B_1 - A_1) + i(B_2 - A_2), \quad \int_a^b T(\tau)\tau d\tau = -(B_3 - A_3), \qquad \ldots(19)$$

$$\int_a^b T(\tau)d\tau = 0, \; T(a) = 0.$$

Now, it is necessary to minimize the functional $\int_a^b T'^2(\tau)d\tau$ under the restrictions (19). Thus, among all the functions $T(\gamma)$ with summable derivative satisfying (19) a function is sought with the smoothest variation on [a, b]. The solution of the problem has the form

$$T(\gamma) = -a_1 \cos\gamma + a_2 \sin\gamma + (a_3/2)\gamma^2 + a_4\gamma + a_5 \qquad \ldots(20)$$

where the coefficients a_1, a_2, a_3, a_4 and a_5 are uniquely determined fom the linear system obtained by substitution of eq. (20) into eq. (19). They have the simplest form when $a = 0$, $b = 2\pi$, $a_2 = (B_2 - A_2)/\pi$, $a_2 = 3[2(B_1 - A_1) + B_3 - A_3]/[2\pi(6 - \pi^2)]$, $a_4 = -\pi a_3$, $a_1 = a_5 = 2a_3 - (B_1 - A_1)/\pi$. Now, if we discard the restriction $T(0) = 0$ and require additionally that $\min_{Q\in U} J_1(Q, P) = J_1(P_*, P)$ then we obtain $T(\gamma)$ in the form $T(\gamma) = (B_3/2 + B_1\cos\gamma + B_2\sin\gamma)/\pi$. This is equivalent to the result by Mangler[17] and coincides with the quasisolution constructed for a functional J_1 when $[a, b] = [0, 2\pi]$.

A Subsonic Gas Flow

There are many paper dealing with the solution of inverse boundary-value problems of aerohydrodynamics for a gas flow[10,11,19-21,26,28-30]. They also employ the plane comparison method when the equations of gas motion are analytically transferred from an unknown region into the auxiliary plane and the resulting direct problem is numerically solved there. As a rule, the questions of guaranteeing the closedness of airfoil contour are not discussed. Sometimes empirical methods are proposed to satisfy these conditions (eg.[28]).

If in solution of the problem the motion of a real gas is replaced by the motion of Chaplygin's gas, then the study of solvability conditions becomes possible. Under this assumption the inverse boundary-value problem of aerohydrodynamics was considered first by Tumashev[26], the analytical solution and solvability conditions being obtained. In order to satisfy these conditions Strand[23] applied a numerical method[24] and presented examples of calculations. The essence of the above mentioned substitution is as follows.

The plane steady potential motion of a gas is known to be described by equations

$$\frac{\partial S}{\partial \psi}=K(\lambda)^{1/2}\frac{\partial \theta}{\partial \varphi},\ \frac{\partial \theta}{\partial \psi}=-K(\lambda)^{1/2}\frac{\partial S}{\partial \varphi}, \qquad \ldots(21)$$

$$S(\lambda)=\ln\left[\frac{2h\lambda}{\left(h^2-\lambda^2\right)^{1/2}+h\left(1-\lambda^2\right)^{1/2}}\left(\frac{\left(h^2-\lambda^2\right)^{1/2}+\left(1-\lambda^2\right)^{1/2}}{h+1}\right)^h\right],$$
$$K(\lambda)=\left(1-\lambda^2\right)\left(1-\lambda^2/h^2\right)^{-h^2},\quad h=[(\kappa+1)/(\kappa-1)]^{1/2}, \qquad \ldots(22)$$

κ is the isoentropic exponent, $\lambda = v/a_*$ is the dimensionless velocity, a_* is the critical velocity and θ is the argument of the velocity vector.

The approximation of the Chaplygin's gas is based on the replacement of the adiabatic curve $p\rho^{-k} = const$ (p is a pressure and ρ is a density) by a linear dependence in the plane $(p, 1/\rho)$. The replacement leads to the following formula for transition from the z-plane to the w-plane

$$dz = e^{-\chi}\, dw - \sigma \overline{e^{\chi}\, dw} \qquad \ldots(23)$$

or

$$dz = e^{i\theta}\left[\left(e^{-S}-\sigma e^{S}\right) d\varphi + i\left(e^{-S}+\sigma e^{S}\right) d\varphi\right], \qquad \ldots(24)$$

where σ is a positive constant. From the condition that dz is a total differential and from eqs. (23), (24) it follows that the function $S(\lambda)$ must have the form

$$S(\lambda) = \ln \{2 \,|\lambda|\, /[1 + 4\sigma\lambda^2)^{1/2}]\} \qquad ...(25)$$

In addition, the constant σ can be chosen from the considerations of the best approximation of the dependence[22] by the function[25]. For example, Stepanov[22] proposed the following form $\sigma = [2(\kappa + 1)(1 - \lambda_\infty^2)]^{-1}$.

The transition from eq. (21) to the equations of motion the Chaplygin's gas reduces the function $K(\lambda)$ to a constant. This in turn means that $\chi(w)$ is an analytic function. This property allows to construct the solution of the problem by the scheme presented on P. Nos. 209-212. The difference between the formulae of transition from the z-plane to the w-plane leads to a modification of B_1, B_2, B_3 in (15) and (16):

$$B_1 = \pi(1-\varepsilon) + 4\pi \sin^2\alpha_0 \tilde{\lambda}_\infty^2 \sigma\left(1+\sigma\tilde{\lambda}_\infty^2\right)^{-1}, \quad B_2 = -2\pi\tilde{\lambda}_\infty \sigma\left(1+\sigma\tilde{\lambda}_\infty^2\right)^{-1} \sin(2\alpha_0),$$

$$B_3 = 2\pi \ln \tilde{\lambda}_\infty$$

where $\tilde{\lambda}_\infty$ exp$S(\tilde{\lambda}_\infty)$, $\tilde{\lambda}_\infty = \; = v_\infty/a_*$ is a given free-stream dimensionless velocity. Now the quasisolution is constructed similarly to the above presentation.

A Viscous Fluid

The idea of the solution of the inverse boundary-value problem of aerohydrodynamics in view of viscosity is based on the following well-known assumptions, namely, the influence of viscosity is manifesting itself only in a relatively thin boundary layer; the pressure distribution on the airfoil contour flowed by a viscous fluid coincides with the pressure distribution on the thickened airfoil flowed by an ideal fluid. The thickened airfoil is obtained by accumulation of the boundary layer thickness displacement δ^* on the airfoil contour and the streamline coming from the trailing edge. In addition to these assumptions for an approximate solution of the problem one more is necessary which determines the shape of the airfoil wake.. Thus, to solve the inverse boundary-value problem of aerohydrodynamics by a given velocity hodograph, (*See* eg.[22], Sec 56 it was supposed that the wake is bounded by two streamlines of constant velocity. Lebedev (1983) considered the problem with $v(s)$ in the symmetrical case on the assumption that the wake is bounded by two parallel straight lines. In order to solve the problem for an arbitrary airfoil we shall assume that the wake is bounded by two congruent steamlines. Obviously, the wake can be regarded to be not loaded.

With these conjectures the inverse boundary-value problem of aerohydrodynamics was solved[12]. By a given pressure distribution along the contour (hence, by a velocity distribution along the thickened airfoil) we solve the problem for an ideal fluid. As a result we find the shape of the thickened airfoil contour, knowing which we can construct airfoil contour. For the contour to be closed it is necessary that the thickened profile has a gap at the trailing edge which value is $\delta_B^* = \delta_1^* + \delta_2^*$ (δ_1^* and δ_2^* are the values of δ^* at B for the upper and lower surfaces of the airfoil respectively). The form of the solvability conditions is preserved; only the constants B_1 and B_2 change. For $\varepsilon = 2$, we obtain $B_1 + iB_2 = -\pi + \delta_B^* \exp[i(\alpha_0 + \theta_B)]/(2U_0)$, where θ_B is a value of θ at B.

From the above argument, it is obvious that values of B_1 and B_2 depend not only on Γ and α_0 but also on δ_B^* and θ_B which vary in the process of construction of the quasisolution. Therefore, we must construct the closed contour in the case of a viscous fluid by iterations. The calculations showed that the process converges if the iterated velocity distributions are non-separating.

To take into account both viscosity and compressibility of the flow we can again use the Chaplygin's gas model. Then the solution of the problem is constructed by the scheme of part 5 and quantities B_1 and B_2 take the form

$$B_1 + iB_2 = -\pi - \frac{\tilde{\lambda}_\infty e^{i\alpha_0}}{1+\sigma\tilde{\lambda}_\infty^2}\left[4\pi i\sigma\tilde{\lambda}_\infty^2 \sin\alpha_0 - (2U_0)^{-1}\delta_B^*\left(e^{i\theta_B} - \sigma\tilde{\lambda}_\infty^2 e^{i\theta_B}\right)\left(1-\sigma\tilde{\lambda}_\infty^2\right)^{-1}\right],$$

$$B_3 = 2\pi \ln \tilde{\lambda}_\infty$$

In conclusion of the part we note that an iterative method of the solution of the inverse boundary-value problem of aerohydrodynamics in view of viscosity by the model of the boundary layer based on a successive solution of the direct problems was proposed in[4].

Results of Calculations

(a) Airfoil Design

We shall give an example of application of the presented theory to the design of airfoil with maximal lift coefficient $C_{L_{max}} \geq 1.5$ and with high aerodynamic ratio $K \approx 150$, under the following restriction on the airfoil thickness $0.17 \leq t \leq 0.19$. The designing process consists of four steps:

- a class of velocity distributions is described which ensures the given characteristics;
- an inverse problem is solved for distributions $v(s)$ from this class;
- the airfoil with maximal K is selected and its other characteristics are studied;
- the airfoil is modified to improve the characteristics.

To attain high values of K it is necessary that the flow should be non-stalling for large C_L. We shall ensure that the flow is non-stalling on the upper surface by giving $v(s)$ as on P. Nos. 207-209. We consider the completely turbulent boundary layer and take $f = \mu f_0$, $\mu \in (0,1)$ to make an additional margin of the unseparation. On the lower surface, except the acceleration zone with a linear distribution of $v(s)$, we shall prescribe a constant $v(s) = v_*$. Thus the chosen class of $v(s)$ has the form

$$\bar{v} = \begin{cases} -1, & s \in [0,\ s_2], \\ \bar{v}_0\,(s - s_*), & s \in [s_2,\ s_3], \\ \bar{v}_0, & s \in [s_3,\ s_0], \\ \bar{v}_0\,[1 + K_0(s - s_0)]^{-m}, & s \in [s_0,\ 1], \end{cases} \quad \dots (26)$$

where $\bar{v} = v/v_*$. This class is determined by four parameters $\bar{v}_0$, s_*, s_3 and μ.

The margin of the unseparation was chosen to be equal to $\mu = 0.3$, while the other three parameters varied. It is ascertained that:

- for the lengths of upper and lower surfaces of the airfoil to be compatible the value s_*, as a rule, should be taken from the interval [0.46, 0.49];
- the thickness t is determined by $v'(s_*)$ or by the length $s = s_3 - s_*$ of the acceleration zone, the thickness of the airfoil is growing when $v'(s_*)$ decreases or when σ increases;
- the value $\bar{v}_0$ affects greatly C_L.

Then on the base of the iterative procedure mentioned on P. Nos. 217-218, accounting compressibility and viscosity of the flow, we designed airfoils for different combination of $\bar{v}_0$, $s_* s_3$. Comparison of aerodynamic characteristics calculated for these combinations of $\bar{v}_0, s_*, s_3$ with desired one showed that the best result corresponds to $s_* = 0.47$, $s_3 = 0.506$, $\bar{v}_0 = 1.775$. This combination defines the velocity distribution $v(s)$, presented by line 1 on Fig. 2. The airfoil contour is, unfortunately, unclosed. Therefore, a quasisolution is designed and airfoil is found (see contour 3 on Fig. 2) with $t = 0.188$ and $C_D = 1.05$, $C_D = 0.0091$, $K = 115.4$ at $\alpha = 6.6^{\circ}$, $M_\infty = 0$, $Re_\infty = 10^6$. Line 2 in Fig. 2 represents the velocity distribution along the airfoil. It was ascertained that for these free stream parameters the airfoil was flowed without stalling, the boundary layer being of laminar character along the most part of the surface. This explains the fact that K is large.

The airfoil characteristics for other values of α were studied as follows. The distributions $v(s)$ were calculated for several α according to the ideal incompressible fluid scheme. These $v(s)$ were used to compute the boundary layer characteristics as well as C_L, C_D and K. The dependencies $C_L(C_D)$ and $K(C_L)$ at $Re_\infty = 10^6$ are presented on Figs. 3 and 4 by line 1. It is easily seen that the airfoil has the low drag ($C_D < 0.01$) and high K ($K_{max} = 117.6$ at $\alpha = 6^{\circ}$) for $C_L < 1.1$. For a larger C_L the value K decreases.

The airfoil was modified to improve K for $C_L > 1.1$. With this purpose, a peak for the velocity distribution at $\alpha = 11^{\circ}$ (see curve 1 on Fig. 5) was cut on the level $v/v_\infty = 1.7$. After designing of the quasisolution the new distribution

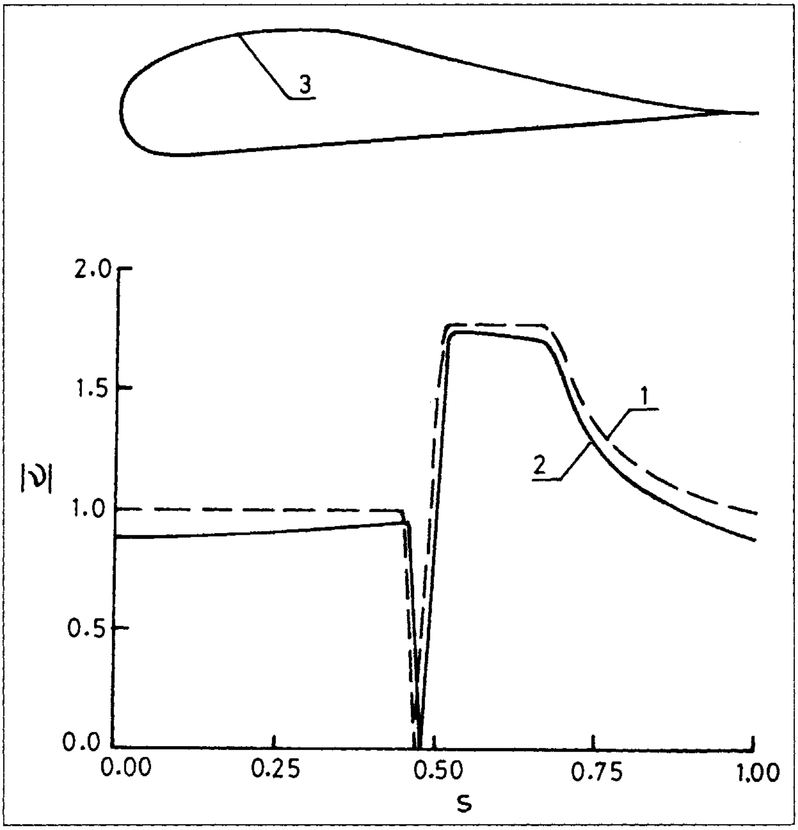
3
2.0
1.5
1.0
0.5
0.0
|v|
0.00
0.25
0.50
0.75
1.00
s
1
2

Fig 2

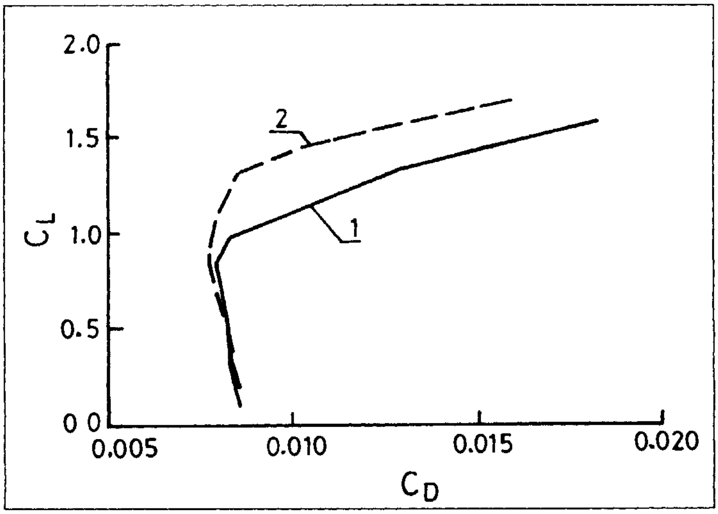
2.0
1.5
1.0
0.5
0 0
C_L
0.005
0.010
0.015
0.020
C_D
1
2

Fig 3

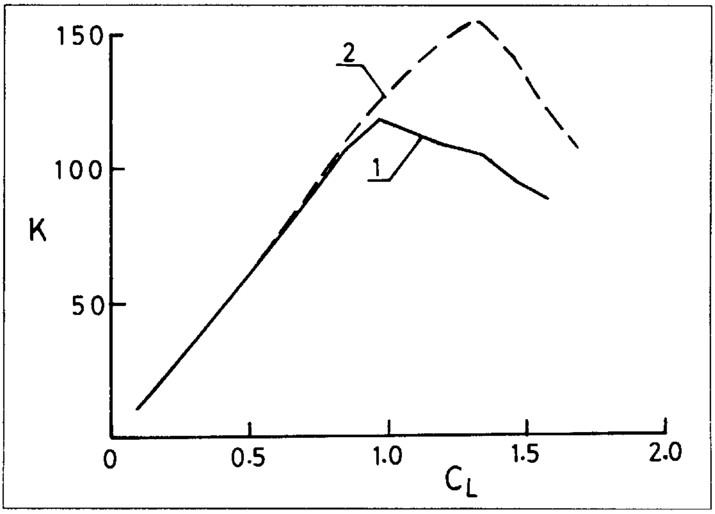

Fig 4

corresponds to the airfoil 3 on Fig. 5 with $t = 0.19$. For $\alpha = 9.5^0$ we have here $C_L = 1.50$, $C_D = 0.0114$ and $K = 132$. The velocity distribution on its surface is presented by line 2 on Fig. 5.

Calculation for the airfoil characteristics for other α (line 2 on Figs. 3 & 4) showed that the modification resulted in an extension of the low drag region

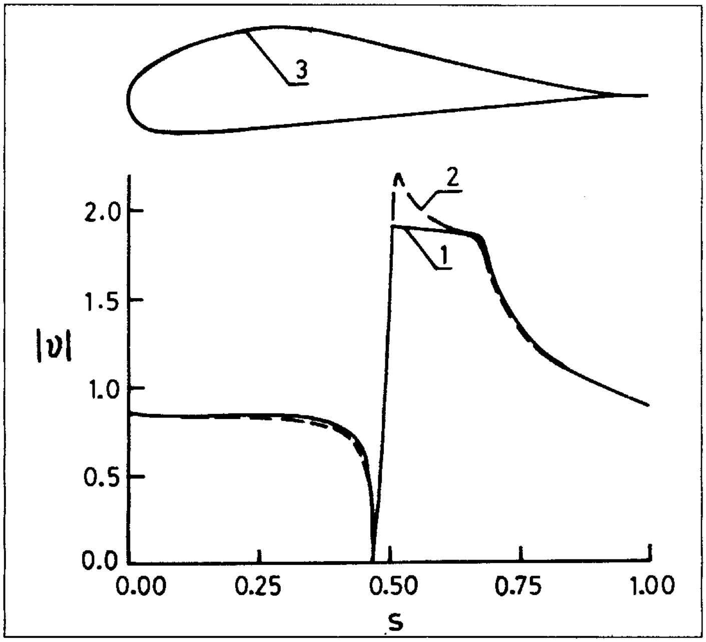

Fig 5

as well as in an increase of K_{max} = 155 which was attained at α = 8° . Thus the goal is achieved.

(b) Maximization of the Lift

We consider design of a separation-free high lift airfoil for viscous compressible flow for $Re = 3.5 \cdot 10^{*}$, $M_{\infty} = 0.17$, using the iterative procedure mentioned on P. Nos. 217-218, to account compressibility and viscosity distribution on the upper surface with maximal possible square with the account of unseparation conditions. It can be proved that this must be a roof-top velocity distribution of eq. (26) where $\mu = 1$, $\bar{v}_0 = 2.35$. Velocity distribution on the lower surface must have a very small square, provide the fulfilment of unseparation condition along this part of the contour and guarantee the closeness. An example of the design under assumption of fully turbulent boundary-layer on the airfoil is presented in Fig. 6. The initial optimal velocity distribution is presented by solid line 1. Corresponding airfoil is shown by solid line 2. The airfoil's lift is $C_L = 2.99$ at $\alpha = 18.44$. The drag coefficient $C_D = 0.021$. It should be noted that the airfoil shape and the lift coefficient are close to those obtained by Liebeck[14] for high lift maximization in an incompressible inviscid flow. However, the lift/drag ratio

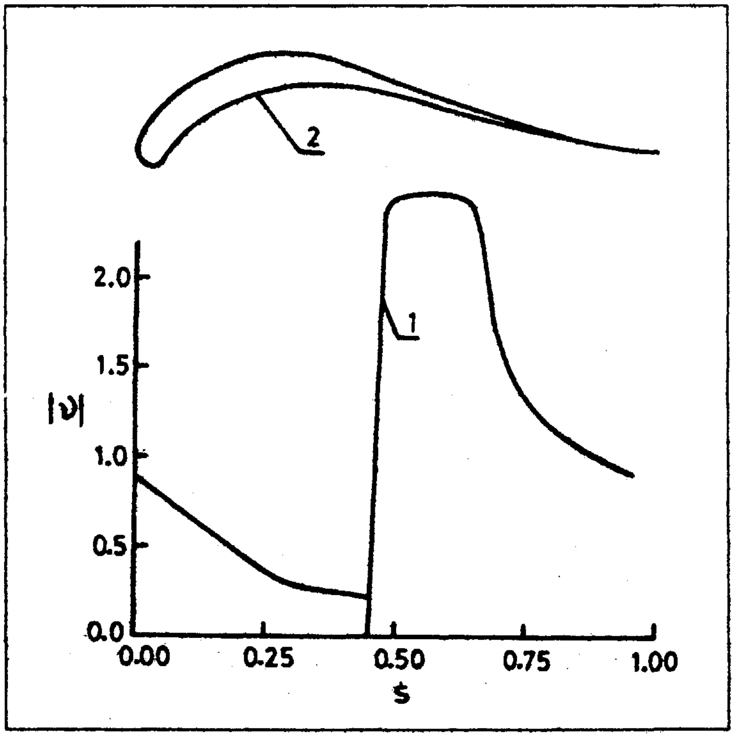

Fig 6

are relatively lower due to lower Reynolds number and the assumption of fully turbulent boundary layer on the upper surface, which was necessary to preserve here the flow unseparation.

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