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THE SPHERICALLY SYMMETRIC COLLAPSE OF AN INHOMOGENEOUS STELLAR DUST BALL INTO A BLACK HOLE*

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A detailed and critical review is done on the spherically symmetric collapse of a pressure-free fluid body into a black hole. Tolman's well-known general solution is adopted for the interior of the fluid body for this purpose. The interior metric and its partial derivatives are matched with the exterior metric. Synge's junction conditions $T^a_b n^b|_{\partial D} = 0$ are satisfied. The Tolman metric is classified into three distinct cases depending on a parameter $\varepsilon = 0, \pm 1$. Three cases are dealt with separately. In each case, five coordinate charts are used to depict the phenomenon of the collapse of a dust ball into a black hole.

Key Words : Spherically Symmetric Collapse; Pressure-Free Fluid Body; Black Hole; Metric—Interior & Exterior; Tolman Metric; Synge's Junction Conditions

1. Introduction

The phenomenon of the collapse of a heavenly body into a black hole according to Einstein's equations is very fascinating. Classical gravitation according to Einstein's field equations can be treated in a mathematically rigorous way. In this arena, black hole solutions like the Kruskal and Kerr metrics have emerged and have been analyzed extensively[1, 2]. But these are solutions of the vacuum equations $G_{ij} = 0$!. The charged black hole solution inherent in the Reissner-Nordstrom-Jeffrey solution is obtained from $G_{ij} = \kappa T_{ij}$, where T_{ij} is the energy-momentum-stress tensor of the electro-magnetic field. It is a totally unrealistic model for a collapsing star. Next, in order of relevance, the Fried-man metric can be considered. It can represent an imploding homogeneous stellar dust ball into a black hole[2,3]. A slightly more realistic model will be the Tolman[4] metric representing a spherically symmetric, inhomogeneous, pressure-free fluid body. Subsequently to Tolman, Oppenheimer and Snyder[5] used a similar model in their pioneering investigation of gravitational collapse. The study of these spacetimes was much more thoroughly taken up by Bondi[6], for which they are often called the Tolman-Bondi spacetimes.

The Tolman-Bondi metrics have been applied primarily to derive cosmological models[7, 8]. In fact, there is a review artile[9] on the application of Friedman-Lemaître metrics (which are special

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cases of Tolman-Bondi metrics) to cosmology. There have been also applications of Tolman-Bondi metrics to construct counter-examples to cosmic censorship hypothesis[10-12]. However, more than fifty years after Tolman's initial discovery, it is difficult to find a mathematically clean description of gravitational collapse in a Tolman-Bondi spacetime.

This paper sets out to fill the gap in the literature. We provide an almost complete global analysis describing gravitational collapse of an inhomogeneous, pressure-free fluid body into a black hole. This includes the three possible cases (parabolic, elliptic, or hyperbolic)[7, 13] complete with the necessary junction conditions[14] applied at the boundary of the body. The analysis is done in painstaking detail, particularly in matching the metrics at the boundary and the transformations of the external metrics into Schwarzschild's form. Included also is a complete analysis that looks at the external metric in Kruskal-Synge-Szekeres form. Shell-crossing singularities are not considered and there is no investigation of boundary layer (single or double).

The most important problem that still remains open is to find doubly-null coordinates for the interior of the fluid body and to match these with the external Kruskal coordinates. (Without this matching, we do not know how to draw the compactification of global spacetime in a Penrose diagram depicting a fluid body imploding into a black hole).

2. Notations and Field Equations

We adopt notations and conventions from Synge's book,[14] except we shall denote partial and covariant derivatives by ∂_k and ∇_k respectively. Einstein's equations for an incoherent fluid or dust are given by

$$\varepsilon_{ij} := G_{ij} + \kappa T_{ij} = 0, \qquad \ldots (2.1a)$$

$$T^{ab} := \mu u^a u^b, \qquad \ldots (2.1b)$$

$$T^a := \nabla_b T^{ab} = 0, \qquad \ldots (2.1c)$$

$$\mathcal{U} := u^a u_a + 1 = 0, \qquad \ldots (2.1d)$$

$$C^a (g_{cd}, \partial_k g_{cd}) = 0 \qquad \ldots (2.1e)$$

and

$$\kappa := 8\pi. \qquad \ldots (2.1f)$$

The C^a stand for four possible coordinate conditions.

Let us count the number of unknown functions versus the number of algebraically and differentially independent equations. The total number of unknown function is

$$10(g_{ab}) + 1(\mu) + 4(u^a) = 15.$$

The total number of algebraic and partial differential equations is

$$16(\varepsilon_{ab} = 0) + 4(T^a = 0) + 1(\mathcal{U}) + 4(C^a = 0) = 25.$$

There exist six algebraic identities $\varepsilon_{ab} - \varepsilon_{ba} \equiv 0$ and four differential identities $\nabla_b \varepsilon^{ab} - \kappa T^a \equiv 0$. Therefore, the total number of independent equations is 25 – (6 + 4) = 15. Then the system of eq. (2.1*a*, *b*, *c*, *d*, *e*) is determinate.

For the algebraic properties of T^a_b, we derive from (2.1*b*, *d*)

$$T^a_b u^b = -\mu u^a. \qquad \text{... (2.2)}$$

Therefore, $-\mu$ is an eigenvalue of the 4×4 matrix $[T^a_b]$. The corresponding eigenvector components are given by u^a. The other three eigenvalues are exactly zero.

By (2.1*b*), the conservation eqs. (2.1*c*) imply that

$$T^a := u^a \nabla_b (\mu u^b) + \mu u^b \nabla_b u^a = 0. \qquad \text{... (2.3)}$$

Multiplying through $(-u_a)$ and substituting (2.1*d*), the above equation yields

$$-u_a T^a = \nabla_b (\mu u^b) = 0. \qquad \text{... (2.4)}$$

This is the continuity equation for the four-dimensional fluid flow. Eqs. (2.4) and (2.3) and an additional requirement $\mu \neq 0$ imply the geodesic equations

$$u^b \nabla_b u^a = 0. \qquad \text{... (2.5)}$$

"Therefore, in an incoherent fluid, the streamlines are geodesics, a rather remarkable result"[14].

The three energy conditions of Hawking[1] yield in this case just one weak inequality, namely

$$\mu \geq 0. \qquad \text{... (2.6)}$$

The unknown functions g_{ab} in the field equations (2.1*a*, *c*, *e*) are assumed to be of the differentiability class C^3_p. By this, we mean that the functions have continuous third order derivatives in a some domain, say $\tilde{D}_1 \cup \tilde{D}_E$, and may have jump continuous on the intermediate boundary $\partial\tilde{D}$. Moreover, the functions μ and u^a are of differnetiability class C^1_p. We adopt the four junction conditions of Synge[14]

$$\{[T^a_b]\, n^b\}_{|\,\partial D} = \{[\mu u^a u_b]\, n^b\}_{|\,\partial D} = 0. \qquad \text{... (2.7)}$$

Here, the jump discontinuity is denoted by the square bracket and n^a are the components of a "unit" normal to the boundary $\partial\tilde{D}$.

3. The Spherically Symmetric Case

In a spherically symmetric comoving coordinate system, we can write the line element as [15]

$$ds^2 = e^{\lambda} d\rho^2 + r^2 d\Omega^2 - d\tau^2, \qquad \text{... (3.1}a\text{)}$$

$$\lambda = \Lambda(\rho, \tau), \qquad \text{... (3.1}b\text{)}$$

$$r = R(\rho, \tau) \qquad \text{... (3.1}c\text{)}$$

and

$$d\Omega^2 := d\theta^2 + \sin^2\theta\, d\phi^2. \qquad \text{... (3.1}d\text{)}$$

From the signature axiom of the Lorentz metric[16], we obtain for an "orthogonal" metric the strict inequalities

$$g_{11} > 0,\ g_{22} > 0,\ g_{33} > 0,\ g_{44} < 0. \qquad \ldots (3.2)$$

By (3.1*a*, *c*, *d*), we must have

$$R(\rho, \tau) \neq 0 \qquad \ldots (3.3)$$

and $\sin \theta \neq 0.$

Out of the four possibilities in (3.3), we choose

$$R(\rho, \tau) > 0 \qquad \ldots (3.4a)$$

and $$\sin \theta > 0. \qquad \ldots (3.4b)$$

Out of denumerically infinite number of intervals $(2n\pi, (2n+1)\pi)$, $n \in Z$, which satisfy (3.4*b*), we choose $\theta \in (0, \pi)$. (Denumerably infinite number of charts which are not considered here will contribute to the maximal atlas for the space-time manifold). In a coordinate system which is consistent with (3.4*a*) and $\theta \in (0, \pi)$, typical domains of validity are

$$\tilde{D}_I := \{(\rho, \theta, \phi, \tau) : \rho_c < \rho < \rho_b,\ 0 < \theta < \pi,\ -\pi < \phi < \pi,\ \tau_0(\rho) < \tau < \tau_1(\rho)\}], \qquad \ldots (3.5a)$$

$$\tilde{D}_E := \{(\rho, \theta, \phi, \tau) : \rho_b < \rho,\ 0 < \theta < \pi,\ -\pi < \phi < \pi,\ \tau_0(\rho) < \tau < \tau_1(\rho)\} \qquad \ldots (3.5b)$$

and $$\partial\tilde{D} := \{(\rho, \theta, \phi, \tau) : \rho = \rho_b,\ 0 < \theta < \pi,\ -\pi < \phi < \pi,\ \tau_0(\rho_b) < \tau < \tau_1(\rho_b)\}. \qquad \ldots (3.5c)$$

We can compute the Christoffel symbols of the second kind from the metric (3.1*a*) in the domain $\tilde{D}_I \cup \tilde{D}_E$ and these are furnished by

$$\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \frac{1}{2}\partial_\rho \lambda,\ \left\{ \begin{matrix} 1 \\ 14 \end{matrix} \right\} = \frac{1}{2}\partial_\tau \lambda,\ \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -(\partial_\rho r)\, e^{-\lambda},$$

$$\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} = -r(\partial_\rho r)\, e^{-\lambda} \sin^2\theta,\ \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\} = \frac{1}{r}\partial_\rho r,$$

$$\left\{ \begin{matrix} 2 \\ 24 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 34 \end{matrix} \right\} = \frac{1}{r}\partial_\tau r,\ \left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} = -\sin\theta\cos\theta,\ \left\{ \begin{matrix} 3 \\ 33 \end{matrix} \right\} = \cot\theta$$

and $$\left\{ \begin{matrix} 4 \\ 11 \end{matrix} \right\} = \frac{1}{2} e^{\lambda}\partial_\tau \lambda,\ \left\{ \begin{matrix} 4 \\ 22 \end{matrix} \right\} = r\partial_\tau r,\ \left\{ \begin{matrix} 4 \\ 33 \end{matrix} \right\} = r(\partial_\tau r)\sin^2\theta. \qquad \ldots (3.6)$$

The time-like geodesic equations in the metric (3.1*a*) can be obtained from (3.6) as

$$\ddot{\rho} + (\partial_\tau \lambda)\dot{\rho}\,\dot{\tau} + \frac{1}{2}(\partial_\rho \lambda)(\dot{\rho})^2 - r(\partial_\rho r)\, e^{-\lambda}[(\dot{\theta})^2 + \sin^2\theta\,(\dot{\phi})^2] = 0,$$

$$\ddot{\theta} + 2(\partial_\tau \ln r)\,\dot{\theta}\dot{\tau} + 2(\partial_\rho \ln r)\,\dot{\theta}\dot{\rho} - (\sin\theta\cos\theta)(\dot{\phi})^2 = 0,$$

$$\ddot{\phi} + 2(\partial_\rho \ln r)\,\dot{\phi}\dot{\rho} + 2(\partial_\tau \ln r)\,\dot{\phi}\dot{\tau} + 2(\cot\theta)\,\dot{\theta}\,\dot{\phi}) = 0$$

and $$\ddot{\tau} + \frac{1}{2} e^{\lambda}(\partial_\tau \lambda)(\dot{\rho})^2 + r(\partial_\tau r)[(\dot{\theta})^2 + (\sin^2\theta)(\dot{\phi})^2] = 0. \qquad \ldots (3.7)$$

Here, the dot refers to the derivative with respect to the proper time parameter s. A special class of exact solutions of (3.7) is furnished by

$$\dot{\rho} \equiv 0,\ \dot{\theta} \equiv 0,\ \dot{\phi} \equiv 0,\ \dot{\tau} \equiv 1, \qquad \ldots (3.8)$$

$$\rho(s) = \text{const.},\ \theta(s) = \text{const.},\ \phi(s) = \text{const.},\ \tau(s) = s.$$

Eqs. (3.8) represent a time-like radial geodesic. For a collapsing dust cloud, we choose fluid velocities along such geodesics. Therefore, the components u^a are expressed in the comoving coordinate system as

$$u^1 = u^2 = u^3 \equiv 0,\ u^4 \equiv 1 \equiv -u_4. \qquad \ldots (3.9)$$

By use of (3.1*a*, *d*), (3.6) and (3.9), Einstein's field equations (2.1*a*, *b*) reduce to the following four non-trivial equations :

$$\varepsilon_1^1 := G_1^1 + \kappa T_1^1$$

$$= \left(\frac{2}{r}\right)(\partial_\tau^2 r) + (\partial_\tau \ln r)^2 (\partial_\rho \ln r)^2 e^{-\lambda} + r^{-2} = 0, \qquad \ldots (3.10a)$$

$$\varepsilon_2^2 \equiv \varepsilon_3^3 := G_3^3 + \kappa T_3^3$$

$$= \frac{1}{r}(\partial_\tau^2 r) + \frac{1}{2}(\partial_\tau^2 \lambda)\left(\frac{1}{2}\partial_\tau \lambda\right)^2 + \left(\frac{1}{2}(\partial_\rho \ln r)(\partial_\rho \lambda) - r^{-1}(\partial_\rho^2 r)\right)e^{-\lambda}$$

$$+ \frac{1}{2r}(\partial_\tau r)(\partial_\tau \lambda) = 0, \qquad \ldots (3.10b)$$

$$\varepsilon_4^4 := G_4^4 + \kappa T_4^4$$

$$= \left((\partial_\rho \ln r)(\partial_\rho \lambda) - \frac{2}{r}(\partial_\rho^2 r) - (\partial \rho \ln r)^2\right)e^{-\lambda}$$

$$+ (\partial_\tau \ln r)^2 + (\partial_\tau \ln r) + (\partial_\tau \lambda) + r^{-2} - \kappa\mu = 0 \qquad \ldots (3.10c)$$

and

$$\varepsilon_4^1 := G_4^1 + \kappa T_4^1$$

$$= r^{-1}(2(\partial_\tau \partial_\rho r) - (\partial_\tau \lambda)(\partial_\rho r))e^{-\lambda} = 0. \qquad \ldots (3.10d)$$

The conservation eq. (2.1*c*) go over into one non-trivial equation :

$$T_4 = \partial_\rho T_4^1 + \left(\frac{1}{2}(\partial_\rho \lambda) + 2\partial \rho (\ln r)\right)T_4^1 - \left(\frac{1}{2}(\partial_\tau \lambda) T_1^1 + 2(\partial_\tau \ln r) T_2^2\right)$$

$$+ \partial_\tau T_4^4 + \left(\frac{1}{2}(\partial_\tau \lambda) + 2\partial_\tau \ln r)\right)T_4^4$$

$$= -\left\{\partial_\tau \mu + \mu \partial_\tau \left(\frac{\lambda}{2} + 2 \ln r\right)\right\} = 0. \qquad \ldots (3.11)$$

The differential identities $\nabla_a \varepsilon_b^a - \kappa T_b \equiv 0$ reduce to two non-trivial identities :

$$\partial_\rho \varepsilon_1^1 + 2(\partial_\rho \ln r)\varepsilon_1^1 + \partial_\tau \varepsilon_1^4 + \left(\partial_\tau\left(\frac{\lambda}{2} + 2\ln r\right)\right)\varepsilon_1^4 - 2(\partial_\rho \ln r)\varepsilon_2^2 \equiv 0 \qquad \ldots (3.12a)$$

and

$$\partial_\rho \varepsilon_4^1 + \left(\partial_\rho\left(\frac{\lambda}{2} + 2\ln r\right)\right)\varepsilon_1^4 - \left(\frac{1}{2}(\partial_\tau \lambda)\varepsilon_1^1 + 2(\partial_\tau \ln r)\varepsilon_2^2\right) + \partial_\tau \varepsilon_4^4 + \left(\partial_\tau\left(\frac{\lambda}{2} + 2\ln r\right)\right)\varepsilon_4^4 - \kappa T_4 \equiv 0. \qquad \ldots (3.12b)$$

In this case, we have three unknown functions Λ, R and μ and five partial differential equations (3.10*a*, *b*, *c*, *d*) and (3.11). Moreover, we have two differential identities (3.12*a*, *b*). Therefore, again we have a determinate system of {italic partial} differential equations in a two-dimensional domain.

The strategy for solving this system is given in the following steps :-

1. We solve the two equations $\varepsilon_4^1 = 0$ and $\varepsilon_1^1 = 0$.
2. At this stage, by the differential identity (3.12*a*) and an additional assumption

   $$\partial_\rho r \neq 0, \qquad \ldots (3.13)$$

   the equation $\varepsilon_2^2 = 0$ must hold. For the case $\partial_\rho r = 0$, see Kramer *et al.*[15].
3. Then, we solve the equation $\varepsilon_4^4 = 0$ by defining $\mu(\rho, \tau)$.
4. Now, by the differential identity (3.12*b*), the equation $T_4 = 0$ must be satisfied.

Thus, we will solve the whole system of equations. We can further note that by solving the first order partial differential equation $\varepsilon_4^1 = 0$ for Λ and the second order partial differential equation $\varepsilon_1^1 = 0$ for R, three arbitrary functions will appear in the general solution. (These arbitrary functions will be denoted in the sequel by f, F and τ_0).

Following this strategy for finding the solution, eq. (3.10*d*) yields

$$-re^{\lambda}(\partial_\rho r)^{-1}\varepsilon_4^1 = \partial_\tau(\lambda - 2\ln|\partial_\rho r|) = 0.$$

Integrating the above equation with respect to τ in a convex domain of the ρ-τ plane, we obtain

$$\lambda = \Lambda(\rho, \tau) = 2\ln\left|\frac{\partial R}{\partial \rho}(\rho, \tau)\right| + h(\rho). \qquad \ldots (3.14)$$

Here, h is an arbitrary function of integration and it belongs to the class C_p^3. It turns out that these solutions can be conveniently classified into three distinct cases according to

(*i*) $1 < \exp[-h(\rho)]$, (*ii*) $\exp[-h(\rho)] \equiv 1$ and (*iii*) $0 < \exp[-h(\rho)] < 1$.

Another way to describe these three cases is to set

$$\exp[-h(\rho)] = 1 - \varepsilon[f(\rho)]^2 > 0, \qquad \ldots (3.15)$$

$$\varepsilon = 0, \pm 1.$$

Here, f is an arbitrary function of class C_p^3. There is a loss of generality of local solutions in the choice (3.15), since three cases cannot intermingle in a single solution. By (3.15), eq. (3.14) yields

$$g_{11} = \exp[\Lambda(\rho, \tau)] = \frac{\left[\frac{\partial R}{\partial \rho}(\rho, \tau)\right]^2}{1 - \varepsilon [f(\rho)]^2} > 0, \qquad \text{... (3.16}a\text{)}$$

$$\frac{\partial R}{\partial \rho}(\rho, \tau) \neq 0 \qquad \text{... (3.16}b\text{)}$$

and

$$\varepsilon [f(\rho)]^2 < 1. \qquad \text{... (3.16}c\text{)}$$

Substituing (3.16a) into the field equation (3.10a), we get

$$\varepsilon_1^1 = r^{-2}(2r\partial_\tau^2 r + (\partial_\tau r)^2 + 1) - r^{-2}\{1 - \varepsilon [f(\rho)]^2\} = 0.$$

Cancelling r^{-2} and multiplying with $\partial_\tau r$, we have

$$\partial_\tau \{r[\partial_\tau r]^2\} = -\varepsilon [f(\rho)]^2 \partial_\tau r. \qquad \text{... (3.17)}$$

Integrating with respect to τ, we obtain

$$[\partial_\tau r]^2 = \left[\frac{\partial R}{\partial \tau}(\rho, \tau)\right]^2 = \frac{F(\rho)}{R(\rho, \tau)} - \varepsilon [f(\rho)]^2 \qquad \text{... (3.18}a\text{)}$$

and

$$F(\rho) \geq \varepsilon R(\rho, \tau)[f(\rho)]^2. \qquad \text{... (3.18}b\text{)}$$

Here, F is an arbitrary function of integration subject to the constraint (3.18b) and it is of class C_p^3.

By (3.16a) and (3.18a), we get

$$-\varepsilon [f(\rho)]^2 = e^{-\lambda}(\partial_\rho r)^2 - 1 = (\partial_\tau r)^2 - F(\rho)/r. \qquad \text{... (3.19)}$$

Therefore, by (3.19), we have

$$F(\rho) = r + r(\partial_\tau r)^2 - re^{-\lambda}(\partial_\rho r)^2 \qquad \text{... (3.20}a\text{)}$$

and

$$F'(\rho) = r^2 \partial_\rho r \{e^{-\lambda}((\partial_\rho \ln r)(\partial_\rho \lambda) - 2r^{-1}\partial_\rho^2 r - (\partial_\rho \ln r)^2) + (\partial_\tau \ln r)^2 + (\partial_\tau \ln r)(\partial_\tau \lambda) + r^{-2}\}$$

$$= (r^2 \partial_\rho r) G_4^4. \qquad \text{... (3.20}b\text{)}$$

Here and in the sequel, a prime denotes the total derivative. Now we can satisfy the field equation $\varepsilon_4^4 = 0$ in (3.10c) by defining

$$\mu(\rho, \tau) := \kappa^{-1} G_4^4 = \frac{F'(\rho)}{8\pi [R(\rho, \tau)]^2 \left[\frac{\partial R}{\partial \rho} (\rho, \tau) \right]}. \quad \ldots (3.21)$$

By the energy condition (2.6) and the equations (3.21) and (3.16*b*), we must have

$$F'(\rho) \left[\frac{\partial R}{\partial \rho} (\rho, \tau) \right]^{-1} \geq 0, \quad \ldots (3.22a)$$

and $$\mu(\rho, \tau) \equiv \Leftrightarrow F(\rho) = \text{constant}. \quad \ldots (3.22b)$$

Out of two logical possibilities to satisfy the weak inequality (3.22*a*), we select

$$F'(\rho) \geq 0, \quad \ldots (3.23a)$$

and $$\frac{\partial R}{\partial \rho} (\rho, \tau) > 0. \quad \ldots (3.23b)$$

In the interior of the collapsing star (corresponding to the interval $\rho_c < \rho < \rho_b$), we assume that the proper mass density μ is strictly positive. Therefore, by (3.21), (3.23*b*), (3.4*a*) and (3.18*b*), we obtain

$$F'(\rho) > 0. \quad \ldots (3.24a)$$

We also assume that

$$F(\rho_c+) := \lim_{\rho \downarrow \rho_c} F(\rho) = 0. \quad \ldots (3.24b)$$

We shall try to interpret physically the function F. We define the "total effective mass function" $M^{\#}$ by

$$M^{\#}(\rho, \tau) := \int_{\rho_c}^{\rho} \int_{0}^{\pi} \int_{-\pi}^{\pi} \mu(x, \tau) [R(x, \tau)]^2 \sin\theta \frac{\partial R}{\partial x} (x, \tau) \, d\phi \, dx \, dx \quad \ldots (3.25)$$

$$= \int_{0}^{R(\rho, \tau)} \int_{0}^{\pi} \int_{-\pi}^{\pi} \mu r^2 \sin\theta \, d\phi \, d\theta \, dr$$

Using (3.21) and (3.24*b*), the integral (3.25) can be evaluated as

$$M^{\#}(\rho, \tau) = \frac{1}{2} [F(\rho) - F(\rho_c+)] = \frac{1}{2} F(\rho) > 0, \quad \ldots (3.26a)$$

and $$\frac{\partial M^{\#}}{\partial \tau} (\rho, \tau) \equiv 0. \quad \ldots (3.26b)$$

Therefore, by (3.26*b*), the total mass of a spherical core corresponding to the interval (ρ_c, ρ) is considered. We can denote the total effective mass function[6] by $M(\rho) := M^{\#}(\rho, \tau_c)$, where τ_c is a constant, Thus, using (3.24*a*), (3.25), (3.26*a*), (3.17) and (3.18*a*), we may express

$$F(\rho) = 2M(\rho) > 0, \quad \ldots (3.27a)$$

$$\lim_{\rho \downarrow \rho_c} M(\rho) = \frac{1}{2} \lim_{\rho \downarrow \rho_c} F(\rho) = 0 \qquad \ldots (3.27b)$$

and

$$\frac{\partial^2 R}{\partial \tau^2}(\rho, \tau) = -\frac{M(\rho)}{[R(\rho, \tau)]^2} < 0. \qquad \ldots (3.27c)$$

The points corresponding to $\rho = \rho_c$ represent the world line of the centre of the star. The condition (3.27*b*) prevents a singularity from appearing at the centre before the final collapse. Also, notice that the equation (3.27*c*) strongly resembles the inverse square law of Newtonian gravitation. This is a glaring example of an equation of motion emerging from Einstein's field equations.

The invariant volume element of the spatial hypersurface inherent in the metric (3.1*a*) is given $e^{\lambda/2} r^2 \sin\theta \, d\rho \, d\phi \, d\theta$. We define the "total proper mass" function[6] M_p by

$$M_p(\rho) = \int_{\rho c}^{\rho} \int_{0}^{\pi} \int_{-\pi}^{\pi} \mu(x, \tau) \exp[\Lambda(x, \tau)/2] [R(x, \tau)]^2 \sin\theta \, d\phi \, d\theta \, dx \qquad \ldots (3.28)$$

$$= \frac{1}{2} \int_{\rho c}^{\rho} \frac{F'(x)}{\sqrt{1 - \varepsilon [f(x)]^2}} dx.$$

The "gravitational binding energy"[17] is given by[6]

$$M_p(\rho) - M(\rho) = \frac{1}{2} \int_{\rho c}^{\rho} \left[\frac{1}{\sqrt{1 - \varepsilon [f(x)]^2}} - 1 \right] F'(x) \, dx. \qquad \ldots (3.29)$$

The physical meaning of the arbitrary function f appearing in (3.16*a*) and (3.18*a*) can be understood by studying a time-like radial geodesic in the Schwarzschild metric. It is given by the ordinary differential equation[18]

$$\frac{1}{2} \left[\frac{dR}{ds}(s) \right]^2 - \frac{m}{R(s)} = \frac{1}{2} [E^2 - 1]. \qquad \ldots (3.30)$$

Here, s is the proper time parameter, m is the total mass of the spherically symmetric central body and E is the conserved total energy (including the rest energy) of a unit mass test particle pursuing the geodesic. The left-hand side of (3.30) has the resemblence of the "kinetic energy" plus the "potential energy" of the particle according to Newtonian physics. Eq. (3.18*a*) with (3.27*a*) at a particular value $\rho = \rho_0$ can be expressed as

$$\frac{1}{2} \left[\frac{\partial R}{\partial \tau}(\rho_0, \tau) \right]^2 - \frac{M(\rho_0)}{R(\rho_0, \tau)} = -\left(\frac{\varepsilon}{2} \right) [f(\rho_0)]^2 =: \frac{1}{2} \{ [E(\rho_0)]^2 - 1 \}. \qquad \ldots (3.31)$$

Eqs. (3.30) and (3.31) have uncanny resemblances. Therefore, we conclude that physically speaking, $\sqrt{1 - \varepsilon [f(\rho_0)]^2}$ represents the total energy $E(\rho_0)$ (including the rest energy) of a unit mass dust particle following the radial geodesics characterized by $\rho = \rho_0$. (We have ignored the negative root $-\sqrt{1 - \varepsilon [f(\rho_0)]^2}$). It can be noted that in eq. (3.31), the "potential energy" is contributed by

the spherical core corresponding to the interval (ρ_c, ρ_0). The total mass of the external spherical shell corresponding to the interval (ρ_0, ρ_b) does not affect the motion of the particle at $\rho = \rho_0$.

Let us investigate now the curvature of the metric in (3.1*a*). The corresponding "orthonormal" tetrad is given by

$$e^k_{(1)} = e^{-\lambda/2}\,\delta^k_{(1)},\ e^k_{(2)} = r^{-1}\,\delta^k_{(2)},\ e^k_{(3)} = r^{-1}(\sin\theta)^{-1}\,\delta^k_{(3)},\ e^k_{(4)} = \delta^k_{(4)}\,. \qquad \ldots (3.32)$$

The non-zero Riemann invariants of the metric (3.1*a*) relative to the tetrad in (3.32) are the following :

$$R_{(2)(3)(2)(3)} = r^{-2}\,[1 - e^{-\lambda}(\partial_\rho r)^2 + (\partial_\tau r)^2] = \frac{F(\rho)}{r^3} = \frac{2M(\rho)}{r^3}\,, \qquad \ldots (3.33a)$$

$$R_{(1)(2)(1)(2)} = \frac{e^{-\lambda}}{r}\left(\frac{1}{2}(\partial_\rho r)(\partial_\rho \lambda) - (\partial^2_\rho r)\right) + \frac{1}{2}(\partial_\tau \ln r)(\partial_\tau \lambda)$$

$$= \frac{\kappa}{2}\mu - \frac{M(\rho)}{r^3} = R_{(1)(3)(1)(3)} \qquad \ldots (3.33b)$$

$$R_{(1)(2)(2)(4)} = \frac{e^{-(\lambda/2)}}{r}\left(\partial_\rho \partial_\tau r - \frac{1}{2}(\partial_\rho r)(\partial_\tau \lambda)\right) \equiv 0 = R_{(1)(3)(3)(4)} \qquad \ldots (3.33c)$$

$$R_{(1)(4)(1)(4)} = -e^{-(\lambda/2)}\,\partial^2_\tau (e^{(\lambda/2)}) = \frac{\kappa}{2}\mu - \frac{2M(\rho)}{r^3}\,, \qquad \ldots (3.33d)$$

$$R_{(2)(4)(2)(4)} = -r^{-1}\,\partial^2_\tau r = \frac{M(\rho)}{r^3} = R_{(3)(4)(3)(4)} \qquad \ldots (3.33e)$$

and

$$R^{abcd}R_{abcd} = R^{(a)(b)(c)(d)}R_{(a)(b)(c)(d)}$$

$$= 3(\kappa\mu)^2 - 16\kappa\mu\,\frac{M(\rho)}{[R(\rho,\tau)]^3} + \frac{48[M(\rho)]^2}{R(\rho,\tau)]^6}\,. \qquad \ldots (3.33f)$$

For the reduction of the Riemann invariants, we have used (3.10*a*, *b*, *c*, *d*) and also the equations (3.16*a*), (3.18*a*), (3.21) and (3.27*a*, *c*). It is interesting to note that the Riemann invariants are algebraic functions of μ, M and r.

The non-zero conformal invariants are furnished by

$$C_{(1)(2)(1)(2)} = -\frac{1}{2}\,C(\rho,\tau) := \frac{\kappa\mu}{6} - \frac{M(\rho)}{r^3} = C_{(1)(3)(1)(3)} \qquad \ldots (3.34a)$$

$$C_{(1)(4)(1)(4)} = C(\rho,\tau) := \frac{\kappa\mu}{3} - \frac{2M(\rho)}{r^3}\,, \qquad \ldots (3.34b)$$

$$C_{(2)(3)(2)(3)} = C(\rho,\tau) = \frac{2M(\rho)}{r^3} - \frac{\kappa\mu}{3}\,, \qquad \ldots (3.34c)$$

$$C_{(2)(4)(2)(4)} = \frac{1}{2}\,C(\rho,\tau) = \frac{M(\rho)}{r^3} - \frac{\kappa\mu}{6} = C_{(3)(4)(3)(4)} \qquad \ldots (3.34d)$$

and $$C(\rho, \tau) = 1/12\, r^2\ \{2re^{-\lambda}[2(\partial_\rho^2 r) - (\partial_\rho r)(\partial_\rho \lambda) - 2r^{-1}(\partial_\rho r)^2] + 4 + 4(\partial_\tau r)^2$$

$$- 2r(\partial_\tau r)(\partial_\tau \lambda) - 4r\partial_\tau^2 r + 2r^2 \partial_\tau^2 \lambda + r^2 (\partial_\tau \lambda)^2\} \quad \ldots (3.34e)$$

$$= \frac{2M(\rho)}{r^3} - \frac{\kappa\mu}{3}.$$

When the metric (3.1*a*) is also conformally flat ($C_{abcd} \equiv 0$), we obtain from (3.34*a*, *b*, *c*, *d*) a single condition

$$\frac{M(\rho)}{r^3} - \frac{\kappa\mu}{6} = 0 \quad \ldots (3.35a)$$

or $$M(\rho) = \frac{4\pi}{3}\mu(\rho, \tau)\,[R(\rho, \tau)]^3. \quad \ldots (3.35b)$$

Eq. (3.35*b*) can be interpreted as "the total mass of the spherical core of dust is identical to the mass density at the boundary times the volume of the spherical core."

Using (3.21) and (3.27*a*), we get from (3.35*a*) the differential equation

$$\partial_\rho\ \{(1/3)\ \ln\ F(\rho) - \ln R(\rho, \tau)\} = 0. \quad \ldots (3.36)$$

Integrating the above equation, we have

$$R(\rho, \tau) = [F(\rho)]^{1/3}\, K(\tau) > 0, \quad \ldots (3.37)$$

where K is an arbitrary, positive valued function of class C_p^3. In (3.37), the variables are separated. Further computations show that, by (3.37) and (3.21), the mass density $\mu(\rho, \tau) = (3/\kappa)\,[K(\tau)]^{-3}$. Therefore, in this case, $\frac{\partial}{\partial\rho}\mu(\rho, \tau) \equiv 0$. The mass density is homogeneous. Substituting (3.37) and (3.16*a*) into the metric (3.1*a*), we obtain

$$ds^2 = [K(\tau)]^2 \left\{ \frac{\left[\frac{d}{d\rho}[F(\rho)]^{1/3}\right]^2}{1 - \varepsilon [f(\rho)]^2} d\rho^2 + [F(\rho)]^{2/3}\, d\Omega^2 \right\} - d\tau^2. \quad \ldots (3.38)$$

Making a coordinate transformation

$$\hat{\rho} = [F(\rho)]^{1/3}, \quad \hat{\tau} = \tau,$$

$$F'(\rho) > 0,$$

$$(\hat{\theta} = \theta, \hat{\phi} = \phi),$$

and $$\hat{f}(\hat{\rho}) := f(\rho), \quad \hat{K}(\hat{\tau}) := K(\tau),$$

we can reduce eq. (3.38) into

$$ds^2 = [\hat{K}(\hat{\tau})]^2 \left\{ \frac{d\hat{\rho}^2}{1 - \varepsilon [\hat{f}(\hat{\rho})]^2} + (\hat{\rho})^2\, d\hat{\Omega}^2 \right\} - d\hat{\tau}^2. \quad \ldots (3.39)$$

The three-dimensional metric defined by

$$dl^2 = \frac{d\hat{\rho}^2}{1-\varepsilon[\hat{f}(\hat{\rho})]^2} + (\hat{\rho})^2 d\hat{\Omega}^2 := \hat{g}_{\alpha\beta} d\hat{x}^\alpha d\hat{x}^\beta, \qquad \ldots (3.40)$$

has for its non-zero Ricci tensor components

$$\hat{R}_{11} = -\left\{\frac{2\varepsilon \hat{f}(\hat{\rho})\hat{f}'(\hat{\rho})}{\hat{\rho}}\right\}\hat{g}_{11},$$

$$\hat{R}_{22} = -\varepsilon \hat{f}(\hat{\rho})\left\{\frac{\hat{f}'(\hat{\rho})}{\hat{\rho}} + \frac{\hat{f}(\hat{\rho})}{\hat{\rho}^2}\right\}\hat{g}_{11}, \qquad \ldots (3.41)$$

and

$$\hat{R}_{33} = -\varepsilon \hat{f}(\hat{\rho})\left\{\frac{\hat{f}'(\hat{\rho})}{\hat{\rho}} + \frac{\hat{f}(\hat{\rho})}{\hat{\rho}^2}\right\}\hat{g}_{33}.$$

In case the three-geometry is locally an Einstein space or has locally constant curvature, the Ricci tensor must be of the form

$$\hat{R}_{\alpha\beta} = \frac{\hat{R}}{3}\hat{g}_{\alpha\beta}, \quad \text{or} \quad \frac{2\varepsilon \hat{f}(\hat{\rho})\hat{f}'(\hat{\rho})}{\hat{\rho}} = \varepsilon \hat{f}(\hat{\rho})\left\{\frac{\hat{f}'(\hat{\rho})}{\hat{\rho}} + \frac{\hat{f}(\hat{\rho})}{\hat{\rho}^2}\right\}. \qquad \ldots (3.42)$$

In the subcase $\varepsilon \hat{f}(\hat{\rho}) \equiv 0$, we have the Euclidean space. However, in the subcase $\varepsilon \hat{f}(\hat{\rho}) \neq 0$, the equation (3.42) leads to the differential equation,

$$\frac{\hat{f}'(\hat{\rho})}{\hat{f}(\hat{\rho})} = \frac{1}{\hat{\rho}}.$$

Integrating the above equation, we obtain

$$\hat{f}(\hat{\rho}) = c\hat{\rho}, \qquad \ldots (3.43)$$

where $c \neq 0$ is the constant of integration. Eqs. (3.43), (3.40) and (3.39) yield the well-known Friedmann-Robertson-Walker metrics. It should be noted that the metric (3.39), though it is conformally flat, is not necessarily a Friedmann-Robertson-Walker metric.

After this short digression, let us now go back to the problem of solving the field equations (3.10*a*, *b*, *c*, *d*). All the equations have been reduced to one remaining nonlinear, first order, second degree equation (3.18*a*). It implies two distinct first order partial order differential equations

$$\frac{\partial R}{\partial \tau}(\rho, \tau) = \pm\sqrt{\frac{F(\rho)}{R(\rho, \tau)} - \varepsilon[(\rho)]^2}. \qquad \ldots (3.44)$$

For the gravitational collapse of a fluid body, we choose the negative sign. (For the expanding phase of the cosmological model, we must choose the positive sign.) We shall consider the differential equation (3.44) in two-dimensional domains corresponding to the interior, exterior and the intermediate boundary (compare (3.5*a*, *b*, *c*)) given by

$$D_I := \{(\rho, \tau) : \rho_c < \rho < \rho_b, \ \tau_0(\rho) < \tau < \tau_1(\rho)\}, \qquad \ldots (3.45a)$$

$$D_E := \{(\rho, \tau) : \rho_b < \rho, \ \tau_0(\rho) < \tau < \tau_1(\rho)\} \qquad \ldots (3.45b)$$

and

$$\partial D := \{(\rho, \tau) : \rho = \rho_b, \ \tau_0(\rho_b) < \tau < \tau_1(\rho_b)\}. \qquad \ldots (3.45c)$$

The boundary ∂D can also be expressed as

$$\partial D := \{(\rho, \tau) : B(\rho, \tau) := \rho - \rho_b = 0, \quad \tau_0(\rho_b) < \tau < \tau_1(\rho_b)\}, \qquad \text{... (3.46}a\text{)}$$

$$\partial_\rho B \equiv 1, \ \partial_\tau B \equiv 0. \qquad \text{... (3.46}b\text{)}$$

The junction conditions (2.7) of Synge reduce to

$$\{[T_b^a]\, \partial_a B\}_{|_{\partial D}} = 0. \qquad \text{... (3.47)}$$

In the case of spherical symmetry, junction conditions (3.47) imply (by (3.46b)) the following nontrivial equations :

$$\{[T_1^1]\, \partial_\rho B + [T_1^4]\, \partial_\tau B\}_{|_{\partial D}} = \{[T_1^1]\}_{|_{\partial D}} = 0 \qquad \text{... (3.48}a\text{)}$$

and

$$\{[T_4^1]\, \partial_\rho B + [T_4^4]\, \partial_\tau B\}_{|_{\partial D}} = \{[T_4^1]\}_{|_{\partial D}} = 0. \qquad \text{... (3.48}b\text{)}$$

By (2.1b) and (3.9), the above conditions are identically satisfied.

Before we complete this section, we would like to mention a few physically relevant quantities for the metric (3.1a) and the velocity field in (3.9).

In any spherically symmetric flow, the vorticity tensor is identically zero. The absolute acceleration in case of a dust or incoherent fluid is also identically zero (due to (2.5)). Using (3.16a), the expansion and the rate of shear are respectively given by (Kramer *et al.*, 1983)[15]

$$\Theta = \partial_\tau \{\lambda/2 + 2 \ln r\} = \partial_\tau \{\ln (\partial_\rho [r^3])\} \qquad \text{... (3.49}a\text{)}$$

and

$$\sigma := \sigma_{(2)}^{(2)} = \sigma_{(3)}^{(3)} = -\frac{1}{2}\sigma_{(1)}^{(1)} = -\frac{1}{3}\partial_\tau \left\{ \ln \left(\frac{\partial_\rho r}{r} \right) \right\}. \qquad \text{... (3.49}b\text{)}$$

The metric with the solution can be written as

$$ds^2 = \frac{\left[\dfrac{\partial R}{\partial \rho}(\rho, \tau)\right]^2}{1 - \varepsilon [f(\rho)]^2} d\rho^2 + [R(\rho, \tau)]^2 d\Omega^2 - d\tau^2. \qquad \text{... (3.50)}$$

With a coordinate transformation

$$\hat{\rho} = f(\rho), \ f(\rho) \neq 0,$$

$$\hat{\tau} = \tau, \qquad \text{... (3.51)}$$

$$\hat{\theta} = \theta,$$

$$\hat{\phi} = \phi$$

and

$$\hat{R}(\hat{\rho}, \hat{\tau}) := R(\rho, \tau),$$

we can transform (3.50) into the reduced form

$$ds^2 = \frac{\left[\frac{\partial \hat{R}}{\partial \hat{\rho}}(\hat{\rho}, \hat{\tau})\right]^2}{1-\varepsilon\hat{\rho}^2} d\hat{\rho}^2 + [\hat{R}(\hat{\rho}, \hat{\tau})]^2 d\hat{\Omega}^2 - d\hat{\tau}^2. \qquad \text{... (3.52)}$$

However, the coordinate transformation (3.51) implies that f must be a differentiable mapping as well as a one-to-one mapping. Since the second condition restricts f considerably, we shall maintain in the sequel the explicit occurrence of f. (Another possible transformation[19] is $\hat{\rho} = \sqrt{[f(\rho)]^{-2} - 1}$).

The local solution of the p.d.e's (3.44) can be written in a unified fashion[3] as parametric equations

$$r = \frac{F(\rho)}{2[f(\rho)]^2} h'_\varepsilon(\eta),$$

$$\tau_0(\rho) - \tau = \pm \frac{F(\rho)}{2[f(\rho)]^3} h_\varepsilon(\eta),$$

$$h_\varepsilon(\eta) = \begin{cases} \eta^3/6 & \text{for } \varepsilon = 0, \\ \eta \pm \sin\eta & \text{for } \varepsilon = +1, \\ \sinh\eta - \eta & \text{for } \varepsilon = -1, \end{cases} \qquad \text{... (3.53)}$$

$$f(\rho) \neq 0$$

and $$r = 0 \Rightarrow F(\rho) = 0 \quad \text{or} \quad h'_\varepsilon(\eta) = 0.$$

For global or semi-global analysis, unified equations (3.53) are not convenient. By equations (3.24*a*, *b*, *c*), the only root of the equation $F(\rho) = 0$ in $[\rho_c, \rho_b)$ is $\rho = \rho_c$, which corresponds to the central world line.

If we want to express the line element (3.50) in terms of the unified solutions (3.53), then the equations become very complicated indeed. However, we note that

$$\frac{\partial R}{\partial \rho}(\rho, \tau)\, d\rho = dR(\rho, \tau) - \frac{\partial R}{\partial \tau}(\rho, \tau)\, d\tau$$

$$= dR(\rho, \tau) \pm \sqrt{\frac{F(\rho)}{R(\rho, \tau)} - \varepsilon[f(\rho)]^2}\, d\tau. \qquad \text{... (3.54)}$$

Introducing a coordinate transformation

$$\bar{r} = R(\rho, \tau), \quad \bar{\theta} = \theta, \quad \bar{\phi} = \phi, \quad \bar{\tau} = \tau,$$

$$\bar{F}(\bar{r}, \bar{\tau}) \equiv F(\rho), \quad \bar{f}(\bar{r}, \bar{\tau}) \equiv f(\rho),$$

and using the equation (3.54), the line element in (3.50) goes over into[26]

$$ds^2 = \frac{\left[d\bar{r} \pm \sqrt{\frac{\bar{F}(\bar{r}, \bar{\tau})}{\bar{r}} - \varepsilon[\bar{f}(\bar{r}, \bar{\tau})]^2}\, d\bar{\tau}\right]^2}{1 - \varepsilon[\bar{f}(\bar{r}, \bar{\tau})]^2} + \bar{r}^2 d\bar{\Omega}^2 - d\bar{\tau}^2 \qquad \text{... (3.55}a\text{)}$$

$$= \frac{\left[d\bar{r} \pm \sqrt{\frac{2\bar{M}(\bar{r},\bar{\tau})}{\bar{r}} + [\bar{E}(\bar{r},\bar{\tau})]^2 - 1}\, d\bar{\tau} \right]^2}{\bar{E}(\bar{r},\bar{\tau})} + \bar{r}^2 d\bar{\Omega}^2 - d\bar{\tau}^2. \qquad \text{... (3.55}b\text{)}$$

In the case $\varepsilon = 0$, at three-dimensional spacelike hypersurface given by $\bar{\tau} = k$ (a constant), has the induced line element from (3.55*a*, *b*) as

$$ds^2_{|_{\bar{\tau}=k}} = (d\bar{r})^2 + (\bar{r})^2 (d\bar{\Omega})^2, \qquad \text{... (3.55}c\text{)}$$

and it is a Euclidean metric ! This result is the best Einstein's equation can do for his Equivalence Principle. Consider another three-dimensional[10] hypersurface given by $R(\rho, \tau) = k$, a constant. By equations (3.50), (3.16*a*) and (3.18*a*), we obtain

$$g^{ab} (\partial_a r) (\partial_b r) = e^{-\lambda} (\partial_\rho r)^2 - (\partial_\tau r)^2 = 1 - \frac{2M(\rho)}{R(\rho, \tau)}. \qquad \text{... (3.56)}$$

Therefore, it follows that the hypersurface $R(\rho, \tau) = k$ is timelike or else spacelike according as it is inside the domain $R(\rho, \tau) > 2M(\rho)$ or else it is inside the domain $R(\rho, \tau) < 2M(\rho)$. The surface $R(\rho, \tau) = 2M(\rho)$ indicates an apparent horizon.

Hellaby and Lake[13] have derived some conditions for preventing criss-crossing of spherical shells in the collapsing incoherent fluid. In the (parabolic) case $\varepsilon = 0$, the conditions are

$$\tau_0'(\rho) \geq 0, \quad F'(\rho) \geq 0, \qquad \text{... (3.57)}$$

but not $\tau_0'(\rho) = F'(\rho) = 0$. In the (hyperbolic) case $\varepsilon = -1$, the conditions are

$$\tau_0'(\rho) \geq 0, \quad F'(\rho) \geq 0, f(\rho) f'(\rho) \geq 0, \qquad \text{... (3.58)}$$

but no more than two equalities at once. In the (elliptic) case $\varepsilon = +1$, the conditions are

$$0 \leq \tau_0'(\rho) \leq \frac{\pi F(\rho)}{[f(\rho)]^3} \left[\frac{F'(\rho)}{F(\rho)} - \frac{3f'(\rho)}{f(\rho)} \right]$$

and $$F'(\rho) \geq 0, \qquad \text{... (3.59)}$$

but no more than two equalities at once. We shall satisfy these conditions in the sequel.

In the next three sections, we shall solve the equation (3.44) for the distinct cases respectively $\varepsilon = 0, +1, -1$ (which are called parabolic, elliptic and hyperbolic). The resulting metric tensor components will be analyzed in detail.

4. The Parabolic Case $\varepsilon = 0$

In this case, by (3.31),

$$\frac{1}{2} \left(\frac{\partial R}{\partial \tau} (\rho_0, \tau) \right)^2 - \frac{M(\rho_0)}{R(\rho_0, \tau)} \equiv 0.$$

Therefore, the "kinetic energy" plus the "potential energy" of a dust particle is exactly zero. Thus dust particles are "coasting".

Eq. (3.44) for the collapsing dust particles is given by

$$\frac{\partial R}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F(\rho)}{R(\rho, \tau)}} \cdot \qquad \ldots (4.1)$$

By (3.4*a*) and (3.24*a*), the right hand side of (4.1) is defined. Integrating (4.1) with respect to τ, we obtain

$$r = R(\rho, \tau) = \left(\frac{3}{2}\right)^{2/3} [F(\rho)]^{1/3} (\tau_1(\rho) - \tau)^{2/3} \qquad \ldots (4.2a)$$

and

$$\tau_1'(\rho) > 0, \qquad \ldots (4.2b)$$

where τ_1 is the arbitrary function of integration and must be of class C_p^3. Note that from (4.2*a*), we get

$$\lim_{\tau \to -\infty} R(\rho, \tau) \to \infty$$

and

$$\lim_{\tau \to -\infty} \frac{\partial R}{\partial \tau}(\rho, \tau) = 0.$$

Moreover, from (4.2*a*), we have

$$R(\rho, \tau) = 0 \quad \Rightarrow \quad F(\rho) = 0, \qquad \ldots (4.3)$$

or, $\tau = \tau_1(\rho)$.

By assumptions (3.24*a*, *b*, *c*), $\rho = \rho_c$ is the only root of the equation $F(\rho) = 0$ in $[\rho_c, \rho_b)$.

We will compute some physically relevant quantities from the solution in (4.2*a*). Using eqs. (3.21), (3.49*a*, *b*) and (4.2*a*, *b*), we obtain

$$\kappa\mu(\rho, \tau) = \frac{4F'(\rho)}{3[F'(\rho)(\tau_1(\rho) - \tau) + 2F(\rho)\tau_1'(\rho)](\tau_1(\rho) - \tau)} > 0, \qquad \ldots (4.4a)$$

$$\lim_{\rho \downarrow \rho_c} \kappa\mu(\rho, \tau) = \frac{4}{3}(\tau_1(\rho) - \tau)^{-2} > 0, \qquad \ldots (4.4b)$$

$$\lim_{\tau \to -\infty} \mu(\rho, \tau) = 0, \quad \lim_{\tau \uparrow \tau_1(\rho)} \mu(\rho, \tau) \to \infty, \qquad \ldots (4.4c)$$

$$\Theta = -\frac{2}{(\tau_1(\rho) - \tau)} \left(\frac{F'(\rho)(\tau_1(\rho) - \tau) + F(\rho)\tau_1'(\rho)}{F'(\rho)(\tau_1(\rho) - \tau) + 2F(\rho)\tau_1'(\rho)} \right) < 0, \qquad \ldots (4.4d)$$

$$\lim_{\tau \to -\infty} \Theta = 0, \quad \lim_{\tau \uparrow \tau_1(\rho)} \Theta \to \infty, \qquad \ldots (4.4e)$$

$$\sigma := \sigma_{(2)}^{(2)} = -\frac{2}{3(\tau_1(\rho) - \tau)} \left(\frac{F(\rho)\tau_1'(\rho)}{F'(\rho)(\tau_1(\rho) - \tau) + 2F(\rho)\tau_1'(\rho)} \right) < 0 \qquad \ldots (4.4f)$$

and

$$\lim_{\tau \to -\infty} \sigma = 0, \qquad \lim_{\tau \uparrow \tau_1(\rho)} \sigma \to \infty. \qquad \ldots (4.4g)$$

The interior, the exterior and the boundary of the star are taken to correspond (compare eqs. (3.45*a*, *b*, *c*)) to

$$D_I := \{(\rho, \tau) : \rho_c < \rho < \rho_b, \ -\infty < \tau < \tau_1(\rho)\}, \qquad \ldots (4.5a)$$

$$D_E := \{(\rho, \tau) : \rho_b < \rho, \ -\infty < \tau < \tau_1(\rho)\} \qquad \ldots (4.5b)$$

and

$$\partial D := \{(\rho, \tau) : \rho = \rho_b, \ -\infty < \tau < \tau_1(\rho_b)\}. \qquad \ldots (4.5c)$$

The functions F and τ_1 occurring in the metric tensor components are of class C_p^3 in the unbounded domain $D_I \cup D_E \cup \partial D$. However, jump discontinuities are permissible on ∂D. We shall express the very smooth pieces of these function by the following equations

$$F(\rho) = \begin{cases} F_I(\rho) & \text{for } \rho_c < \rho < \rho_b, \\ F_E(\rho) & \text{for } \rho_b < \rho, \end{cases} \qquad \ldots (4.6a)$$

$$\tau_1(\rho) = \begin{cases} \tau_I(\rho) & \text{for } \rho_c < \rho < \rho_b, \\ \tau_E(\rho) & \text{for } \rho_b < \rho, \end{cases} \qquad \ldots (4.6b)$$

and

$$R(\rho, \tau) = \begin{cases} R_I(\rho, \tau) & \text{for } (\rho, \tau) \in D_I, \\ R_E(\rho, \tau) & \text{for } (\rho, \tau) \in D_E. \end{cases} \qquad \ldots (4.6c)$$

In the exterior domain, we take the mass density $\mu(\rho, \tau) \equiv 0$. Therefore, by eq. (3.22*b*),

$$F_E(\rho) = 2m > 0. \qquad \ldots (4.7)$$

Here, m is a positive constant. The total mass function M, by (3.26*a*), (4.6*a*) and (4.7), is given by

$$M(\rho) = \frac{F(\rho)}{2} = \begin{cases} \frac{1}{2} F_I(\rho) & \text{for } \rho_c < \rho < \rho_b, \\ m & \text{for } \rho_b < \rho. \end{cases} \qquad \ldots (4.8)$$

Therefore, $m > 0$ stands for the total (Schwarzschild) mass of the star.

Now, we shall try to match continuously the metric tensor components and their first partial derivatives across the boundary ∂D. Consider a function H belonging to the class C_p^3 in $D_I \cup D_E$. The jump across the the boundary ∂D is defined in the usual manner as

$$[\Delta H(\rho_b, \tau)] := \lim_{\rho \downarrow \rho_b} H(\rho, \tau) - \lim_{\rho \uparrow \rho_b} H(\rho, \tau)$$

$$\equiv H(\rho_b{}^+, \tau) - H(\rho_b{}^-, \tau) \qquad \ldots (4.9)$$

for all $-\infty < \tau < \tau_1(\rho_b)$. In the case

$$[\Delta H(\rho_b, \tau)] \equiv 0, \qquad \ldots (4.10)$$

the function H is said to have a removable discontinuity on ∂D. In the sequel, a removable discontinuity will always be eliminated by definitions like

$$H(\rho_b, \tau) := \lim_{\rho \downarrow \rho_b} H(\rho, \tau) = \lim_{\rho \uparrow \rho_b} H(\rho, \tau). \qquad \ldots (4.11)$$

With (4.10) and (4.11), the function H is now continuous across the boundary.

The interior and the exterior metric tensor components are summarized (using (3.1*a*), (3.4*a*), (3.16*a*), (3.23*b*), $\varepsilon = 0$ and (4.6*b*)) by

$$ds_I^2 = \left[\frac{\partial R_I}{\partial \rho}(\rho, \tau)\right]^2 d\rho^2 + [R_I(\rho, \tau)]^2 d\Omega^2 - d\tau^2, \qquad \ldots (4.12a)$$

$$ds_E^2 = \left[\frac{\partial R_E}{\partial \rho}(\rho, \tau)\right]^2 d\rho^2 + [R_E(\rho, \tau)]^2 d\Omega^2 - d\tau^2, \qquad \ldots (4.12b)$$

$$0 < R_I(\rho, \tau) < R_E(\rho, \tau) \qquad \ldots (4.12c)$$

and

$$\frac{\partial R_I}{\partial \rho}(\rho, \tau) > 0, \quad \frac{\partial R_E}{\partial \rho}(\rho, \tau) > 0. \qquad \ldots (4.12d)$$

By (4.2*a*) and (4.6*a*, *b*, *c*) and denoting a total derivative by a prime, we obtain

$$R_I(\rho, \tau) = \left(\frac{3}{2}\right)^{2/3} [F_I(\rho)]^{1/3} (\tau_I(\rho) - \tau)^{2/3} > 0, \qquad \ldots (4.13a)$$

$$R_E(\rho, \tau) = \left(\frac{3}{2}\right)^{2/3} [2m]^{1/3} (\tau_E(\rho) - \tau)^{2/3} > 0, \qquad \ldots (4.13b)$$

$$\frac{\partial R_I}{\partial \rho}(\rho, \tau) = \frac{1}{3} R_I(\rho, \tau) \left([\ln F_I(\rho)]' + \frac{2\tau_I'(\rho)}{\tau_1(\rho) - \tau} \right) \quad 0, \qquad \ldots (4.13c)$$

$$\frac{\partial R_E}{\partial \rho}(\rho, \tau) = \frac{2}{3} R_E(\rho, \tau) \left(\frac{\tau_E'(\rho)}{\tau_E(\rho) - \tau} \right) > 0. \qquad \ldots (4.13d)$$

$$\frac{\partial R_I}{\partial \tau}(\rho, \tau) = -\frac{2}{3} \frac{R_I(\rho, \tau)}{(\tau_I(\rho) - \tau)} < 0, \qquad \ldots (4.13e)$$

$$\frac{\partial R_E}{\partial \tau}(\rho, \tau) = -\frac{2}{3} \left(\frac{R_E(\rho, \tau)}{\tau_E(\rho) - \tau} \right) < 0, \qquad \ldots (4.13f)$$

$$\frac{\partial^2 R_I}{\partial \rho^2}(\rho, \tau) = \frac{1}{3} R_I(\rho, \tau) \left([\ln F_I(\rho)]'' - 2\left[\frac{\tau_I'(\rho)}{\tau_I(\rho) - \tau} \right]^2 + \frac{2\tau_I''(\rho)}{\tau_I(\rho) - \tau} \right)$$
$$+ \frac{1}{3} \frac{\partial R_I}{\partial \rho}(\rho, \tau) \left([\ln F_I(\rho)]' + \frac{2\tau_I''(\rho)}{\tau_I(\rho) - \tau} \right), \quad \ldots (4.13g)$$

$$\frac{\partial^2 R_E}{\partial \rho^2}(\rho, \tau) = \frac{2}{3} R_E(\rho, \tau) \left(\frac{2\tau_E''(\rho)}{\tau_E(\rho) - \tau} - \left[\frac{\tau_E'(\rho)}{\tau_E(\rho) - \tau} \right]^2 \right)$$
$$+ \frac{2}{3} \left(\frac{\partial R_I}{\partial \rho}(\rho, \tau) \frac{\tau_E'(\rho)}{\tau_I(\rho) - \tau} \right), \quad \ldots (4.13h)$$

$$\frac{\partial^2 R_I}{\partial \tau \partial \rho}(\rho, \tau) = \frac{1}{3} \frac{\partial R_I}{\partial \tau}(\rho, \tau) \left([\ln F_I(\rho)]' + \frac{2\tau_I'(\rho)}{\tau_I(\rho) - \tau} \right) + \frac{2}{3} \frac{R_I(\rho, \tau)\, \tau_I'(\rho)}{(\tau_I(\rho) - \tau)^2} \quad \ldots (4.13i)$$

and

$$\frac{\partial^2 R_E}{\partial \tau \partial \rho}(\rho, \tau) = \frac{2}{3} \frac{\partial R_E}{\partial \tau}(\rho, \tau) \left(\frac{\tau_E'(\rho)}{\tau_E(\rho) - \tau} \right) + \frac{2}{3} \frac{R_E(\rho, \tau)\, \tau_E'(\rho)}{(\tau_E(\rho) - \tau)^2}. \quad \ldots (4.13j)$$

The continuity of the metric tensor component g_{22} across the boundary ∂D implies (on account of (4.12*a*, *b*), (4.9) and (4.10)) that

$$[\Delta R(\rho_b, \tau)] := \lim_{\rho \uparrow \rho_b} [R_I(\rho, \tau)] - \lim_{\rho \downarrow \rho_b} [R_E(\rho, \tau)] \equiv 0. \quad \ldots (4.14)$$

Using (4.13*a*, *b*), the condition (4.14) implies

$$\sqrt{F_I(\rho_b -)}\, [\tau_I(\rho_b -) - \tau] \equiv \sqrt{2m}\, [\tau_E(\rho_b +) - \tau]. \quad \ldots (4.15)$$

Differentiating (4.15) with respect to τ, and recalling (4.6*a*) and (4.8), we have

$$F_I(\rho_b -) = 2m = F_E(\rho_b +). \quad \ldots (4.16)$$

By (4.8) and (3.24*a*, *b*, *c*), the condition (4.16) implies that the total mass function M attains its only extremum in the interval $(\rho_c, \rho_b]$ at the boundary point $\rho = \rho_b$ and its maximum value is m. This conclusion is physically reasonable.

Substituting (4.16) in (4.15), we get

$$\tau_I(\rho_b -) = \tau_E(\rho_b +) := \tau_1(\rho_b). \quad \ldots (4.17)$$

At this stage, the functions F and τ_1 are continuous across the boundary.

Now, let us investigate implications of continuities of the first partial derivatives of g_{22}. By (4.12*a*, *b*) and (4.14), the continuity $\partial_\tau g_{22}$ demands that

$$\lim_{\rho \uparrow \rho_b} \left[\frac{\partial R_I}{\partial \tau} (\rho, \tau) \right] - \lim_{\rho \downarrow \rho_b} \left[\frac{\partial R_E}{\partial \tau} (\rho, \tau) \right] \equiv 0. \qquad \ldots (4.18)$$

By (4.13*a*, *b*), (4.16), (4.17) and (4.13*e*, *f*), the above conditions are automatically satisfied. Therefore, by (4.18), the radial "velocities" along radial geodesics across the boundary of the star are continuous.

For the implications of the continuity of $\partial_\rho g_{22}$, we have to explore the condition

$$\lim_{\rho \uparrow \rho_b} \left[\frac{\partial R_I}{\partial \rho} (\rho, \tau) \right] - \lim_{\rho \downarrow \rho_b} \left[\frac{\partial R_E}{\partial \rho} (\rho, \tau) \right] \equiv 0. \qquad \ldots (4.19)$$

By (4.13*a*, *c*, *d*), (4.14), (4.16) and (4.17), we derive from (4.19)

$$[\tau_1(\rho_b) - \tau] \lim_{\rho \uparrow \rho_b} [\ln F_I(\rho)]' + 2 \left(\lim_{\rho \uparrow \rho_b} [\tau_I'(\rho)] - \lim_{\rho \downarrow \rho_b} [\tau_E'(\rho)] \right) \equiv 0. \qquad \ldots (4.20)$$

Differentiating with respect to τ and using (3.21) and (3.22*b*), we obtain

$$\lim_{\rho \uparrow \rho_b} F_I'(\rho) = 0, \qquad \ldots (4.21a)$$

$$\lim_{\rho \uparrow \rho_b} \mu(\rho, \tau) \equiv 0 \qquad \ldots (4.21b)$$

and $$\mu(\rho, \tau) \equiv \text{ for } \rho_b \leq \rho. \qquad \ldots (4.21c)$$

Conditions (4.21*a*, *b*, *c*) and (4.7) make the functions F' and μ continuous across the boundary. Substituting (4.21*a*) into (4.20), we get

$$\lim_{\rho \uparrow \rho_b} [\tau_I'(\rho)] - \lim_{\rho \downarrow \rho_b} [\tau_E'(\rho)] \equiv 0. \qquad \ldots (4.22)$$

Therefore, by (4.22), the function τ_1' is also continuous across the boundary.

Now, by (4.12*a*, *b*), the continuity of g_{11} implies the continuity of $\frac{\partial R}{\partial \rho}(\rho, \tau)$. From (4.19), we conclude that no additional information emerges. The continuity of $\partial_\rho g_{11}$ implies by (4.12*a*, *b*) and (4.19) that

$$\lim_{\rho \uparrow \rho_b} \left[\frac{\partial^2 R_I}{\partial \rho^2} (\rho, \tau) \right] - \lim_{\rho \downarrow \rho_b} \left[\frac{\partial^2 R_E}{\partial \rho^2} (\rho, \tau) \right] \equiv 0. \qquad \ldots (4.23)$$

By (4.13*g*, *h*), (4.14), (4.19), (4.16), (4.17), (4.21*a*) and (4.22), we get

$$(\tau_1(\rho_b) - \tau) \left(\lim_{\rho \uparrow \rho_b} [\ln F_I(\rho)]'' \right) + 2 \left(\lim_{\rho \uparrow \rho_b} [\tau_I''(\rho)] - \lim_{\rho \downarrow \rho_b} [\tau_E''(\rho)] \right) \equiv 0. \qquad \ldots (4.24)$$

Differentiating the above with respect to τ, we have

$$\lim_{\rho \uparrow \rho_b} [\ln F_I(\rho)]'' = 0. \qquad \ldots (4.25)$$

Substituting (4.25) into (4.24), we obtain

$$\lim_{\rho \uparrow \rho_b} [\tau_I''(\rho)] - \lim_{\rho \downarrow \rho_b} [\tau_E''(\rho)] \equiv 0. \qquad \ldots (4.26)$$

We conclude from (4.25) and (4.26) that the functions F'' and τ_1'' are continuous across the boundary. The continuity of $\partial_\tau g_{22}$ does not yield new conditions because of equations (4.13*i*, *j*), (4.14), (4.16), (4.17), (4.18), (4.21*a*) and (4.22). The continuities of g_{33} are equivalent to those of g_{22}. The continuities of $g_{44} = -1$ are trivial. Therefore, we have proved that the continuities of the metric tensor components and of their first partial derivatives imply the continuities of $F, F', F'', \tau_1, \tau_1'$ and τ_1''. Conversely, we can prove that the continuities of $F, F', F'', \tau_1, \tau_1'$ and τ_1'' imply the continuities of the metric tensor components and their first partial derivatives across the boundary ∂D.

The metric in equations (4.12*a*, *b*) can be expressed with the help of equations (4.13*c*, *d*) as

$$ds_I^2 = [R_I(\rho, \tau)]^2 \left((1/9) \left[\frac{F_I'(\rho)}{F_I(\rho)} + \frac{2\tau_I'(\rho)}{\tau_I(\rho) - \tau} \right]^2 d\rho^2 + d\Omega^2 \right) - d\tau^2, \qquad \ldots (4.27a)$$

and

$$ds_E^2 = \left(\frac{4m}{3(\tau_E(\rho) - \tau)} \right)^{2/3} [\tau_E'(\rho)]^2 d\rho^2 + \left(\frac{3}{2} \sqrt{2m}\, (\tau_E(\rho) - \tau) \right)^{4/3} d\Omega^2 - d\tau^2. \qquad \ldots (4.27b)$$

(Recall from the equation (4.2*b*), $\tau_1'(\rho) > 0$, $\tau_I'(\rho) > 0$, $\tau_E'(\rho) > 0$.)

Introducing another coordinate system (see Fig. 2) by

$$\rho^\# = \tau_1(\rho), \qquad \ldots (4.28a)$$

$$\tau^\# = \tau, \qquad \ldots (4.28b)$$

$$\frac{\partial(\rho^\#, \tau^\#)}{\partial(\rho, \tau)} = \tau_1'(\rho) > 0, \quad (\rho, \tau) \in D_I \cup D_E \cup \partial D, \qquad \ldots (4.28c)$$

$$D^\# := D_I^\# \cup D_E^\# \cup \partial D^\# \qquad \ldots (4.28d)$$

$$D_I^\# := \{(\rho^\#, \tau^\#) : \rho_b^\# < \rho^\# < \rho_b^\#, \ -\infty < \tau^\# < \rho^\#\}, \qquad \ldots (4.28e)$$

$$D_E^\# := \{(\rho^\#, \tau^\#) : \rho_b^\# < \rho^\#, \ -\infty < \tau^\# < \rho^\#\}, \qquad \ldots (4.28f)$$

$$\partial D^{\#} := \{(\rho^{\#}, \tau^{\#}) : \rho^{\#} = \rho_b^{\#},\ -\infty < \tau^{\#} < \rho_b^{\#}\} \qquad \ldots (4.28g)$$

and
$$\rho_c^{\#} := \tau_1(\rho_c),\ \rho_b^{\#} := \tau_1(\rho_b). \qquad \ldots (4.28h)$$

In this transformation, $\theta^{\#} = \theta$ and $\phi^{\#} = \phi$ and thus play passive roles. The line elements (4.27*a*, *b*) transform (by use of (4.28*a-h*) and (4.13*a*, *b*, *c*, *d*)) into

$$ds_I^2 = \frac{2M^{\#}(\rho^{\#})}{\left[\frac{3}{2}\sqrt{2M^{\#}(\rho^{\#})}\,(\rho^{\#} - \tau^{\#})\right]^{2/3}}\left(1 + [\ln\sqrt{2M^{\#}(\rho^{\#})}]'\,(\rho^{\#} - \tau^{\#})\right)^2 (d\rho^{\#})^2$$

$$+ \left[\frac{3}{2}\sqrt{2M^{\#}(\rho^{\#})}\,(\rho^{\#} - \tau^{\#})\right]^{4/3} (d\Omega^{\#})^2 - (d\tau^{\#})^2, \qquad \ldots (4.29a)$$

$$2M^{\#}(\rho^{\#}) \equiv F^{\#}(\rho^{\#}) := F(\rho) \qquad \ldots (4.29b)$$

and
$$ds_E^2 = \frac{2m}{[(3/2)\sqrt{2m}\,(\rho^{\#} - \tau^{\#})]^{2/3}}(d\rho^{\#})^2 + \left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{4/3}(d\Omega^{\#})^2 - (d\tau^{\#})^2.$$

$$\ldots (4.29c)$$

The metric (4.29*a*) will be called the interior Lemaître metric. The line element (4.29*c*) is the well-known Lemaître metric[3] for the spherically symmetric vacuum solution.

There is a real singularity of the exterior metric at $\rho^{\#} = \tau^{\#}$, which can be concluded from the Kretchmann scalar in (3.33*f*). However, the metric is defined and differentiable on the hypersurface $[(3/2)\sqrt{2m}\,(\rho^{\#} - \tau^{\#})]^{2/3} = 2m$. Moreover, on this hypersurface (which is called the event horizon), the metric (4.29*c*) reduces to

$$ds_{EH}^2 = (2m)^2 (d\Omega^{\#})^2.$$

There is a loss of dimensionality in the above expression indicating that the hypersurface is null.

Now, we shall derive the Schwarzschild metric from (4.29*c*). We perform another coordinate transformation[3] defined by the equations

$$\hat{r} = \hat{R}(\rho^{\#}, \tau^{\#}) := \left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{2}{3}}, \qquad \ldots (4.30a)$$

$$\hat{t} = \hat{T}(\rho^{\#}, \tau^{\#}) := \tau^{\#} - 2\sqrt{2m}\left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{1}{3}}, \qquad \ldots (4.30b)$$

$$+ 2m \ln\left[\frac{\left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{1}{3}} + \sqrt{2m}}{\left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{1}{3}} - \sqrt{2m}}\right],$$

$$\frac{\partial(\hat{r},\hat{t})}{\partial(\rho^{\#},\tau^{\#})}=\left[\frac{4m}{3(\rho^{\#}-\tau^{\#})}\right]^{\frac{1}{3}}=\sqrt{\frac{2m}{\hat{r}}}>0, \qquad \ldots (4.30c)$$

$$\rho^{\#}=\mathcal{P}^{\#}(\hat{r},\hat{t}):=\hat{t}+2\sqrt{2m\hat{r}}-2m\ln\left[\frac{\sqrt{\hat{r}}+\sqrt{2m}}{\sqrt{\hat{r}}-\sqrt{2m}}\right]+\frac{2}{3}\left(\frac{r^3}{2m}\right)^{\frac{1}{2}} \qquad \ldots (4.30d)$$

$$\tau^{\#}=\mathcal{T}^{\#}(\hat{r},\hat{t}):=\hat{t}+2\sqrt{2m\hat{r}}-2m\ln\left[\frac{\sqrt{\hat{r}}+\sqrt{2m}}{\sqrt{\hat{r}}-\sqrt{2m}}\right], \qquad \ldots (4.30e)$$

$$\frac{\partial\mathcal{P}^{\#}}{\partial\hat{r}}(\hat{r},\hat{t})=\sqrt{\frac{\hat{r}}{2m}}\left(1-\frac{2m}{\hat{r}}\right)^{-1}, \qquad \ldots (4.30f)$$

$$\frac{\partial\mathcal{T}^{\#}}{\partial\hat{r}}(\hat{r},\hat{t})=\sqrt{\frac{2m}{\hat{r}}}\left(1-\frac{2m}{\hat{r}}\right)^{-1}, \qquad \ldots (4.30g)$$

$$d\rho^{\#}=d\hat{t}\sqrt{\frac{\hat{r}}{2m}}\left(1-\frac{2m}{\hat{r}}\right)^{-1}d\hat{r} \qquad \ldots (4.30h)$$

and

$$d\tau^{\#}=d\hat{t}+\sqrt{\frac{2m}{\hat{r}}}\left(1-\frac{2m}{\hat{r}}\right)^{-1}d\hat{r}. \qquad \ldots (4.30i)$$

In this transformation, $\hat{\theta}=\theta^{\#}$, $\hat{\phi}=\phi^{\#}$, and thus play passive roles. It can be noted that the proper time along a class of radial geodesics[3] in the Schwarzschild universe satisfy eq. (4.30*e*). By eqs. (4.30*b*, *d*, *e*, *f*, *g*, *h*, *i*), it is clear that the transformation is undefined at $\hat{r}=2m$ or $\rho^{\#}-\tau^{\#}=(4/3)m$, although the Jacobian (4.30c) is deceptively regular there.

It will be instructive to study the function in (4.30*a*) defined by

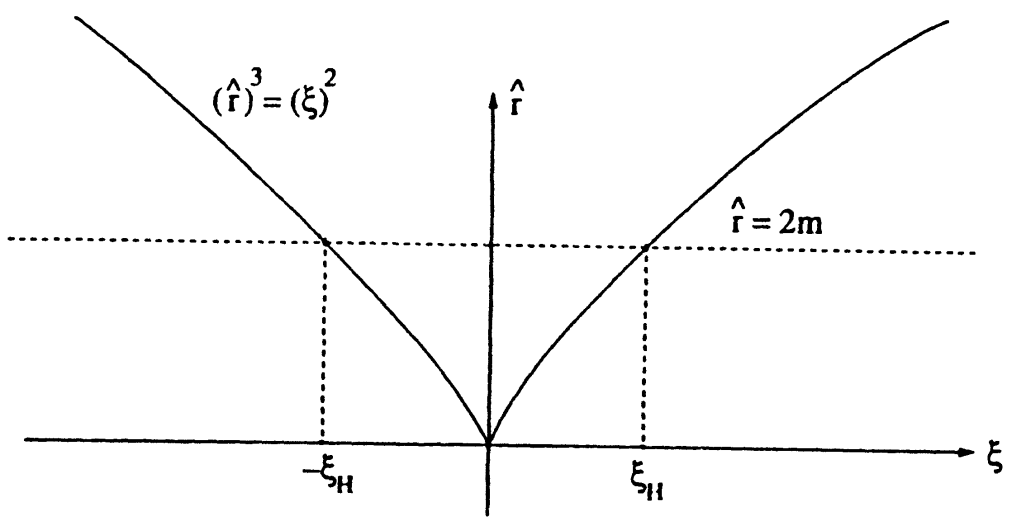

Fig. 1 Qualitative graph of $(\hat{r})^3=(\xi)^2$

$$\hat{r} = [(3/2)\sqrt{2m}\,(\rho^{\#} - \tau^{\#})]^{2/3} =: (\xi)^{2/3},$$

$$(\hat{r}) = (\xi)^2, \qquad \ldots (4.31)$$

$$\xi \in I\!R.$$

The graph of the above function is a semi-cubical parabola shown in Fig. 1. The function in (4.31) is even, continuous and piecewise analytic in $I\!R$. Moreover, the derivative is undefined at the origin. The horizontal graph of $\hat{r} = 2m$ intersects the cubical parabola at two points $\xi = \pm \xi_H$, $\xi_H := (2m)^{3/2}$. By (4.31), the point ξ_H corresponds to $\rho^{\#} - \tau^{\#} = (4/3)m$. The line $\rho^{\#} - \tau^{\#} = (4/3)m$ indicates the event horizon (see Fig. 2). Our coordinate transformation must be restricted to $\hat{r} > 2m$ or, $\rho^{\#} - \tau^{\#} > (4/3)m$. However, from Fig. 1, we would speculate about four different external domains of vacuum metrics corresponding to (*i*) $\xi > (2m)^{3/2}$, (*ii*) $0 < \xi < (2m)^{3/2}$, (*iii*) $-(2m)^{3/2} < \xi < 0$ and (*iv*) $\xi < (2m)^{3/2}$. These domains correspond to domains I, II, III and IV of the Kruskal universe[1-3, 17, 18].

For physical insight in regard to Fig. 1, let us suppose that from the surface of the earth, a massive ball is vertically ejected exactly with the escape velocity. The height of the centre of mass of the ball plotted against time, according to Newtonian physics, will trace a curve just like that in the right half of Fig. 1. The time reversed phenomenon will be depicted by the left half of the diagram.

After this minor digression, we go back to the transformation (4.30*a*, *b*) and the inverse transformation (4.30*d*, *e*) and specify the appropriate domains and ranges :

$$D_S^{\#} := \left\{ (\rho^{\#}, \tau^{\#}) : \rho_b < \rho^{\#}, \frac{4}{3} m < \rho^{\#} - \tau^{\#} \right\} \subset D_E^{\#} \qquad \ldots (4.32a)$$

and

$$\hat{D}_S := \{(\hat{r}, \hat{t}) : \rho_b^{\#} < \hat{t} + 2\sqrt{2m\hat{r}} + (2/3)(\hat{r}^3/2m)^{1/2} - 2m \ln [(\sqrt{\hat{r}} + \sqrt{2m})/(\sqrt{\hat{r}} - \sqrt{2m})]\ \hat{t} \in I\!R\}. \qquad \ldots (4.32b)$$

We should note from (4.30*b*) that in the limit as $\rho^{\#} - \tau^{\#} \to [(4/3)m]_+$ or $\hat{r} \to (2m)_+$, the time $\hat{t} \to \infty$. Therefore, this chart is unable to describe the events on the event horizon $\hat{r} = 2m$.

From (4.29*c*), (4.30*h*) and (4.30*i*), we obtain

$$ds_E^2 = \left(\frac{2m}{\hat{r}}\right)\left[d\hat{t} + \sqrt{\frac{\hat{r}}{2m}}\left(1 - \frac{2m}{\hat{r}}\right)^{-1} d\hat{r}\right] + \hat{r}^2 d\hat{\Omega}^2 - \left[d\hat{t} + \sqrt{\frac{2m}{\hat{r}}}\left(1 - \frac{2m}{\hat{r}}\right)^{-1} d\hat{r}\right]^2$$

$$= \left(1 - \frac{2m}{\hat{r}}\right)^{-1} (d\hat{r})^2 + \hat{r}^2 d\hat{\Omega}^2 - \left(1 - \frac{2m}{\hat{r}}\right)^{-1} (d\hat{t})^2. \qquad \ldots (4.33)$$

This is of course the Schwarzschild metric.

There exists another coordinate chart (corresponding to $0 < \xi < (2m)^{3/2}$ in Fig. 1) from a domain $D_T^{\#} \subset D_E^{\#}$. The range of this mapping is denoted $\hat{D}_T$. This transformation is furnished by the following equations :

$$X = \tau^{\#} - 2\sqrt{2m}\left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{1}{3}} + 2m \ln\left[\frac{\sqrt{2m} + \left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{1}{3}}}{\sqrt{2m} - \left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{1}{3}}}\right], \quad \text{... (4.34}a\text{)}$$

$$T = \left[\frac{3}{2}\sqrt{2m}\,(\rho^{\#} - \tau^{\#})\right]^{\frac{2}{3}}, \quad \text{... (4.34}b\text{)}$$

$$\frac{\partial(X, T)}{\partial(\rho^{\#}, \tau^{\#})} = \sqrt{\frac{2m}{T}} < 0, \quad \text{... (4.34}c\text{)}$$

$$\rho^{\#} = X + 2\sqrt{2mT} - 2m \ln\left[\frac{\sqrt{2m} + \sqrt{T}}{\sqrt{2m} - \sqrt{T}}\right] + \frac{2}{3}\left(\frac{T^3}{2m}\right)^{\frac{1}{2}}, \quad \text{... (4.34}d\text{)}$$

$$\tau^{\#} = X + 2\sqrt{2mT} - 2m \ln\left[\frac{\sqrt{2m} + \sqrt{T}}{\sqrt{2m} - \sqrt{T}}\right], \quad \text{... (4.34}e\text{)}$$

$$d\rho^{\#} = dX - \sqrt{\frac{T}{2m}}\left(\frac{2m}{T} - 1\right)^{-1} dT, \quad \text{... (4.34}f\text{)}$$

$$d\tau^{\#} = dX - \sqrt{\frac{2m}{T}}\left(\frac{2m}{T} - 1\right)^{-1} dT, \quad \text{... (4.34}g\text{)}$$

$$D_T^{\#} := \{(\rho^{\#}, \tau^{\#}) : \rho_b^{\#} < \rho^{\#},\ 0 < \rho^{\#} - \tau^{\#} < (4/3)m\} \quad \text{... (4.34}h\text{)}$$

and

$$\hat{D}_T := \{(X, T) : \rho_b^{\#} < X + 2\sqrt{2mT} + (2/3)\,(T^3/2m)^{1/2} - 2m \ln\,[(\sqrt{2m} + \sqrt{T})/(\sqrt{2m} - \sqrt{T})],\ 0 < T < 2m\}. \quad \text{... (4.34}i\text{)}$$

Substituting (4.34*f*, *g*) into the Lemaître metric (4.29*c*), we obtain

$$ds_E^2 = \left(\frac{2m}{T} - 1\right)dX^2 + T^2\,d\Omega^2 - \left(\frac{2m}{T} - 1\right)^{-1} dT^2. \quad \text{... (4.35)}$$

The above metric describes the matter-free part of the interior of a spherically symmetric classical black hole known as the T-domain[20-22].

Synge[23] introduced doubly-null coordinates to extend the spherically symmetric vacuum metric beyond the Schwarzschild patch. His approach was completed by Kruskal[24] and Szekeres[25]. We shall introduce the Kruskal-Synge-Szekeres (KSS in short) doubly-null coordinate chart now. The transformation from the Schwarzschild coordinates to the KSS coordinates is given by

$$u = U(\hat{r}, \hat{t}) := -\sqrt{\frac{\hat{r}}{2m} - 1}\, \exp\left(\frac{\hat{r} - \hat{t}}{4m}\right) < 0, \qquad \ldots (4.36a)$$

$$v = V(\hat{r}, \hat{t}) := -\sqrt{\frac{\hat{r}}{2m} - 1}\, \exp\left(\frac{\hat{r} + \hat{t}}{4m}\right) > 0, \qquad \ldots (4.36b)$$

$$\frac{\partial(u, v)}{\partial(\hat{r}, \hat{t})} = -\frac{\hat{r}}{16m^3} \exp\left(\frac{\hat{r}}{2m}\right) < 0, \qquad \ldots (4.36c)$$

$$\hat{r} = Y(uv) > 2m > 0, \qquad \ldots (4.36d)$$

$$\left(\frac{\hat{r}}{2m} - 1\right) \exp\left(\frac{\hat{r}}{2m}\right) = \left(\frac{Y(uv)}{2m} - 1\right) \exp\left(\frac{U(uv)}{2m}\right) = -uv = |u|v > 0, \qquad \ldots (4.36e)$$

$$\hat{t} = \hat{T}(v/u) := 2m \ln [-v/u], \qquad \ldots (4.36f)$$

$$(4m)\, u\, du = \exp\left[\frac{\hat{r} - \hat{t}}{2m}\right]\left(\frac{\hat{r}}{2m} d\hat{r} - \left(\frac{\hat{r}}{2m} - 1\right) d\hat{t}\right), \qquad \ldots (4.36g)$$

$$(4m)\, v\, dv = \exp\left[\frac{\hat{r} + \hat{t}}{2m}\right]\left(\frac{\hat{r}}{2m} d\hat{r} + \left(\frac{\hat{r}}{2m} - 1\right) d\hat{t}\right) \qquad \ldots (4.36h)$$

and

$$D_{K_I} := \{(u, v) : u < 0,\ v > 0,\ \rho_b^{\#} < 2m \ln [-v/u] + \sqrt{2mY(uv)} + (2/3)\sqrt{[Y(uv)]^3/2m} - 2m \ln [(\sqrt{Y(uv)} + \sqrt{2m})/(\sqrt{Y(uv)} - \sqrt{2m})]\}. \qquad \ldots (4.36i)$$

The function Y in (4.36*e*) is expressed implicitly by necessity.

The Schwarzschild line element (4.33) transforms under (4.36*a*, *b*, *g*, *h*) into

$$ds_E^2 = -\frac{32m^2}{Y(uv)} \exp\left(\frac{-Y(uv)}{2m}\right) du dv + [Y(uv)]^2\, d\Omega^2. \qquad \ldots (4.37)$$

This is the well-known vacuum metric in KSS coordinates. There is another transformation from the black hole metric (4.35) in domain $\hat{D}_T$ into the KSS chart. This transformation is given by

$$u = \hat{U}(X, T) := \sqrt{1 - \frac{T}{2m}}\, \exp\left(\frac{T - X}{4m}\right) > 0, \qquad \ldots (4.38a)$$

$$v = \hat{V}(X, T) := \sqrt{1 - \frac{T}{2m}}\, \exp\left(\frac{T + X}{4m}\right) > 0, \qquad \ldots (4.38b)$$

$$\frac{\partial(u, v)}{\partial(X, T)} = \frac{T}{16m^3} \exp\left(\frac{T}{2m}\right) > 0, \qquad \ldots (4.38c)$$

$$X = 2m \ln [v/u], \qquad \ldots (4.38d)$$

$$T = \hat{Y}(uv) > 0, \qquad \ldots (4.38e)$$

$$\left(1-\frac{T}{2m}\right)\exp\left(\frac{T}{2m}\right)=\left(1-\frac{\hat{Y}(uv)}{2m}\right)\exp\left(\frac{\hat{Y}(uv)}{2m}\right)=uv=|u|v>0, \quad \ldots (4.38f)$$

$$(4m)\,u\,du=-\exp\left(\frac{T-X}{2m}\right)\left(\left(1-\frac{T}{2m}\right)dS+\frac{T}{2m}dT\right), \quad \ldots (4.38g)$$

$$(4m)\,v\,dv=\exp\left(\frac{T+X}{2m}\right)\left(\left(1-\frac{T}{2m}\right)dS-\frac{T}{2m}dT\right), \quad \ldots (4.38h)$$

$$D_{K_{II}}:=\{(u,v):0<u,\ 0<v,\ \rho_b^{\#}<2m\ln\left[\frac{v}{u}\right]+2\sqrt{2m\,\hat{Y}(uv)}+(2/3)\sqrt{[\hat{Y}(uv)]^3/2m}$$
$$-2m\ln[(\sqrt{2m}+\sqrt{\hat{Y}(uv)})/(\sqrt{2m}-\sqrt{\hat{Y}(uv)})]\} \quad \ldots (4.38i)$$

and
$$ds_E^2=-\frac{32m^2}{\hat{Y}(uv)}\exp\left(\frac{-\hat{Y}(uv)}{2m}\right)dudv+[\hat{Y}(uv)]^2\,d\Omega^2. \quad \ldots (4.38j)$$

We note that the defining equations (4.36*e*) and (4.38*f*) for the functions Y and $\hat{Y}$ are exactly the same. Only the corresponding domains of definition D_{K_I} and $D_{K_{II}}$ are different. Moreover, the metric tensor components in (4.37) in D_{K_I} and (4.38*j*) in $D_{K_{II}}$ are identical. Therefore, it is reasonable to look for an analytical extension $\tilde{Y}$ which will cover both domains D_{K_I} and $D_{K_{II}}$ as well as the intermediate boundary points ($\{i.b.\}$ for short).

We notice for the boundary points that

$$\lim_{u\uparrow 0}Y(uv)=\lim_{u\downarrow 0}\hat{Y}(uv)=2m.$$

Furthermore,

$$\lim_{u\uparrow 0}\left|\frac{u}{\sqrt{Y(uv)}-\sqrt{2m}}\right|=\lim_{u\downarrow 0}\left|\frac{u}{\sqrt{\hat{Y}(uv)}-\sqrt{2m}}\right|=\frac{2e}{\sqrt{2m\,|v|}}.$$

Therefore, we can define the extended function $\tilde{Y}$ in the following manner :

$$\tilde{Y}(uv)_{|_{D_{K_I}}}:=Y(uv), \quad \ldots (4.39a)$$

$$\tilde{Y}(uv)_{|_{D_{K_{II}}}}:=\tilde{Y}(uv), \quad \ldots (4.39b)$$

$$\tilde{Y}(uv)_{|\{i.b.\}}:=2m, \quad \ldots (4.39c)$$

$$\left|\frac{\tilde{Y}(uv)}{2m}-1\right|\exp\left[\frac{\tilde{Y}(uv)}{2m}\right]:=|u|v\geq 0, \quad \ldots (4.39d)$$

$$\left|\frac{u}{\sqrt{\tilde{Y}(uv)}-\sqrt{2m}}\right|_{|_{u=0}}:=\frac{2e}{\sqrt{2m\,|v|}}, \quad \ldots (4.39e)$$

$$D_{K_I}\cup D_{K_{II}}\cup\{i.b.\}:=\left\{(u,v):v>0,\ uv<1,\ \rho_b^{\#}<2\sqrt{2m\,\tilde{Y}(uv)}+\frac{2}{3}\sqrt{\frac{[\tilde{Y}(uv)]}{2m}}\right.$$

$$\{ - 2m \ln | [u (\sqrt{2m} + \sqrt{\tilde{Y}(uv)})]/[u (\sqrt{2m} + \sqrt{\tilde{Y}(uv)})] | \} , \quad \ldots (4.39f)$$

and

$$ds_E^2 = - \frac{32m^2}{\tilde{Y}(uv)} \exp\left(\frac{- \tilde{Y}(uv)}{2m} \right) dudv + [\tilde{Y}(uv)]^2 \, d\Omega^2. \quad \ldots (4.39g)$$

The above KSS vacuum metric is regular for all $\tilde{Y}(uv) > 0$.

Now, we shall qualitatively depict the phenomenon of a stellar dust body of type $\varepsilon = 0$ collapsing from infinity into a black hole by five charts in Fig. 2.

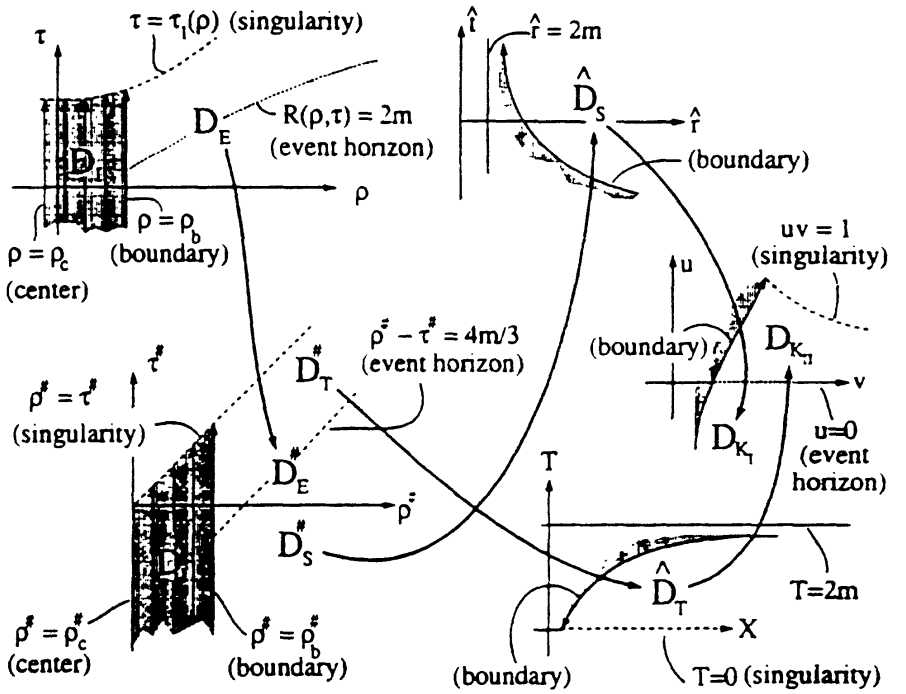

Fig. 2. Gravitational collapse for the parabolic case $\varepsilon = 0$ shown qualitatively in 5 coordinate charts

5. The Elliptic Case $\varepsilon = +1$

Eq. (3.31) yields

$$\frac{1}{2}\left[\frac{\partial R}{\partial \tau}(\rho_0, \tau) \right]^2 - \frac{M(\rho_0)}{R(\rho_0, \tau)} = - \frac{1}{2} [f(\rho_0)]^2 < 0, \quad \ldots (5.1a)$$

and

$$f(\rho_0) \neq 0. \quad \ldots (5.1b)$$

In this case, the absolute value of the "potential energy" exceeds the "kinetic energy". Eq. (3.44) for the gravitational collapse becomes

$$\frac{\partial R}{\partial \tau}(\rho, \tau) = - \sqrt{\frac{F(\rho)}{R(\rho, \tau)} - [f(\rho)]^2} < 0, \quad \ldots (5.2a)$$

and $$0 < [f(\rho)]^2 < \frac{F(\rho)}{R(\rho, \tau)}. \qquad \text{... (5.2}b\text{)}$$

We can conclude from the inequality (5.2*b*) that

$$\lim_{\rho \downarrow \rho_c} F(\rho) = 0 \Rightarrow \lim_{\rho \downarrow \rho_c} R(\rho, \tau) \equiv 0. \qquad \text{... (5.3)}$$

Every solution is (5.2*a*) can define a curve in the ρ-τ plane by the equation

$$\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} - 1 = 0.$$

On this curve, $\frac{\partial R}{\partial \tau}(\rho, \tau) \equiv 0$. If we take this curve as an initial boundary of our domain of consideration $\tau = \tau_0(\rho)$ (see the equations (3.5*a*, *b*, *c*) and (5.6)), then all collapsing dust particles start from rest at finite proper times. Therefore, out of the three cases considered in this review, this case is the most pertinent in describing the collapse of a spherically symmetric star into a black hole. If we choose a particular $\rho_0 \in (\rho_c, \rho_b]$, then by the inequality (5.2*a*) and the first mean value theorem, we have

$$R(\rho_0, \tau) = R(\rho_0, \tau_0(\rho_0)) + (\tau - \tau_0(\rho_0)) \left[\frac{\partial R}{\partial \tau}(\rho_0, \tau) \right]_{\tau_0(\rho_0) + \theta(\tau - \tau_0(\rho_0))} < R(\rho_0, \tau_0(\rho_0)),$$

where $(0 < \theta < 1)$. Therefore, $R(\rho_0, \tau_0(\rho_0))$ is the maximum value of $R(\rho_0, \tau)$ for $\tau \in (\tau_1(\rho_0), \tau_0(\rho_0)]$.

Eqs. (3.16*a*), (3.23*b*) and (5.1*b*) imply that

$$\exp[\Lambda(\rho, \tau)] = \frac{\left[\frac{\partial R}{\partial \rho}(\rho, \tau) \right]^2}{1 - [f(\rho)]^2} > 0, \qquad \text{... (5.4}a\text{)}$$

$$\frac{\partial R}{\partial \rho}(\rho, \tau) > 0 \qquad \text{... (5.4}b\text{)}$$

and $$0 < |f(\rho)| < 1. \qquad \text{... (5.4}c\text{)}$$

By (3.29), (3.24*a*) and (5.4*c*), the "gravitational binding energy" is

$$M_p(\rho) - M(\rho) = \frac{1}{2} \int_{\rho_c^+}^{\rho} \left[\frac{1}{\sqrt{1 - [f(x)]^2}} - 1 \right] F'(x)\, dx > 0. \qquad \text{... (5.5)}$$

The partial differential equation (5.2*a*) can be regarded as an ordinary differential equation; the result of integrating with respect to τ is

$$\tau - \tau_0(\rho) = \frac{R(\rho, \tau)}{|f(\rho)|} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} - 1} + \frac{F(\rho)}{|f(\rho)|^3} \operatorname{Arctan} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} - 1}. \qquad \text{... (5.6)}$$

The arbitrary function τ_0 of class C^3_{pp} arises out of the integration. (The expression under the root sign is always positive by virtue of the inequality (5.2*b*).) The principal branch of the arctan function will be chosen in the sequel without loss of generality. The function R is defined implicitly by necessity.

The interior, exterior and the boundary of the star are assumed to be corresponding to the following domains (compare to equations (4.5*a*, *b*, *c*))

$$D_I := \{(\rho, \tau) : \rho_c < \rho < \rho_b, \quad \tau_0(\rho) < \tau < \tau_1(\rho)\}, \qquad \ldots (5.7a)$$

$$D_E := \{(\rho, \tau) : \rho_b < \rho, \quad \tau_0(\rho) < \tau < \tau_1(\rho)\}, \qquad \ldots (5.7b)$$

$$\partial D := \{(\rho, \tau) : \rho = \rho_b, \quad \tau_0(\rho_b) < \tau < \tau_1(\rho_b)\} \qquad \ldots (5.7c)$$

and

$$\tau_1(\rho) := \tau_0(\rho) + \frac{\pi E(\rho)}{2|f(\rho)|^3}. \qquad \ldots (5.7d)$$

The choice of τ_1 will be essentially explained in eq. (5.33*e*). Note also that D_I is a bounded domain.

Since differentiation of the absolute value function is essentially complicated, we make a simplifying assumption

$$0 < f(\rho) < 1. \qquad \ldots (5.8)$$

We shall denote very, very smooth branches of the functions F, τ_0 and R exactly as in (4.6*a*, *b*, *c*). We shall use a similar notation for the function f. Moreover, eqs. (4.7), (4.8), (3.27*a*) and (3.24*a*) hold. The interior and exterior line elements are given by

$$ds_I^2 = \frac{\left[\dfrac{\partial R_I}{\partial \rho}(\rho, \tau)\right]^2}{1 - [f(\rho)]^2} d\rho^2 + [R_I(\rho, \tau)]^2 \, d\Omega^2 - d\tau^2, \qquad \ldots (5.9a)$$

and

$$ds_E^2 = \frac{\left[\dfrac{\partial R_E}{\partial \rho}(\rho, \tau)\right]^2}{1 - [f_E(\rho)]^2} d\rho^2 + [R_E(\rho, \tau)]^2 \, d\Omega^2 - d\tau^2. \qquad \ldots (5.9b)$$

From (5.6), the interior and exterior branches of the function R satisfy

$$\tau - \tau_I(\rho) = \frac{R_I(\rho, \tau)}{f_I(\rho)} W_I(\rho, \tau) + \frac{F_I(\rho)}{[f_I(\rho)]^3} \operatorname{Arctan}[W_I(\rho, \tau)], \qquad \ldots (5.10a)$$

$$\tau - \tau_E(\rho) = \frac{R_E(\rho, \tau)}{f_E(\rho)} W_E(\rho, \tau) + \frac{2m}{[f_E(\rho)]^3} \operatorname{Arctan}[W_E(\rho, \tau)], \qquad \ldots (5.10b)$$

$$\frac{\partial R_I}{\partial \rho}(\rho, \tau) = \left(\frac{F_I'(\rho)}{F_I(\rho)} - \frac{2f_I'(\rho)}{f_I(\rho)} \right) R_I(\rho, \tau)$$

$$+ W_I(\rho, \tau) \left\{ \tau_I'(\rho) + \left(\frac{F_I'(\rho)}{F_I(\rho)} - \frac{3f_I'(\rho)}{f_I(\rho)} \right) (\tau - \tau_I(\rho)) \right\}$$

$$= \frac{f_I'(\rho)}{f_I(\rho)} R_I(\rho, \tau) + f_I(\rho)\, \tau_I'(\rho)\, W_I(\rho, \tau) + \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho) f_I'(\rho)}{[f_I(\rho)]^3} \right)$$

$$\{1 + W_I(\rho, \tau) \text{ Arctan } [W_I(\rho, \tau)], \quad \ldots (5.10c)$$

$$\frac{\partial R_E}{\partial \rho}(\rho, \tau) = \frac{f_E'(\rho)}{f_E(\rho)} R_E(\rho, \tau) + f_E(\rho)\, \tau_E'\, W_E(\rho, \tau)$$

$$- \frac{6m f_E'(\rho)}{[f_E(\rho)]^3} \{1 + W_E(\rho, \tau) \text{ Arctan } [W_E(\rho, \tau)]\}, \quad \ldots (5.10d)$$

$$\frac{\partial R_I}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F_I(\rho)}{R_I(\rho, \tau)} - [f_I(\rho)]^2} < 0, \quad \ldots (5.10e)$$

$$\frac{\partial R_E}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{2m}{R_E(\rho, \tau)} - [f_E(\rho)]^2} < 0, \quad \ldots (5.10f)$$

$$\frac{\partial^2 R_I}{\partial \tau\, \partial \rho}(\rho, \tau) = \frac{1}{2} \left(\frac{F_I'(\rho)}{R_I(\rho, \tau)} - \frac{F_I(\rho) \dfrac{\partial R_I}{\partial \rho}(\rho, \tau)}{[R_I(\rho, \tau)]^2} - 2 f_I(\rho) f_I'(\rho) \right) \left[\frac{\partial R_I}{\partial \tau}(\rho, \tau) \right]^{-1} \quad \ldots (5.10g)$$

$$\frac{\partial^2 R_E}{\partial \tau\, \partial \rho}(\rho, \tau) = -\left(\frac{m}{[R_E(\rho, \tau)]^2} \frac{\partial R_E}{\partial \rho}(\rho, \tau) + f_E(\rho) f_E'(\rho) \right) \left[\frac{\partial R_E}{\partial \tau}(\rho, \tau) \right]^{-1}, \quad \ldots (5.10h)$$

$$\frac{\partial^2 R_I}{\partial \rho^2}(\rho, \tau) = [\ln f_I(\rho)]' \frac{\partial R_I}{\partial \rho}(\rho, \tau) + R_I(\rho, \tau) [\ln f_I(\rho)]''$$

$$+ \frac{1}{2} \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho) f_I'(\rho)}{[f_I(\rho)]^3} \right) \left(\left[\ln \frac{F_I'(\rho)}{[f_I(\rho)]^2} \right]' - \frac{\dfrac{\partial R_I}{\partial \rho}(\rho, \tau)}{R_I(\rho, \tau)} \right)$$

$$+ \frac{1}{2} \frac{F_I(\rho)}{2[f_I(\rho)]^2 R_I(\rho, \tau) W_I(\rho, \tau)} \left(\left[\ln \frac{F_I'(\rho)}{[f_I(\rho)]^2} \right]' - \frac{\dfrac{\partial R_I}{\partial \rho}(\rho, \tau)}{R_I(\rho, \tau)} \right)$$

$$\times \left(f_I(\rho)\, \dot{\tau}_I(\rho) + \left(\frac{\dot{F}_I(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho)\, \dot{f}_I(\rho)}{[f_I(\rho)]^3} \right) \text{Arctan}\, [W_I(\rho, \tau)] \right)$$

$$+ W_I(\rho, \tau)\, [f_I(\rho)\, \dot{\tau}_I(\rho)]' + (1 + W_I(\rho, \tau)\, \text{Arctan}\, [W_I(\rho, \tau)])$$

$$\times \left(\frac{\ddot{F}_I(\rho)}{[f_I(\rho)]^2} - 5\frac{\dot{F}_I(\rho)\, \dot{f}_I(\rho)}{[f_I(\rho)]^3} - 3\frac{F_I(\rho)\, \ddot{f}_I(\rho)}{[f_I(\rho)]^3} + 9\frac{F_I(\rho)\, [\dot{f}_I(\rho)]^2}{[f_I(\rho)]^4} \right) \quad \ldots (5.10i)$$

$$\frac{\partial^2 R_E}{\partial \rho^2}(\rho, \tau) = [\ln f_E(\rho)]' \frac{\partial R_E}{\partial \rho}(\rho, \tau) + R_E(\rho, \tau)\, [\ln f_E(\rho)]''$$

$$+ \frac{3m\dot{f}_E(\rho)}{[f_E(\rho)]^3} \left(2\, [\ln f_E(\rho)]' + \frac{\frac{\partial R_E}{\partial \rho}(\rho, \tau)}{R_E(\rho, \tau)} \right) + W_E(\rho, \tau)\, [f_E(\rho)\, \dot{\tau}_E(\rho)]'$$

$$- \frac{m}{[f_E(\rho)]^3\, R_E(\rho, \tau)\, W_W(\rho, \tau)} \left(2\, [\ln f_E(\rho)]' + \frac{\frac{\partial R_E}{\partial \rho}(\rho, \tau)}{R_E(\rho, \tau)} \right)$$

$$\times \left(f_E(\rho)\, \dot{\tau}_E(\rho) - \frac{6m\dot{f}_E(\rho)}{[f_E(\rho)]^3} \text{Arctan}\, [W_E(\rho, \tau)] \right)$$

$$- 6m\, (1 + W_E(\rho, \tau)\, \text{Arctan}\, [W_E(\rho, \tau)]) \left(\frac{\ddot{f}_E(\rho)}{[f_E(\rho)]^3} - 3\frac{[\dot{f}_E(\rho)]^2}{[f_E(\rho)]^4} \right) \quad \ldots (5.10j)$$

$$W_I(\rho', \tau) := -\sqrt{\frac{F_I(\rho)}{[f_I(\rho)]^2\, R_I(\rho, \tau)} - 1} \quad \ldots (5.10k)$$

and

$$W_E(\rho, \tau) := \sqrt{\frac{2m}{[f_E(\rho)]^2\, R_E(\rho, \tau)} - 1}\,. \quad \ldots (5.10l)$$

Let us now consider the continuities of the metric tensor components (5.9*a*, *b*) and of their first partial derivatives. We recall that the continuity of g_{22} implies (compare with eq. (4.14))

$$R_E(\rho_b{}^+, \tau) - R_I(\rho_b{}^-, \tau) \equiv 0. \quad \ldots (5.11)$$

The consequence of (5.11) on (5.10*a*, *b*) are hard to elicit and will be found later. The continuity of $\partial_\tau g_{22}$ together with (5.9*a*, *b*), (5.11) and (5.10*e*, *f*), implies that

$$\left[\Delta \left(\frac{\partial R}{\partial \tau}(\rho_b, \tau) \right)^2 \right] \equiv 0. \quad \ldots (5.12a)$$

and

$$\frac{[F_f(\rho_b^- - 2m]}{R(\rho_b, \tau)} \equiv [\Delta(f(\rho_b))^2]. \qquad \ldots (5.12b)$$

Differentiating (5.12*b*) with respect to τ, recalling that $\frac{\partial R}{\partial \tau}(\rho, \tau) < 0$ and using (5.12*b*) again, we obtain

$$F_f(\rho_b^-) - 2m = 0, \qquad \ldots (5.13a)$$

and

$$[\Delta f(\rho_b)] = 0. \qquad \ldots (5.13b)$$

Therefore, the functions F and f are continuous across the boundary. Continuities of R, F, and f imply by (5.10*a*, *b*) that

$$[\Delta \tau_0(\rho_b)] = 0. \qquad \ldots (5.14)$$

(It can be noted that the contituities of F, f and τ_0 imply the continuity of R). The continuity of $\partial_\rho g_{22}$ implies by (5.9*a*, *b*), (5.11) and (5.10*a*, *b*) that

$$\left[\Delta\left(\frac{\partial R}{\partial \tau}(\rho_b, \tau)\right)\right] \equiv 0, \qquad \ldots (5.15a)$$

$$\{f(\rho_b)\,[\Delta \tau_0'(\rho_b)]\}\; w(\tau) + \left\{\frac{2m[\Delta f(\rho_b)]}{[f(\rho_b)]^3}\right\}\{1 + [w(\tau)]^2\}^{-1}$$

$$+ \left\{\frac{[\Delta F'(\rho_b)]}{[f(\rho_b)]^2} - \frac{6m[\Delta f(\rho_b)]}{[f(\rho_b)]^3}\right\}\{1 + w(\tau)\ \text{Arctan}\ [w(\tau)]\} \equiv 0 \qquad \ldots (5.15b)$$

and

$$w(\tau) := -\sqrt{\frac{2m}{[f(\rho_b)]^2 R(\rho_b, \tau)} - 1}\ . \qquad \ldots (5.15c)$$

Now, the Wronskian of the set of functions $\{\theta, (1+\theta^2)^{-1}, 1 + \theta\ \text{Arctan}\ \theta\}$ in an interval is gvien by

$$\begin{vmatrix} \theta & (1+\theta^2)^{-1} & 1 + \theta\,\text{Arctan}\theta \\ 1 & -2\theta(1+\theta^2)^{-2} & \text{Arctan}\theta + \theta(1+\theta^2)^{-1} \\ 0 & -2(1-3\theta^2)\,(1+\theta^2)^{-3} & 2(1+\theta^2)^{-2} \end{vmatrix} = -\,4(1+\theta^2)^{-4} < 0.$$

Therefore, the set of functions $\{w(\tau), (1+[w(\tau)]^2)^{-1}, 1 + w(\tau)\ \text{Arctan}[w(\tau)]\}$ is linearly independent in the interval $\tau_0(\rho) < \tau < \tau_1(\rho)$. It follows from (5.15*b*) that the coefficients of the independent functions must all be zero :

$$f(\rho_b)\,[\Delta \tau_0'(\rho_b)] = 0,$$

$$\frac{2m[\Delta f(\rho_{b)}]}{[f(\rho_b)]^3} = 0$$

and $$\frac{[\Delta F'(\rho_b)]}{[f(\rho_b)]^2} - \frac{6m[\Delta f(\rho_b)]}{[f(\rho_b)]^3} = 0.$$

The above equations imply the three independent continuities :

$$[\Delta \dot{\tau}_0(\rho_b)] = 0, \qquad \ldots (5.16a)$$

$$[\Delta f(\rho_b)] = 0, \qquad \ldots (5.16b)$$

and $$[\Delta F'(\rho_b)] = 0. \qquad \ldots (5.16c)$$

Now, by (5.9*a*, *b*) and (5.13*b*), the continuity of g_{11} implies the continuity of $\frac{\partial R}{\partial \rho}(\rho, \tau)$. By (5.15*a*), no other additional information emerges. The continuity of $\partial_\tau g_{11}$ implies the continuity of $\frac{\partial^2 R}{\partial \tau \partial \rho}(\rho, \tau)$. By (5.10*g*, *h*), (5.11), (5.13*a*, *b*), (5.15*a*) and (5.16*b*, *c*), no other new equation arises. The continuity of $\partial_\rho g_{11}$ implies by (5.9*a*, *b*) and (5.15*a*) that

$$\left[\Delta\left(\frac{\partial^2 R}{\partial \rho^2}(\rho_b, \tau)\right)\right] \equiv 0. \qquad \ldots (5.17)$$

By (5.10*i*, *j*), (5.11), (5.13*a*, *b*), (5.15*a*) and (5.16*a*, *b*, *c*), we obtain from (5.17)

$$2m\frac{[\Delta f'(\rho_b)]}{[f(\rho_b)]^3}\{1 + [w(\tau)]^2\}^{-1} + f(\rho_b)[\Delta \ddot{\tau}_0(\rho_b)]\, w(\tau)$$

$$+ \frac{1}{[f(\rho_b)]^2}\left(F''(\rho_b^-) - 6m\frac{[\Delta f'(\rho_b)]}{f(\rho_b)}\right)\{1 + w(\tau)\,\mathrm{Arctan}[w(\tau)]\} \equiv 0 \qquad \ldots (5.18)$$

Since the set of functions $\{w(\tau), (1 + [w(\tau)]^2)^{-1}, 1 + w(\tau)\,\mathrm{Arctan}[w(\tau)]\}$ is linearly independent in any interval, the identity (5.18) yields

$$F''(\rho_b^-) = 0. \qquad \ldots (5.19a)$$

$$[\Delta f'(\rho_b)] = 0 \qquad \ldots (5.19b)$$

and $$[\Delta \ddot{\tau}_0(\rho_b)] = 0. \qquad \ldots (5.19c)$$

Therefore, we have proved that the continuities of the metric tensor components in (5.9*a*, *b*) and of their first partial derivatives imply that the functions $F, f, \tau_0, F', f', \dot{\tau}_0, F'', f'$ and $\ddot{\tau}_0$ are continuous across the boundary. Conversely, we can prove that the continuities of $F, f, \tau_0, F', f', \dot{\tau}_0, F'', f'$ and $\ddot{\tau}_0$ imply that the metric tensor components in (5.9*a*, *b*) and their first partial derivatives are necessarily continuous also.

Now, we shall transform the exterior metric (5.9*b*) into the Schwarzschild metric (4.33). The transformation of the coordinate chart is furnished by

$$\hat{r} = \hat{R}(\rho, \tau) := R_E(\rho, \tau),$$

$$\hat{t} = \hat{T}(\rho, \tau) \qquad \text{... (5.20)}$$

and $(\hat{\theta} = \theta, \hat{\phi} = \phi)$.

Here, the only unknown function is $\hat{T}$. To derive the partial differential equations for $\hat{T}$, we have to use the tensor transformation rules

$$\hat{g}^{ab}(\hat{x}) = \frac{\partial \hat{X}^a}{\partial x^c}(x) \frac{\partial \hat{X}^b}{\partial x^d}(x)\, g^{cd}(x),$$

where $(\hat{x}^1, \hat{x}^4) := (\hat{r}, \hat{t})$ and $(x^1, x^4) := (\rho, \tau)$. By (4.33) and (5.9*b*), we have

$$0 = \hat{g}^{14} = \frac{\partial R_E}{\partial \rho}(\rho, \tau) \frac{\partial \hat{T}}{\partial \rho}(\rho, \tau) \left(\frac{1 - [f_E(\rho)]^2}{\left[\frac{\partial R_E}{\partial \rho}(\rho, \tau) \right]^2} \right) - \frac{\partial R_E}{\partial \tau}(\rho, \tau) \frac{\partial \hat{T}}{\partial \tau}(\rho, \tau), \qquad \text{... (5.21}a\text{)}$$

$$-\left(1 - \frac{2m}{\hat{r}}\right)^{-1} = \hat{g}^{44} = \left[\frac{\partial \hat{T}}{\partial \rho}(\rho, \tau) \right]^2 \left(\frac{1 - [f_E(\rho)]^2}{\left[\frac{\partial R_E}{\partial \rho}(\rho, \tau) \right]^2} \right) - \left[\frac{\partial \hat{T}}{\partial \tau}(\rho, \tau) \right]^2 \qquad \text{... (5.21}b\text{)}$$

and

$$1 - \frac{2m}{\hat{r}} = \hat{g}^{11} = \left[\frac{\partial R_E}{\partial \rho}(\rho, \tau) \right]^2 \left(\frac{1 - [f_E(\rho)]^2}{\left[\frac{\partial R_E}{\partial \rho}(\rho, \tau) \right]^2} \right) - \left[\frac{\partial R_E}{\partial \tau}(\rho, \tau) \right]^2. \qquad \text{... (5.21}c\text{)}$$

By (5.10*f*) and (5.20), eq. (5.21*c*) is identically satisfied. From (5.21*a*), we get a linear, homogeneous first order partial differential equation

$$\frac{\partial \hat{T}}{\partial \rho}(\rho, \tau) = \frac{\partial R_E}{\partial \rho}(\rho, \tau) \frac{\partial R_E}{\partial \tau}(\rho, \tau) \{1 - [f_E(\rho)]^2\}^{-1} \frac{\partial \hat{T}}{\partial \tau}(\rho, \tau). \qquad \text{... (5.22)}$$

Substituting the above into (5.21*b*) and using (5.10*f*), we obtian the first order, second degree equation

$$\left[\frac{\partial \hat{T}}{\partial \rho}(\rho, \tau) \right]^2 = \left(1 - \left[\frac{\partial R_E}{\partial \tau}(\rho, \tau) \right]^2 (1 - [f_E(\rho)]^2)^{-1} \right) \left(1 - \frac{2m}{\hat{r}} \right)^{-1}$$

$$= \{1 - [f_E(\rho)]^2\} \left(1 - \frac{2m}{\hat{r}} \right)^{-2} > 0. \qquad \text{... (5.23)}$$

Out of two possibilities in (5.23), we choose $\frac{\partial \hat{T}}{\partial \rho} > 0$. Therefore, eq. (5.23) yields

$$\frac{\partial \hat{T}}{\partial \tau} = \sqrt{1 - [f_E(\rho)]^2} \left(1 - \frac{2m}{\hat{r}} \right)^{-2} > 0. \qquad \text{... (5.24)}$$

Substituting (5.24) into (5.22), we have

$$\frac{\partial \hat{T}}{\partial \rho}(\rho, \tau) = \frac{\partial R_E}{\partial \rho}(\rho, \tau)\,\frac{\partial R_E}{\partial \tau}(\rho, \tau)\,\{1 - [f_E(\rho)]^2\}^{-1/2}\left(1 - \frac{2m}{\hat{r}}\right)^{-1} < 0. \qquad \ldots (5.25)$$

The pair of first order partial differential equations (5.24) and (5.25) is soluble provided the right hand sides satisfy an integrability condition. It is extremely cumbersome to try to check the integrability condition in these ρ-τ coordinates. It will be more convenient to introduce an intermediate coordinate chart by the following equations :

$$\xi = \Xi(\rho, \tau) := \rho, \qquad \ldots (5.26a)$$

$$\eta = H(\rho, \tau) := 2 \operatorname{Arctan} \sqrt{\frac{2m}{[f_E(\rho)]^2 R_E(\rho, \tau)} - 1}\,, \qquad \ldots (5.26b)$$

$$\tan^2\left(\frac{\eta}{2}\right) = \frac{2m}{[f_E(\rho)]^2 R_E(\rho, \tau)} - 1 > 0, \qquad \ldots (5.26c)$$

$$\frac{\partial(\xi, \eta)}{\partial(\rho, \tau)} = \frac{\partial H}{\partial \tau}(\rho, \tau) = \frac{f_E(\rho)}{R_E(\rho, \tau)} > 0 \qquad \ldots (5.26d)$$

and

$$D_E = \{(\rho, \tau) : \rho_b < \rho,\ \tau_E(\rho) < \tau < \tau_1(\rho)\}. \qquad \ldots (5.26e)$$

We shall also denote

$$f_E^{\#}(\xi) \equiv f_E(\rho) = f_E(\xi) \qquad \ldots (5.27a)$$

and

$$\tau_E^{\#}(\xi) \equiv \tau_E(\rho) = \tau_E(\xi). \qquad \ldots (5.27b)$$

By (5.26*a*, *b*, *c*), (5.27*a*, *b*) and (5.10*b*), we write the inverse transformation

$$\rho = \mathcal{P}(\xi) := \xi, \qquad \ldots (5.28a)$$

$$\tau = \mathcal{T}(\xi, \eta) := \tau_E(\xi) + \frac{m}{[f_E(\xi)]^3}(\eta + \sin\eta), \qquad \ldots (5.28b)$$

$$\frac{\partial(\rho, \tau)}{\partial(\xi, \eta)} = 2m\,\frac{\cos^2(\eta/2)}{[f_E(\xi)]^3} > 0 \qquad \ldots (5.28c)$$

and

$$D_E^{\#} := \{(\xi, \eta) : \rho_b < \xi,\ 0 < \eta < \pi\}. \qquad \ldots (5.28d)$$

(Although in (5.26*b*) $\eta \in (-\pi, \pi)$, but for $\eta \in (-\pi, 0)$, the inequality (5.2*a*) is violated. So we have to choose $\eta \in (0, \pi)$). For the transformation of ξ-η coordinates into the Schwarzschild coordinates, we write

$$\hat{r} = R_E(\rho, \tau) = 2m\,\frac{\cos^2(\eta/2)}{[f_E(\xi)]^2} = m\,\frac{(1 + \cos\eta)}{[f_E(\xi)]^2} =: R^{\#}(\xi, \eta), \qquad \ldots (5.29a)$$

$$\hat{t} = \hat{T}(\rho, \tau) =: T^{\#}(\xi, \eta), \qquad \ldots (5.29b)$$

$$D_S^{\#} := \{(\xi, \eta) : \rho_b < \xi,\ 0 < \eta < \eta_H(\xi)\} \subset D_E^{\#} \qquad \ldots (5.29c)$$

and $$\eta_H(\xi) := 2 \operatorname{Arccos}[f_E(\xi)]. \qquad \ldots (5.29d)$$

The above choice of $\eta_H(\xi)$ implies by (5.29*a*) that $\hat{r} = R^{\#}(\xi, \eta_H(\xi)) \equiv 2m$. The function $T^{\#}$ in (5.29*b*) is still unknown and we have to determine it.

Using (5.29*b*), the chain rule and equations (5.24), (5.25) and (5.28*b*), we obtain

$$\begin{aligned}
\frac{\partial T^{\#}}{\partial \eta}(\xi, \eta) &= \frac{\partial \hat{T}}{\partial \rho}(\rho, \tau)\frac{\partial P}{\partial \eta}(\xi, \eta) + \frac{\partial \hat{T}}{\partial \tau}(\rho, \tau)\frac{\partial T}{\partial \eta}(\xi, \eta) \\
&= 0 + \sqrt{1 - [f_E(\xi)]^2}\left(1 - \frac{2m}{R^{\#}(\xi, \eta)}\right)^{-1}\left(\frac{m(1 + \cos\eta)}{[f_E(\xi)]^3}\right) \\
&= \frac{2m\Sigma(\xi)}{[f_E(\xi)]^3}\left(\frac{\cos^2(\eta/2)}{1 - [f_E(\xi)]^2/\cos^2(\eta/2)}\right) \\
&= \frac{2m\Sigma(\xi)}{[f_E(\xi)]^3}\left(\cos^2(\eta/2) + [f_E(\xi)]^2 + \frac{[f_E(\xi)]^4}{\cos^2(\eta/2) - f_E(\xi)]^2}\right) \\
&=: P(\xi, \eta), \qquad \ldots (5.30a)
\end{aligned}$$

$$\Sigma(\xi) := \sqrt{1 - [f_E(\xi)]^2}, \qquad \ldots (5.30b)$$

and

$$\begin{aligned}
\frac{\partial T^{\#}}{\partial \xi}(\xi, \eta) &= \frac{\partial \hat{T}}{\partial \rho}(\rho, \tau)\frac{\partial P}{\partial \xi}(\xi, \eta) + \frac{\partial \hat{T}}{\partial \tau}(\rho, \tau)\frac{\partial T}{\partial \xi}(\xi, \eta) \\
&= \frac{\tau_E'(\xi)}{\Sigma(\xi)} + \frac{m f_E'(\xi)}{\Sigma(\xi)[f_E(\xi)]^2}\left(-3\frac{\eta + \sin\eta}{[f_E(\xi)]^2} + 4\frac{\sin(\eta/2)\cos^3(\eta/2)}{\cos^2(\eta/2) - [f_E(\xi)]^2}\right) \\
&=: Q(\xi, \eta). \qquad \ldots (5.30c)
\end{aligned}$$

The integrability condition for the pair of first order p.d.e's (5.30*a*, *c*) is

$$\begin{aligned}
\frac{\partial P}{\partial \xi}(\xi, \eta) &= \frac{-2m f_E'(\xi)\cos^4(\eta/2)}{[f_E(\xi)]^4\,\Sigma(\xi)} \\
&\quad \times \left(\frac{4[f_E(\xi)]^4 - 5[f_E(\xi)]^2 - 2[f_E(\xi)]^2\cos^2(\eta/2) + 3\cos^2(\eta/2)}{(\cos^2(\eta/2) - [f_E(\xi)]^2)^2}\right) \\
&\equiv \frac{\partial Q}{\partial \eta}(\xi, \eta). \qquad \ldots (5.31)
\end{aligned}$$

So, this condition is identically satisfied. Therefore, the solution $T^{\#}(\xi, \eta)$ exists and it is given

explicitly by a line integral (along any continuous and piecewise differentiable curve Γ in $D_S^{\#}$) as

$$\hat{t} = T^{\#}(\xi, \eta) = \int_{(\xi, 0)(\Gamma)}^{(\xi, \eta)} \left[\frac{\partial T^{\#}}{\partial x}(x, y)\, dx + \frac{\partial T^{\#}}{\partial x}(x, y)\, dy \right]$$

$$= \int_{\xi_0}^{\xi} \frac{\tau_E'(x)}{\sqrt{1 - [f_E(x)]^2}}\, dx + m\sqrt{1 - [f_E(x)]^2} \left(\frac{(\eta + \sin \eta)}{[f_E(\xi)]^3} + \frac{2\eta}{f_E(\xi)} \right)$$

$$+ 2m \ln \left[\frac{\tan (\eta_H(\xi)/2) + \tan (\eta/2)}{\tan (\eta_H(\xi)/2) - \tan (\eta/2)} \right] \qquad \ldots (5.32a)$$

and $$\tan (\eta_H(\xi)/2) = [f_E(\xi)]^{-1} \sqrt{1 - [f_E(\xi)]^2}. \qquad \ldots (5.32b)$$

The solution function $T^{\#}$ contains one arbitrary constant $\xi_0 > \rho_b$ and two arbitrary functions f_E and τ_E. Now, by (5.20), (5.10*b*), (5.29*b*), (5.32*a*, *b*) and (5.26*b*, *c*), the transformation from the ρ-τ coordinates into the Schwarzschild coordinates is furnished by

$$\hat{r} = R_E(\rho, \tau), \qquad \ldots (5.33a)$$

$$\hat{t} = \hat{T}(\rho, \tau) = \int_{\rho_0}^{\rho} \frac{\tau_E'(x)}{\sqrt{1 - [f_E(x)]^2}}\, dx$$

$$+ \sqrt{1 - [f_E(\rho)]^2} \left(\tau - \tau_E(\rho) + \frac{4m}{f_E(\rho)} \operatorname{Arctan} \sqrt{\frac{2m}{[f_E(\rho)]^2 R_E(\rho, \tau)} - 1} \right)$$

$$+ 2m \ln \left[\frac{\tan (\eta_H(\rho)/2) + \sqrt{(2m/[f_E(\rho)]^2 R_E(\rho, \tau)) - 1}}{\tan (\eta_H(\rho)/2) - \sqrt{(2m/[f_E(\rho)]^2 R_E(\rho, \tau)) - 1}} \right], \qquad \ldots (5.33b)$$

$$\tan (\eta_H(\rho)/2) = [f_E(\rho)]^{-1} \sqrt{1 - [f_E(\rho)]^2}, \qquad \ldots (5.33c)$$

$$\tau_H(\rho) := \tau_E(\rho) + \frac{m}{[f_E(\rho)]^3} (\eta_H(\rho) + \sin [\eta_H(\rho)]), \qquad \ldots (5.33d)$$

$$\tau_H(\rho) < \tau_1(\rho); \quad \tau_1(\rho) := \tau_E(\rho) + \frac{m\pi}{[f_E(\rho)]^3} \qquad \ldots (5.33e)$$

and $$D_S := \{(\rho, \tau) : \rho_b < \rho, \quad \tau_E(\rho) < \tau < \tau_H(\rho)\}. \qquad \ldots (5.33f)$$

To derive the range $\hat{D}_S$ of the above transformation, we have to consider the boundary curve

$$\hat{r} = \hat{B}(\hat{t}) \qquad \ldots (5.34)$$

defined by the parametric equations

$$\hat{r} = R_E(\rho_b, \tau),$$

$$\hat{t} = \hat{T}(\rho_b, \tau) \qquad \text{... (5.35)}$$

and $\tau_E(\rho_b) < \tau < \tau_H(\rho_b)$.

Schwarzschild coordinates are valid for

$$\hat{D}_S := \{(\hat{r}, \hat{t}) : \hat{B}(\hat{t}) < \hat{r},\ \hat{t}_0 < \hat{t} < \infty\}$$

and

$$\hat{t}_0 := \int_{\rho_0}^{\rho_b} \frac{\tau_E'(x)}{\sqrt{1 - [f_E(x)]^2}}\, dx. \qquad \text{... (5.36)}$$

To provide some physical insights for the transformations (5.29*a*, *b*) and (5.32*a*, *b*), we consider a η-coordinate curve fixed at $x = \xi_1 > \rho_b$. Putting $\tau_E(\xi) \equiv 0$ (for the sake of simplicity), and writing $E := \sqrt{1 - [f_E(\xi_1)]^2}$ (compare with eq. (3.30), eqs. (5.29*a*), (5.32*a*) and (5.28*b*) imply[15] that

$$\hat{r} = R^{\#}(\xi_1, \eta) = \frac{2m}{1 - E^2} \cos^2(\eta/2) =: \hat{r}_i \cos^2(\eta/2), \qquad \text{... (5.37}a\text{)}$$

$$\hat{t} = T^{\#}(\xi_1, \eta) = E \sqrt{\frac{\hat{r}_i^3}{2m}} \left(\frac{1}{2}(\eta + \sin\eta) + (1 - E^2)\eta \right) + 2m \ln\left[\frac{\tan(\eta_H(\xi_1)/2) + \tan(\eta/2)}{\tan(\eta_H(\xi_1)/2) - \tan(\eta/2)} \right], \qquad \text{... (5.37}b\text{)}$$

$$\tau = \mathcal{T}^{\#}(\xi_1, \eta) = \sqrt{\frac{\hat{r}_i^3}{8m}}\, (\eta + \sin\eta), \qquad \text{... (5.37}c\text{)}$$

$$\hat{r} := \left[\frac{\partial R_E}{\partial \tau}(\rho, \tau) \right]_{\Xi(\rho, \eta) = \xi_1} = -\sqrt{1 - E^2} \tan(\eta/2) < 0, \qquad \text{... (5.37}d\text{)}$$

$$\lim_{\eta \downarrow 0} R^{\#}(\xi_1, \eta) = \hat{r}_i = \lim_{\eta \downarrow 0} [R_E(\rho, \tau)]_{\Xi(\rho, \eta) = \xi_1}, \qquad \text{... (5.37}e\text{)}$$

$$\hat{r}_i := \lim_{\eta \downarrow 0} \left[\frac{\partial R_E}{\partial \tau}(\rho, \tau) \right]_{\Xi(\rho, \eta) = \xi_1} = 0 \qquad \text{... (5.37}f\text{)}$$

and

$$\lim_{\eta \uparrow \eta_H(\xi)} T^{\#}(\xi, \eta) \to \infty. \qquad \text{... (5.37}g\text{)}$$

These equations represent a time-like radial geodesic of a test particle with unit mass and total energy E in the Schwarzschild universe. The above equations imply that $t = 0 \Leftrightarrow \tau = 0 \Leftrightarrow \eta = 0$. Therefore, the initial position of the particle is $\hat{r}_i$ and the initial "velocity" is $\hat{r}_i = 0$. In the equations (5.37*a*, *b*, *c*, *d*, *e*, *f*, *g*) the appropriate coordinate intervals are given by

$$2m < \hat{r} < \hat{r}_i, \quad 0 < \hat{t} < \infty, \quad 0 < \tau < \tau_H,$$

$$\tau_H := \sqrt{\frac{\hat{r}_i^3}{8m}}\,[\eta_H(\xi_1) + \sin \eta_H(\xi_1)].$$

If we plot the $\hat{r}$ versus η graph for all real values of η, then we obtain a analytic, periodic graph (see Fig. 3). However, if we plot the $\hat{r}$ versus τ curve, then we obtain a continuous, piecewise analytic, periodic curve known as the cycloid (see Fig. 3).

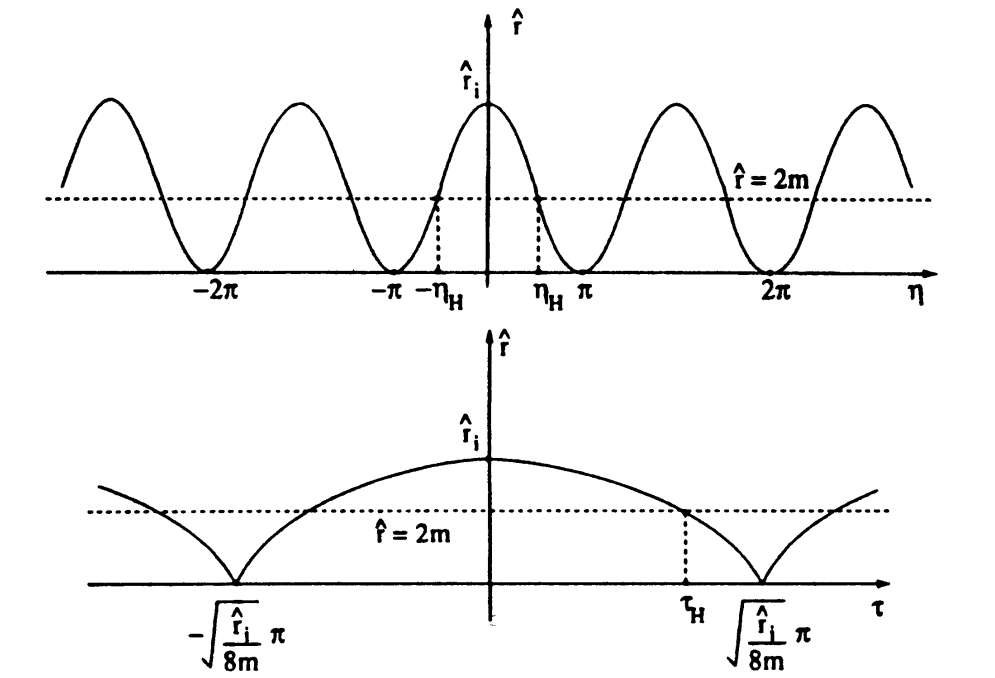

Fig. 3. Qualitative graph of $\hat{r}$ versus η and the corresponding cycloid as a function of τ

Suppose in a thought experiment, we dig a wide hole from the surface of the earth to the centre. If we drop a massive, perfectly elastic ball from rest into the hole, then it will fall due to the earth's gravity to the center and bounce back again and again. According to Newtonian physics, the height of the center of mass of the ball plotted against time will produce a cycloid exactly as in Fig. 3. However, in the general relativistic gravitation, $\hat{r} = 0$ is a real singularity of the space-time curvature [see eq. (3.33*f*)]. In order to avoid such singularities (in the case $\varepsilon = +1$), the coordinate τ must belong to a finite interval $(\tau_0(\rho), \tau_1(\rho))$ for every ρ.

Now, we shall consider the $\xi - \eta$ chart in the complementary domain $D_T^{\#}$ given by

$$D_T^{\#} := \{(\xi, \eta) : \rho b\,, \xi, \eta_H(\xi) < \eta < \pi\} \subset D_E^{\#}\,. \qquad \ldots (5.38)$$

We are allowed to make a transformation from $D_T^{\#}$ into $\mathbb{R}^2$ by

$$T = \Theta^{\#}(\xi, \eta) := \frac{2m}{[f_E(\xi)]^2} \cos^2(\eta/2) \qquad \ldots (5.39a)$$

and $$X = X^{\#}(\xi, \eta)$$

$$:= \int_{\xi_0}^{\xi} \frac{\tau_E'(x)}{\sqrt{1-[f_E(x)]^2}}\,dx + m\sqrt{1-[f_E(\xi)]^2}\left(\frac{(\eta+\sin\eta)}{[f_E(\xi)]^3}+\frac{2\eta}{f_E(\xi)}\right)$$

$$+ 2m\ln\left[\frac{\tan(\eta/2)+\tan(\eta_H(\xi)/2)}{\tan(\eta/2)-\tan(\eta_H(\xi)/2)}\right]. \qquad \ldots (5.39b)$$

To find the range $\hat{D}_T$ of the above transformation, we have to consider the boundary curve

$$X = \beta(T) \qquad \ldots (5.40)$$

given parametrically by the equations

$$X = X^{\#} = (\rho b, \eta), \qquad \ldots (5.41\ a)$$

$$T = \Theta^{\#}(\rho b, \eta) \qquad \ldots (5.41\ b)$$

and $$\eta \in (\eta_H(\rho b), \pi). \qquad \ldots (5.41c)$$

The coordinates X, T are valid in the domain

$$\hat{D}_T := \{(X, T) : \beta(T) < X < \infty,\ 0 < T < 2m\}. \qquad \ldots (5.42)$$

By eqs. (5.28 *a*, *b*) and (5.39 *a*, *b*), the exterior metric in (5.9*b*) goes over into the black hole metric (compare equation 4.35)).

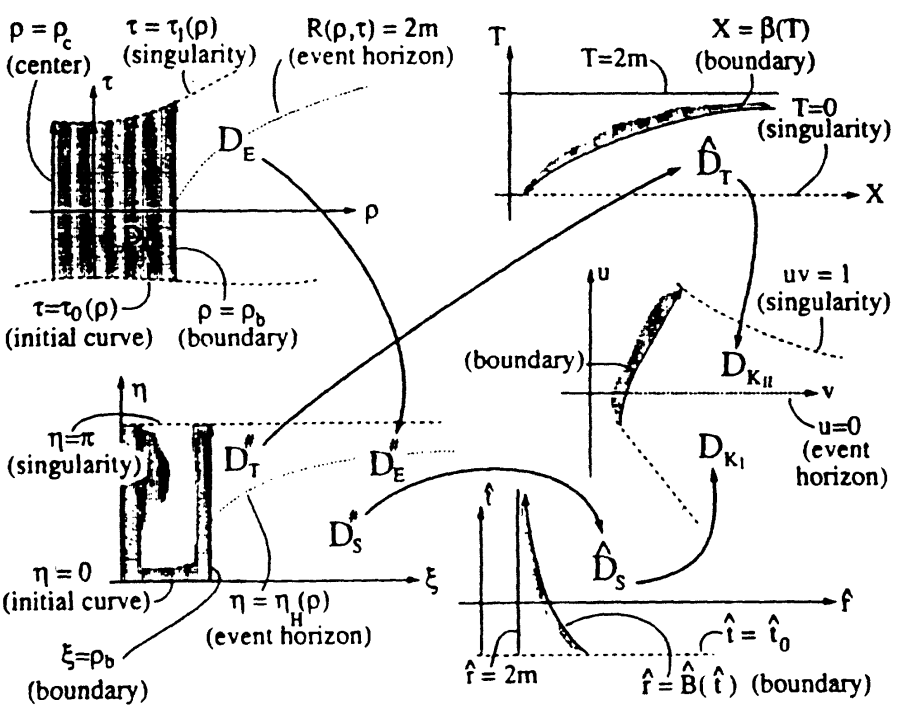

Fig. 4. Gravitational collapse in the elliptic case $\varepsilon = +1$ shown in 5 charts

$$ds_E^2 = \left(\frac{2m}{T} - 1\right) dX^2 + T^2 \, d\Omega^2 - \left(\frac{2m}{T} - 1\right)^{-1} dT^2. \qquad \ldots (5.43)$$

The above metric, as well as the Schwarzschild metric, can be transformed by KSS coordinates in (4.36*a*, *b*) and (4.38*a*, *b*) of the preceding section. (However, it should be noted that domains and ranges are different in different sections).

Now, we shall depict qualitatively the collapse of a stellar dust cloud of type $\varepsilon = +1$ into a black hole in Fig. 4. Five coordinate systems or charts will be used.

6. The Hyperbolic Case $\varepsilon = -1$

Eq. (3.31) for the collapsing radial velocity yields

$$\frac{\partial R}{\partial \tau}(ro, \tau) = -\sqrt{\frac{F(\rho)}{R(\rho, \tau)} + [f(\rho)]^2} < 0, \qquad \ldots (6.1a)$$

$$\frac{F(\rho)}{R(\rho, \tau)} + [f(\rho)]^2 > 0 \qquad \ldots (6.1b)$$

and $$f(\rho) \neq 0. \qquad \ldots (6.1c)$$

Eqs. (3.16 *a*) and (3.23*b*) imply that

$$\exp\,[\Lambda(\rho, \tau)] = \frac{\left[\frac{\partial R}{\partial \rho}(\rho, \tau)\right]^2}{1 + [f(\rho)]^2} > 0\,, \qquad \ldots (6.2a)$$

and $$\frac{\partial R}{\partial \rho}(\rho, \tau) > 0. \qquad \ldots (6.2b)$$

By (3.24*a*), (3.28), (3.29) and (6.1*c*), the "gravitational binding energy" is

$$M_p(\rho) - M(\rho) = \frac{1}{2} \int_{\rho c^+}^{\rho} \left[\frac{1}{\sqrt{1 + [f(x)]^2} - 1}\right] F'(c)\, dx < 0 \qquad \ldots (6.3)$$

Integrating the partial differential equation (6.1*a*) with respect to τ, we obtain

$$\tau - \tau_1(\rho) = -\frac{R(\rho, \tau)}{|f(\rho)|}\sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} + 1} + \frac{F(\rho)}{|f(\rho)|^3} \operatorname{argtanh} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} + 1}$$

$$\ldots (6.4a)$$

for $\sqrt{\frac{F(\rho)}{R(\rho, \tau)} + [f(\rho)]^2} < |f(\rho)|$, and

$$\tau - \tau_1(\rho) = -\frac{R(\rho, \tau)}{|f(\rho)|}\sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} + 1} + \frac{F(\rho)}{|f(\rho)|^3} \operatorname{argcoth} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} + 1}$$

$$\ldots (6.4b)$$

for $|f(\rho)| < \sqrt{\frac{F(\rho)}{R(\rho, \tau)} + [f(\rho)]^2}$. Here the function τ_1 is the arbitrary C_p^3 function of integration. Now, in case of (6.4*a*), $F(\rho)/R(\rho, \tau) < 0$, and it is not considered here (see eq. (3.4*a*) and (3.27*a*)). Therefore, we consider the equation (6.4*b*) only. Moreover, for the sake of simplicity, we choose

$$f(\rho) > 0. \qquad \ldots (6.5)$$

Thus, (6.4*b*) yields

$$\tau_1(\rho) - \tau = \frac{R(\rho, \tau)}{f(\rho)} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} + 1} - \frac{F(\rho)}{[f(\rho)]^3} \operatorname{argcoth} \sqrt{\frac{F(\rho)}{[f(\rho)]^2 R(\rho, \tau)} + 1} \qquad \ldots (6.6)$$

Here, the function R is defined implicitly.

Assuming the boundary of the start at $\rho = \rho_b$, the line elements in the interior and exterior are given by

$$ds_I^2 = \frac{\left[\frac{\partial R_I}{\partial \rho}(\rho, \tau)\right]^2}{1 + [f_I(\rho)]^2} d\rho^2 + [R_I(\rho, \tau)]^2 \, d\Omega^2 - d\tau^2, \qquad \ldots (6.7a)$$

and

$$ds_E^2 = \frac{\left[\frac{\partial R_E}{\partial \rho}(\rho, \tau)\right]^2}{1 + [f_E(\rho)]^2} d\rho^2 + [R_\varepsilon(\rho, \tau)]^2 \, d\Omega^2 \, d\tau^2 \; . \qquad \ldots (6.7b)$$

By (6.6), the interior and the exterior very smooth branches of the function R satisfy.

$$\tau_I(\rho) - \tau = \frac{R_I(\rho, \tau)}{f_I(\rho)} V_1(\rho, \tau) - \frac{F_I(\rho)}{[f_I(\rho)]^3} \operatorname{argcoth} [V_I(\rho, \tau)], \qquad \ldots (6.8a)$$

$$\tau_E(\rho) - \tau = \frac{R_E(\rho, \tau)}{f_E(\rho)} V_E(\rho, \tau) - \frac{2m}{[f_1(\rho)]^3} \operatorname{argcoth} [V_E(\rho, \tau)], \qquad \ldots (6.8b)$$

$$D_I := \{(\rho, \tau) : \rho_c < \rho < \rho b, -\infty < \tau < \tau_I(\rho)\}, \qquad \ldots (6.8c)$$

$$D_E := \{(\rho, \tau) : \rho b < \rho, -\infty < \tau < \tau_E(\rho)\}, \qquad \ldots (6.8d)$$

$$\partial D := \{(\rho, \tau) : \rho = \rho b, -\infty < \tau < \tau_1(\rho b)\}, \qquad \ldots (6.8e)$$

$$V_I(\rho, \tau) := \sqrt{\frac{F_I(\rho)}{[f_I(\rho)^2 R_I(\rho, \tau)} + 1} \qquad \ldots (6.8f)$$

and

$$V_E(\rho, \tau) := \sqrt{\frac{2m}{[f_E(\rho)]^2 R_E(\rho, \tau)} + 1} \; . \qquad \ldots (6.8g)$$

Differentiating eqs. (6.8*a*, *b*), we get

$$\frac{\partial R_I}{\partial \rho}(\rho, \tau) = \left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^2} \right]' R_I(\rho, \tau) + V_I(\rho, \tau) \left(\tau_I'(\rho) + \left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^3} \right]' (\tau - \tau_I(\rho) \right)$$

$$= \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho) f_I'(\rho)}{[f_I(\rho)]^3} \right) V_I(\rho, \tau) \text{ argcoth } [V_I(\rho, \tau)] - 1)$$

$$+ \frac{f_I'(\rho)}{f_I(\rho)} R_I(\rho, \tau) + f_I(\rho)\, \tau_I'(\rho)\, V_I(\rho, \tau), \quad \ldots (6.9a)$$

$$\frac{\partial R_E}{\partial \rho}(\rho, \tau) = \frac{6m f_E'(\rho)}{[f_E(\rho)]^3} (1 - V_E(\rho, \tau) \text{ argcoth } [V_E(\rho, \tau)])$$

$$+ \frac{f_E'(\rho)}{f_E(\rho)} R_E(\rho, \tau) + f_E(\rho)\, \mathcal{T}_E'(\rho)\, V_E(\rho, \tau) \quad \ldots (6.9b)$$

$$\frac{\partial R_I}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{F_I(\rho)}{R_I(\rho, \tau)} + [f_I(\rho)]^2} < 0, \quad \ldots (6.9c)$$

$$\frac{\partial R_E}{\partial \tau}(\rho, \tau) = -\sqrt{\frac{2m}{R_E(\rho, \tau)} + [f_E(\rho)]^2} < 0\ , \quad \ldots (6.9d)$$

$$\frac{\partial^2 R_I}{\partial \tau \partial \rho}(\rho, \tau) = \frac{1}{1} \left(\frac{F_I'(\rho)}{R_I(\rho, \tau)} - \frac{F_I(\rho) \dfrac{\partial R_I}{\partial \rho}(\rho, \tau)}{[R_I(\rho, \tau)]^2} + 2 f_I(\rho) f_I'(\rho) \right) \left(\frac{\partial R_I}{\partial \tau}(\rho, \tau) \right)^{-1} \quad \ldots (6.9e)$$

$$\frac{\partial^2 R_E}{\partial \tau \partial \rho}(\rho, \tau) = -\left(\frac{m}{[R_E(\rho, \tau)]^2} \frac{\partial R_E}{\partial \rho}(\rho, \tau) = f_E(\rho) f_E'(\rho) \right) \left(\frac{\partial R_E}{\partial \tau}(\rho, \tau) \right)^{-1} \quad \ldots (6.9f)$$

$$\frac{\partial^2 R_I}{\partial \rho^2}(\rho, \tau) = [\ln f_I(\rho)]' \frac{\partial R_I}{\partial \rho}(\rho, \tau) + R_I(\rho, \tau)\, [\ln f_I(\rho)]''$$

$$- \frac{1}{2} \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho) f_I'(\rho)}{[f_I(\rho)]^3} \right) \left(\left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^2} \right]' - \frac{\dfrac{\partial R_I}{\partial \rho}(\rho, \tau)}{R_I(\rho, \tau)} \right)$$

$$+ \frac{1}{2} \frac{F_I(\rho)}{2[f_I(\rho)]^2 R_I(\rho, \tau)\, V_I(\rho, \tau)} \left(\left[\ln \frac{F_I(\rho)}{[f_I(\rho)]^2} \right]' - \frac{\dfrac{\partial R_I}{\partial \rho}(\rho, \tau)}{R_I(\rho, \tau)} \right)$$

$$\times \left\{ f_I(\rho) \tau_I'(\rho) + \left(\frac{F_I'(\rho)}{[f_I(\rho)]^2} - \frac{3F_I(\rho) f_I'(\rho)}{[f_I(\rho)]^3} \right) \text{argcoth } [V_I(\rho, \tau)] \right\}$$

$$+ \; V_I(\rho, \tau)\,[f_I(\rho)\,\tau_I'(\rho)]' + (V_I(\rho, \tau)\ \text{argcoth}\ [V_I(\rho, \tau)] - 1)$$

$$\times \left(\frac{F_I''(\rho)}{[f_I(\rho)]^2} - 5\,\frac{F_I'(\rho)\,f_I'(\rho)}{[f_I(\rho)]^3} - 3\,\frac{F_I(\rho)\,f_I''(\rho)}{f_I(\rho)]^3} + 9\,\frac{F_I(\rho)\,[f_I'(\rho)]^2}{[f_I(\rho)]^4} \right) \quad \ldots (6.9g)$$

and

$$\frac{\partial^3 R_E}{\partial \rho^2}(\rho, \tau) = [\ln f_E(\rho)]'\,\frac{\partial R_E}{\partial \rho}(\rho, \tau) + R_E(\rho, \tau)\,[\ln f_E(\rho)]''$$

$$- \frac{3m f_E'(\rho)}{[f_E(\rho)]^3}\left(2\,[\ln f_E(\rho)]' + \frac{\partial R_E(\rho, \tau)}{R_E(\rho, \tau)} \right) + V_E(\rho, \tau)\,[f_E(\rho)\,\tau_E'(\rho)]'$$

$$- \frac{m}{[f_E(\rho)]^2\,R_E(\rho, \tau)\,V_E(\rho, \tau)}\left(2\,[\ln f_E(\rho)]' + \frac{\frac{\partial R_E}{\partial \rho}(\rho, \tau)}{R_E(\rho, \tau)} \right)$$

$$\times \left\{ f_E(\rho)\tau_E'(\rho) - \frac{6m f_E'(\rho)}{[f_E(\rho)]^3}\ \text{argcoth}\ [V_E(\rho, \tau)] \right\}$$

$$- 6m\,(V_E(\rho, \tau)\ \text{argcoth}\ [V_E(\rho, \tau)] - 1)\left(\frac{f_E''(\rho)}{[f_E(\rho)]^3} - \frac{3f_E'(\rho)]^2}{[f_E(\rho)} \right). \ldots (6.9h)$$

Now, suppose that the functions, $F, f, \tau_1, F', f', \tau_1', F'', f''$ and τ_1'' are continuous across the boundary ∂D. Then by eqs. (6.8*a*, *b*), we obtain

$$R_1(\rho_b -, \tau)\sqrt{\frac{2m}{[f(\rho_b)]^2\,R_I(\rho_b - \tau)} + 1} - \frac{2m}{[f(\rho_b)]^2}\ \text{argcoth}\ \sqrt{\frac{2m}{[f(\rho_b)]^2\,R_I(\rho_b - \tau)} + 1}$$

$$\equiv R_E(\rho_b +, \tau)\sqrt{\frac{2m}{[f(\rho_b)]^2\,R_E(\rho_b - \tau)} + 1} - \frac{2m}{[f(\rho_b)]^2}\ \text{argcoth}\ \sqrt{\frac{2m}{[f(\rho_b)]^2\,R_I(\rho_b - \tau)} + 1}$$

$$\ldots (6.10)$$

Now, we define the functions

$$\omega_I = \mathcal{W}_I(\tau) := \sqrt{\frac{2m}{[f(\rho_b)]^2\,R_I(\rho_b - \tau)} + 1} > 1, \quad \ldots (6.11a)$$

$$\omega_E = \mathcal{W}_E(\tau) := \sqrt{\frac{2m}{[f(\rho_b)]^2\,R_E(\rho_b +, \tau)} + 1} > 1 \quad \ldots (6.11b)$$

and

$$J(\omega) := \omega(\omega^2 - 1)^{-1} - \text{argcoth}\ \omega, \omega > 1. \quad \ldots (6.11c)$$

The derivative of J is

$$J'(\omega) = -\frac{2}{(\omega^2 - 1)^2} < 0. \quad \ldots (6.12)$$

The strict inequality (6.12) shows that the function is one-to-one. By (6.11 *a*, *b*, *c*), the identity (6.10) reduces to

$$J(\omega_1) = J(\omega_E). \qquad \ldots (6.13)$$

By the one-to-one property of the function J, we conclude that

$$\left.\begin{aligned} &\mathcal{W}_1(\tau) = \omega_1 = \omega_E = \mathcal{W}_E(\tau), \\ &[\mathcal{W}_1(\tau)]^2 - 1 \equiv [\mathcal{W}_E(\tau)]^2 - 1 \end{aligned}\right\} \qquad \ldots (6.14)$$

and $[\Delta R(\rho b\ \tau)] \equiv 0.$

Therefore, the continuity of R across the boundary ∂D is established. So, the metric tensor component g_{22} is continuous. By the continuities of F, f and R and the equations (6.9c, *d*), the continuity of $\frac{\partial R}{\partial \tau}(\rho, \tau)$ is proved. Thus, $\partial_\tau g_{22}$ is continuous. By the continuities of F, f, R, F' and f', the continuities of $\frac{\partial R}{\partial \rho}$ and $\partial_\rho g_{22}$ are proved. By the continuities of F, f, R, F', f', $\frac{\partial R}{\partial \rho}(\rho, \tau)$ and $\frac{\partial R}{\partial \tau}(\rho, \tau)$, the equations (6.9 *a*, *b*) yield the continuity of $\partial_\tau g_{11}$. From the equations (6.9*g*, *h*) and the continuities of F, f, R, F', f', τ_1', $\frac{\partial R}{\partial \rho}(\rho, \tau)$, $\frac{\partial R}{\partial \tau}$, F'', f'' and τ_1'', we conclude that $\partial_\tau g_{11}$ is continuous. Therefore, we have proved that the continuities of F, f, τ_1, F', f', τ_1', F'', f'' and τ_1'' imply the continuities of the metric tensor components and of their first partial derivatives. The converse result can also be proved.

Now, we shall transform the exterior line element (6.7*b*) into the Schwarzschild line element by the transformation

$$\left.\begin{aligned} &\hat{r} = \hat{R}(\rho, \tau) := R_E(\rho, \tau), \\ &\hat{t} = \hat{T}(\rho, \tau), \\ \text{and} \quad &(\hat{\theta} = \theta, \hat{\phi} = \phi), \end{aligned}\right\} \qquad \ldots (6.15)$$

where R_E is given implicitly by (6.8*b*). The function $\hat{T}$ has to be determined. Analagous to (5.24) and (5.25), we have to satisfy a pair of first order,linear p.d.e's

$$\frac{\partial \hat{T}}{\partial \tau}(\rho, \tau) = \sqrt{1 + [f_E(\rho)]^2}\left(1 - \frac{2m}{R_E(\rho, \tau)}\right)^{-1} > 0 \qquad \ldots (6.16a)$$

and

$$\frac{\partial \hat{T}}{\partial \rho}(\rho, \tau) = \frac{\partial R_E}{\partial \rho}(\rho, \tau)\frac{\partial R_E}{\partial \tau}(\rho, \tau)\ \{1 + [f_E(\rho)]^2\}^{-\frac{1}{2}}\left(1 - \frac{2m}{R_E(\rho, \tau)}\right)^{-1} < 0 \qquad \ldots (6.16b)$$

It is very complicated to check the integrability condition for (6.16 *a*, *b*). It will be more convenient to introduce another coordinate system by the following equations:

$$\xi = \Xi(\rho, \tau) := \rho, \qquad \ldots (6.17a)$$

$$\zeta = Z(\rho, \tau) := 2\ \text{arcoth}\ \sqrt{\frac{2m}{[f_E(\rho)]^2 R_E(\rho, \tau)} + 1} > 0, \qquad \ldots (6.17b)$$

$$\coth^2\left(\frac{\zeta}{2}\right)=\frac{2m}{[f_E(\rho)]^2 R_E(\rho,\tau)}+1, \qquad \ldots (6.17c)$$

$$\frac{\partial(\xi,\zeta)}{\partial(\rho,\tau)}=\frac{\partial Z}{\partial\tau}(\rho,\tau)=-\frac{f_E(\rho)}{R_E(\rho,\tau)}<0 \qquad \ldots (6.17d)$$

and $$D_E=\{(\rho,\tau):\rho_b<\rho,-\infty<\tau<t_E(\rho)\}. \qquad \ldots (6.17e)$$

We shall also denote

$$f_E^{\#}(\xi)\equiv f_E(\rho)=f_E(\xi) \qquad \ldots (6.18a)$$

and $$\tau_E^{\#}(\xi)=\tau_E(\rho)=\tau_E(\xi). \qquad \ldots (6.18b)$$

By (6.17*a*, *b*, *c*), (6.18*a*, *b*) ad (6.8*b*), we obtain the inverse transformation as

$$\rho=\mathcal{P}^{\#}(\xi):=\xi, \qquad \ldots (6.19a)$$

$$\tau=\mathcal{T}^{\#}(\xi,.\,\zeta):=\tau_E(\xi)+\frac{m}{[f_E(\xi)]^3}(\zeta-\sinh\zeta), \qquad \ldots (6.19b)$$

$$\frac{\partial(\rho,\tau)}{\partial(\xi,\zeta)}=-2m\frac{\sinh^2(\zeta/2)}{[f_E(\xi)]^3}<0 \qquad \ldots (6.19c)$$

and $$D_E^{\#}:=\{(\xi,\zeta):\rho b<\xi,0<\zeta<\infty\}. \qquad \ldots (6.19d)$$

For the transformation of $\xi-\zeta$ coordinates into the Schwarschild coordinates (using the equations (6.17*c*, 6.18*a* and 6.15), we write

$$\hat{r}=R_E(\rho,\tau)=2m\frac{\sinh^2(\zeta/2)}{[f_E(\xi)]^2}=\frac{m}{[f_E(\xi)]^2}(\cosh\zeta-1)=:R^{\#}(\xi,\zeta) \qquad \ldots (6.20a)$$

$$\hat{t}=\hat{T}(\rho,\tau)=T^{\#}(\xi,\zeta) \qquad \ldots (6.20b)$$

$$D_s^{\#}:=\{(\xi,\zeta):\rho_b<\xi,\zeta_H(\xi)<\zeta\} \qquad \ldots (6.20c)$$

and $$\zeta_H(\xi):=2\ \text{argsinh}\ [f_E(\xi)]. \qquad \ldots (6.20d)$$

Here, the function $T^{\#}$ is unknown. We use (6.20*b*), chain-rules, (6.16*a*, *b*) and (6.19*b*) to derive the following p.d.e's of $T^{\#}$:

$$\frac{\partial T^{\#}}{\partial\zeta}(\xi,\zeta)=\frac{-2m\,\Sigma^{\#}(\xi)}{[f_E(\xi)]^3}\frac{\sinh^4(\zeta/2)}{\sinh^2(\zeta/2)-[f_E(\xi)]^2}$$

$$=:P^{\#}(\xi,\zeta) \qquad \ldots (6.21a)$$

$$\frac{\partial T^{\#}}{\partial \xi}(\xi, \zeta) = \frac{\partial \hat{T}}{\partial \rho}(\rho, \tau)\frac{\partial P}{\partial \xi}(\xi, \zeta) + \frac{\partial \hat{T}}{\partial \tau}(\rho, \tau)\frac{\partial T}{\partial \xi}(\xi, \zeta)$$

$$= \frac{\tau_E'(\xi)}{\Sigma^{\#}(\xi)} + \frac{m f_E'(\xi)}{\Sigma^{\#}(\xi)\,[f_E(\xi)]^2}\left(3\frac{\sinh\zeta - \zeta}{[f_E(\xi)]^2} + 4\frac{\cosh(\zeta/2)\sinh^3(\zeta/2)}{\sinh^2(\zeta/2) - [f_E(\xi)]^2}\right)$$

$$=: Q^{\#}(\xi, \zeta). \qquad \ldots (6.21b)$$

and $$\Sigma^{\#}(\xi) := \sqrt{1 + [f_E(\xi)]^2}. \qquad \ldots (6.21c)$$

The integrability condition for (6.21*a*, *b*)

$$\frac{\partial P^{\#}}{\partial \xi}(zi, \zeta) = \frac{-2m f_E'(\xi)\sinh^4(\zeta/2)}{\Sigma^{\#}(\xi)\,[f_E(\xi)]^4}$$

$$\times\left(\frac{4[f_E(\xi)]^4 + 5[f_E(\xi)]^2 - 2[f_E(\xi)]^2\sinh^2(\zeta/2) - 3\sinh^2(\zeta/2)}{(\sinh^2(\zeta/2) - [f_E(\xi)]^2)^2}\right)$$

$$\equiv \frac{\partial Q^{\#}}{\partial \zeta}(\xi, \zeta) \qquad \ldots (6.22)$$

is identically satisfied. Therefore, we get the solution $T^{\#}(\xi, \zeta)$ by a line integral (and eq. (6.20*d*)) as

$$\hat{t} = T^{\#}(\xi, \zeta) = \int_{(\xi_0, 0)[\Gamma^{\#}]}^{(\xi, \zeta)}\left[\frac{\partial T^{\#}}{\partial x}(x, y)\,dx + \frac{\partial T^{\#}}{\partial y}(x, y)\,dy\right]$$

$$= \int_{\xi_0}^{\xi}\frac{T_E'(x)}{\sqrt{1 + [f_E(x)]^2}}\,dx - m\sqrt{1 + [f_E(\xi)]^2}\left(\frac{(\sinh\zeta - \zeta)}{[f_E(\xi)]^3} + \frac{2\zeta}{f_E(\xi)}\right)$$

$$+ 2m\ln\left[\frac{\coth[\zeta_H(\xi)/2] + \coth[\zeta/2]}{\coth[\zeta_H(\xi)/2] - \coth[\zeta/2]}\right], \qquad \ldots (6.23a)$$

and $$\coth[\zeta_H(\xi)/2] = \sqrt{1 + [f_E(\xi)]^{-2}}. \qquad \ldots (6.23b)$$

The solution function $T^{\#}$ contains one arbitrary constant $\xi_0 > \rho_b$ and two arbitrary functions f_E and τ_E. Now, by (6.15), (6.20*a*, *b*, *c*) and (6.23*a*), the transformation from the $\rho - \tau$ coordinates into the Schwarzschild coordinates is furnished by

$$\hat{r} = R_E(\rho, \tau), \qquad \ldots (6.24a)$$

$$\hat{t} = \hat{T}(\rho, \tau) = \int_{\rho 0}^{\rho}\frac{T_E'(x)}{\sqrt{1 + [f_E(x)]^2}}\,dx$$

$$+ \sqrt{1+[f_E(\rho)]^2}\left(\tau - \tau_E(\rho) - \frac{4m}{f_E(\rho)} \operatorname{argcoth} \sqrt{\frac{2m}{[f_E(\rho)]^2 R_E(\rho,\tau)} + 1}\right)$$

$$+ 2m \ln\left[\frac{\coth[\zeta_H(\rho)/2] + \sqrt{(2m/[f_E(\rho)]^2 R_E(\rho,\tau)) + 1}}{\coth[\zeta_H(\rho)/2] - \sqrt{(2m/[f_E(\rho)]^2 R_E(\rho,\tau) + 1}}\right], \quad \ldots (6.24b)$$

$$\coth[\zeta_H(\rho)/2] = \sqrt{1+[f_E(\rho)]^{-2}} \quad \ldots (6.24c)$$

and $$D_S := \{(\rho,\tau) : \rho b < \rho, -\infty < \tau < \tau_H(\rho)\}. \quad \ldots (6.24d)$$

Also $$\tau_H(\rho) := \tau_E(\rho) - \frac{m}{[f_E(\rho)]^3}\{\sinh[\zeta_H(\rho)] - \zeta_H(\rho)\} \quad \ldots (6.24e)$$

and $$R_E(\rho, \tau_H(\rho)) \equiv 2m. \quad \ldots (6.24f)$$

For the range $\hat{D}_S$ of the above transformation, we have to consider the boundary curve

$$\hat{r} = \hat{B}(t)$$

given by the parametric equations

$$\hat{r} = R_E(\rho b, \tau),$$

$$\hat{t} = \hat{T}(\rho b, \tau) \quad \ldots (6.25)$$

and $$-\infty < \tau < \tau_H(\rho_b).$$

The Schwarzschild coordinate chart is valid for

$$\hat{D}_S := \{(\hat{r}, \hat{t}) : \hat{B}(t) < \hat{r}, \hat{t} \in R\}. \quad \ldots (6.26)$$

To gain some physical insight, let us draw graphs for the equations for the special values $\xi = \xi_1 > \rho b, m = 1, f_E(\xi_1) \equiv 1$ and $\tau_E(\xi_1) = 0$ for all $\zeta \in \mathbb{R}$ and for all $\tau \in \mathbb{R}$. We obtain the $\hat{r}$ versus ζ and $\hat{r}$ versus $\hat{t}$ graphs in Fig. 5.

If we eject a massive ball vertically upwards with a velocity greater than the escape velocity, then, according to Newtonian gravity, the centre of the ball will follow the right-hand side of the $\hat{r}$ - τ graph (with respect to the Newtonian time). For the time reversed collapse from infinity, the left-hand side of the graph provides the proper representation. This can be called a hyperbolic cycloid.

Let us consider again (6.20*a*) and (6.23*a*) in an alternate domain,

$$D_T^{\#} := \{(\xi, \zeta(: \rho_b < \xi, 0 < \zeta < \zeta_H(\xi)\} \subset D_E^{\#}. \quad \ldots (6.27)$$

The transformation is furnished by

$$T = \Theta^{\#}(\xi, \zeta) := \frac{2m}{f_E(\xi)]^2} \sinh^2(\zeta/2) \quad \ldots (6.28a)$$

and $$X = X^{\#}(\xi, \zeta)$$

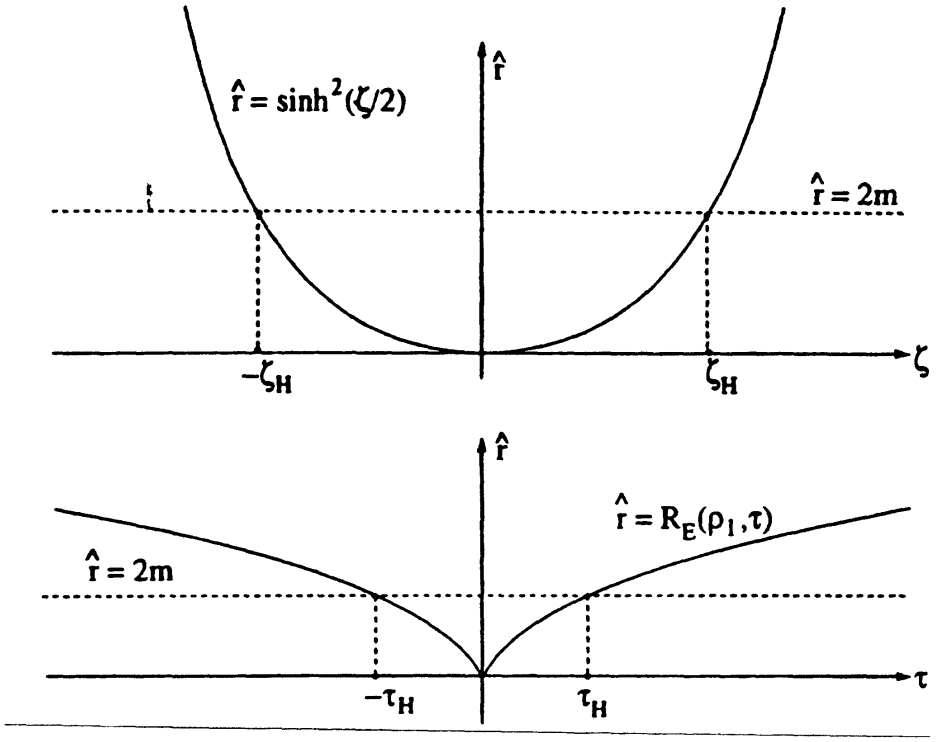

Fig. 5. Qualitative graph of $\hat{r}$ *vs.* ζ and the corresponding hyperbolic cycloid as a function of τ

$$:= \int_{\xi_0}^{\xi} \frac{\tau_E'(x)}{\sqrt{1+[f_E(x)]^2}}\,dx - m\sqrt{1+[f_E(\xi)]^2}\left(\frac{(\sinh\zeta-\zeta)}{[f_E(\xi)]^3}+\frac{2\zeta}{f_E(\xi)}\right)$$

$$+ 2m \ln\left[\frac{\coth(\zeta/2)+\coth(\zeta_H(\xi)/2)}{\coth(\zeta/2)-\coth(\zeta_H(\xi)/2)}\right]. \qquad \ldots (6.28b)$$

By (6.7*b*), (6.17*a*, *b*, *c*) and (6.28*a*, *b*), the exterior line element becomes (compare with eq. 4.32)

$$ds_E^2 = \left(\frac{2m}{T}-1\right)dX^2 + T^2\,d\Omega^2 - \left(\frac{2m}{t}-1\right)^{-1} dT^2.$$

This is again the black hole metric. The domain $\hat{D}_T$ of validity for the X-T coordinate chart is given by

$$\hat{D}_T := \{(X,T) : \hat{\beta}(T) < X < \infty, 0 < T < 2m\}. \qquad \ldots (6.29)$$

Here, the boundary curve $X = \hat{\beta}(T)$ is provided by the parametric equations

$$X = X^{\#}(\rho_b, \zeta)$$

$$T = \mathcal{T}^{\#}(\rho_b, \zeta) \qquad \ldots (6.30)$$

and $0, \zeta < \zeta_H(\rho_b)$.

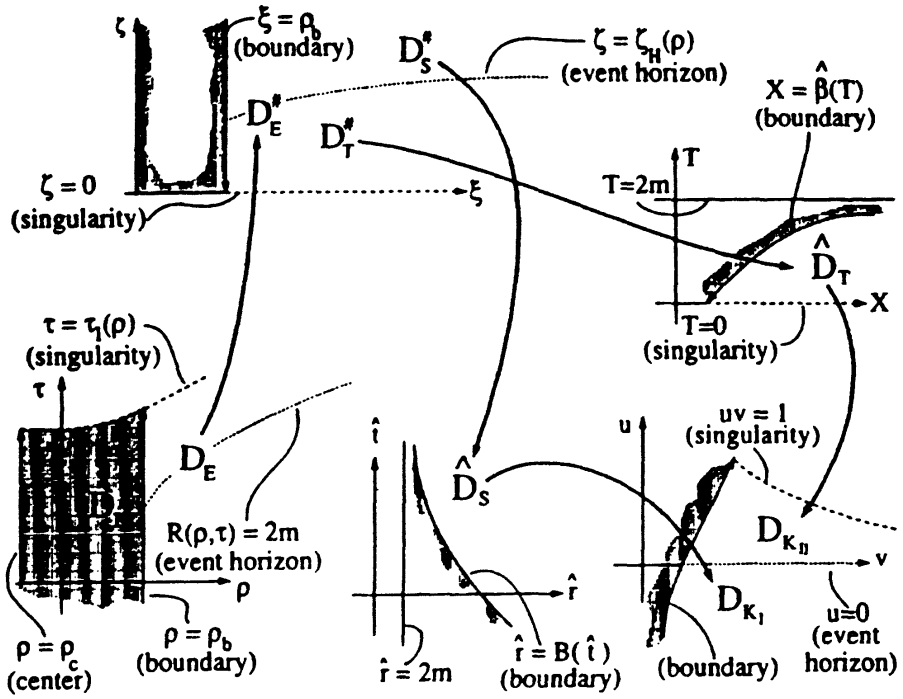

Fig. 6. Gravitational collapse in the hyperbolic case $\varepsilon = -1$ shown qualitatively in 5 charts

The Schwarzschild metric and the black hole metric can be transformed (see the equation (4.39*f*)) by KSS coordinates into

$$ds_E^2 = -\frac{32m^2}{\tilde{Y}(uv)} \exp\left(\frac{-\tilde{Y}(uv)}{2m}\right) du\, dv + [\tilde{Y}(uv)]^2\, d\Omega^2.$$

Note that the domains and ranges in different sections differ even though the same notations have occasionally been used for the sake of simplicity.

Now, we shall prove Fig. 6 which depicts the collapse of a stellar dust ball of type $\varepsilon = -1$ into a black hole with the help of five coordinate charts.

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