PINSA, **65, A**, No. 2, March 1999, pp. 149-166
© Printed in India.

A REVIEW ON SOME PROBLEMS OF ELASTO-DYNAMICS UNDER THE INFLUENCE OF GRAVITY

P R SENGUPTA[@], SUDIPTA SENGUPTA[†] AND SUMANTRA SENGUPTA[†]

[@]*Department of Mathematics, University of Kalyani, Kalyani-741 235, India*

[†]*Indian Institute of Mechanics of Continua, 201, Manicktala Main Road, Flat No. 42, Calcutta-700 054, India*

(Received 05 February 1998; Accepted 27 January 1999)

The aim of this review is to investigate the effect of gravity on some problems of propagation of waves in an isotropic elastic solid medium. The reported problems are directly related to engineering problems of dynamics of elastic systems as well as the problems of Seismology and Earthquake waves. The problems of surface waves of earthquakes in presence of earth's gravity field has been considered paying special attention to Rayleigh, Love and Stoneley type of waves. The gravity field has been included in this theory of surface waves by considering Biot's theory of initial stress. The effect of gravity on this propagation of waves and vibration in an isotropic elastic layer has been reported here. The effect of gravity on plane Lamb's problem in an isotropic elastic layer and semi-space has also been highlighted. The effect of gravity on steady-state response to moving loads as well as the effect of gravity on thermo-elastic Rayleigh waves are also considered in this article. In all these problems basic equations of motion of the solid under the influence of gravity has been focused and this theory has been applied to the relevant problems to find out distribution of stresses and the wave velocity equations. It has been shown that the wave velocity equations and the distribution of stress are in fair agreement with the corresponding classical problem in absence of gravity. Moreover, a good number of research problems as investigated by one of the authors (PRS) have been reported in this article. In all these investigated problems as reviewed here, the theories of initial stress of Biot have been taken into consideration.

Key Words: Effect of Gravity; Surface Waves; Elastic Layer; Plane Lamb's Problem; Steady State Response to Moving Loads

Introduction

The elastic surface waves propagated over the boundary surfaces of an isotropic elastic solid medium have been studied by a good number of investigators and presented in details in the informative monographs of Love[1], Bullen[2],. Jeffreys[3] as well as Ewing, Jardetsky and Press[4] and others[5-8]. In fact, Jefferys[9-11] studied thoroughly the surface waves of Earthquakes and the elastic waves in a continuous stratified medium. Stoneley[12-13] studied elastic waves at the surface of separation of two solids and Rayleigh waves in a medium with two surface layers. Stoneley[14] also studied Love waves in a triple surface layer. Moreover, we may refer the work of Rayleigh[15] on waves propagated along the plane surface of an elastic solid medium. These types of problems and its' further developments are exhibited in "Some Problems of Geodynamics" by Love[16]. Similar such problems associated with the propagation of waves in layered media have been presented thoroughly by Ewing *et al*[4]. Numerous references and literature, dealing with the investigations of a good number of investigators are presented in this monograph. It is highly useful to study these types of problems in the realm of researches of Earth sciences system.

However, the effect of gravity on such problem has not been studied thoroughly. It needs further developments and thorough investigations. It was Bromwich[17] who first investigated the influence of gravity on elastic waves and in particular on the vibration of an elastic globe. Biot[18] investigated the effect of gravity on elastic surface waves by taking the force of gravity to create an initial stress of

hydrostatic nature, the material being incompressible. This was done by applying his theory of initial stress. Although the effect of gravity is sometimes small for small wavelength of the waves, it is not always so. In discussing waves and vibrations on the surface of the earth, the effect of gravity and it's variation should not be ignored in order to accommodate the actual situation of the problem. Ignoring the effect of gravity the elastic equations of motion are drastically simplified. Even if we assume a uniform gravity field on the surface of the earth, the matter resting on it will create some initial stress owing to it's weight. This initial stress will vary from layer to layer and then becomes a function of the depth. It is found that the equilibrium initial stress is hydrostatic in nature and as such the classical investigation of waves on the surface of the earth create some deviations from the actual situation and these deviations are not always important. As such the authors of the present paper have attempted to show the influence of gravity on some problems of elastic waves and vibrations assuming the force of gravity create a type of initial stress of hydrostatic nature.

In investigating the problems of elastic waves and vibrations in various fields the monumental work in the form of monograph "Elasto-dynamics" by Eringen and Suhubi[19] is noteworthy. In their books a thorough treatment accompanined by the basic ideas and theories is presented. The research papers of Eringen and Suhubi[20] and Suhubi[21-25] are also very illuminating in this area. While preparing this paper the authors have consulted the ideas contained in these books and the research papers.

Further, the various investigations related to the study of effect of gravity on the various problems of elastic waves and vibrations by one of the authors (PRS) have also been presented in this research review article. De and Sengupta[33] investigated the effect of gravity on steady state response to moving loads on the free plane surface of a semi-infinite elastic solid considering various circumstances of sub-sonic, super-sonic and transonic speeds of the loads. De and Sengupta[34] further studied plane Lamb's problem under the influence of gravity. Different types of surface waves of earth-quakes, such as Rayleigh, Love and Stoneley types under the influence of gravity had also been investigated by De and Sengupta[35]. They have also considered[36,37] theromo-elastic Rayleigh waves under the influence of gravity and thermo-elastic plane Lamb's problem under the same influence. Thermo-elastic surface waves under the influence of gravity was thoroughly investigated by Sengupta and Acharya[38]. Effects of anisotropy on different types of surface waves in connection with Earthquakes under the influence of gravity was studied by Sengupta and De[39]. Lamb's plane problem in micro-polar elastic medium under the influence of temperature was studied by Acharya and Sengupta[40]. Magneto-elastic plane Lamb's problem under the influence of gravity was investigated by Mal and Sengupta[41]. Sengupta and Chel[42] studied the effects of couple stress and gravity on plane Lamb's problem in thermo-elastic solid. Further, the cumulative effect of gravity, magnetic field and thermal field on plane Lamb's problem in general visco-elastic solid medium was studied by Pal and Sengupta[43]. Further, surface waves in connection with earth-quakes in micro-polar thermo-elastic solid under the influence of gravity was investigated by Das and Sengupta[44]. This is followed by the problem of Rayleigh waves in higher order thermo-visco-elastic solids under the influence of couple stress and gravity by Das *et al.*[45]. Also, surface waves in magneto-thermo-visco-elastic media under the influence of gravity considering strain-rate and stress-rate of higher order had been studied by Biswas and Sengupta[46]. In micro-polar theory the effect of gravity had also been studied. Das and Sengupta[47] studied the effect of gravity on the propagation of wave in micro-polar thermo-elastic half space under the action of various heat exposure on the free surface. Different types of Earthquake waves of Rayleigh, Love and Stoneley types were investigated in thermo-visco-elastic medium under the same effect by Mukherjee and Sengupta[48]. Subsequently, Biswas *et al.*[49] attempted to investigate surface waves in thermo-visco-elastic media under the influence of gravity considering strain-rate and stress-rate of higher order. Surface waves in higher order thermo-visco-elastic media under the influence of gravity have been studied by Mukherjee and Sengupta[50]. In this problem the visco-elasticity of order n had been considered. Further, Kundu *et al.*[51] gave a mathematical solution to analyze the

influence of gravity on surface waves in general visco-elastic solid media involving time-rate of strain and time-rate of stress. Rayleigh type of surface wves in thermo-visco-elastic solid under the influence of couple stress and gravity had been considered by Das and Sengupta[52]. They presented a closed form wave velocity equation to study the velocity of waves in such types of solids in which all the effects are possible to be present. Further, Mukherjee *et al.*[53] investigated an interesting problem of surface waves in higher order visco-elastic media under the influence of gravity. This is followed by the case of surface waves in thermo-visco-elastic media under the influence of gravity[54]. Das and Sengupta[55] studied the same effect on wave propagation in a general higher order visco-elastic layer including strain-rate and stress-rate. The problem of uneven heating in a half-space in micro-polar elastic media under the influence of gravity was theoretically studied by Biswas and Sengupta[56]. The influence of gravity on visco-elastic surface waves in solids involving time rate of strain and stress of first order had been mathematically studied by Das and Sengupta[57]. Lamb's plane problem in a thermo-visco-elastic micro-polar medium with effect of gravity was thoroughly studied by Das *et al.*[58] The problem of surface waves of earth-quakes in an inhomogeneous elastic medium under the influence of gravity was considered by Das *et al.*[59]. An interesting problem of two dimensional wave propagation in a higher order micro-polar thermo-visco-elastic layer under the influence of gravity was studied by Sengupta and Dey(Das)[60]. Similar such problem on propagation of waves under the effect of gravity was also investigated by De and Sengupta[61].

To be precise, we state that in studying the effect of gravity, following the concept of Biot[18] in his monograph as well as in his research papers[26-29], the various authors had investigated different types of problems of Geodynamics and Seismology. In particular, surface waves of earth-quake, waves in an elastic layer, plane Lamb's problem, steady state response to moving loads and thermo-elastic Rayleigh waves had been presented in this review article.

Effect of Gravity on Surface Waves

Let us consider that isotropic, homogeneous elastic solid media M_1 and M_2, M_2 being above M_1 are welded in contact in the influence of gravity and the media are separated by a plane horizontal boundary, extending to infinitely great distance from the origin. We suppose a orthogonal cartesian axes OXYZ, where origin O is at any point on the boundary and OZ points normally into M_2.

We consider the possibility of a type of wave travelling along the OX direction in such a fashion that the disturbance produced is largely confined to the neighborhood of the boundary and at any instant all particles in any line parallel to OY have equal displacements. So, these two assumptions conclude that the wave type is a surface one and all partial derivatives with respect to the Y co-ordinate are zero.

We suppose that g is the acceleration due to gravity and acts in the OZ direction and hence the components of the body force are X=0, Y=0, Z=−g. We assume that the initial stress field due to gravity is hydrostatic; the state of initial stress S_{ij} are

$$S_{ij} = S, \quad i = j;$$
$$S_{ij} = 0, \quad i \neq j; \quad i, j = 1, 2, 3 \qquad \ldots (1)$$

where S is function of depth. The equilibrium conditions of the initial stress field will be

$$\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0, \quad \frac{\partial S}{\partial Z} - \rho\, g = 0, \qquad \ldots (2)$$

ρ being the density of the medium. The dynamical equations of motions of three dimensional problems under the above initial stress are[18]

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{13}}{\partial z} - \rho g \frac{\partial w}{\partial x} = \rho \frac{\partial^2 u}{dt^2},$$
$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} - \rho g \frac{\partial w}{\partial x} = \rho \frac{\partial^2 v}{dt^2},$$
$$\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{32}}{\partial y} + \frac{\partial s_{33}}{\partial z} - \rho g \rho \frac{\partial^2 v}{dt^2}, \rho \frac{\partial^2 w}{\partial t^2} \qquad \ldots (3)$$

where, (u, v, w) are the displacement components and s_{ij}, the stress tensor, connected with the strain tensor e_{ij} by Hooke's law

$$s_{ij} = 2\mu e_{ij} + \lambda \Delta \delta_{ij} \qquad \ldots (4)$$

(the symbols have their usual significance)

Let ρ, λ, μ and ρ', λ', μ' be the properties of the media M_1 and M_2 respectively. Then from eqs (3), (4) and the previous assumption, we can demand that the displacements (u, v, w) in M_1 will satisfy the equations

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u-\rho g\frac{\partial w}{\partial x}=\rho\frac{\partial^2 u}{dt^2},$$

$$\mu\nabla^2 v=\rho\frac{\partial^2 v}{dt^2},$$

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 w+\rho g\frac{\partial u}{\partial x}=\rho\frac{\partial^2 w}{dt^2} \quad \ldots (5)$$

where, $\Delta=\frac{\partial u}{\partial x}+\frac{\partial w}{dz}, \nabla^2=\frac{\partial^2}{\partial x^2}+\rho\frac{\partial^2}{dz^2}$

Now, we introduce the potential functions which are defined as[2]

$$u=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z},\quad w=\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial x}, \quad \ldots (6)$$

so that $\nabla^2\phi=\Delta, \Delta^2\psi=\frac{\partial w}{\partial x}-\frac{\partial u}{dz}$ and then we observe that the systems (5) in M_1 will be satisfied if ϕ, ψ and v satisfy the wave equations

$$\nabla^2\phi-\frac{g}{c_1^2}\frac{\partial\psi}{\partial x}=\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2},$$

$$\nabla^2\psi+\frac{g}{c_2^2}\frac{\partial\phi}{\partial x}=\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2},$$

$$\nabla^2 v=\frac{1}{c_2^2}\frac{\partial^2 v}{\partial t^2}, \quad \ldots (7)$$

where, $c_1^2=\frac{\lambda+2\mu}{\rho}, \quad c_2^2=\frac{\mu}{\rho}$

We can write down the similar relations in M_2 substituting ρ, λ, μ by ρ', λ', μ'. For solving the equations of (7) we put

$$\phi=P(z)\,e^{i\xi(x-ct)},$$
$$\psi=Q(z)\,e^{i\xi(x-ct)}, \quad \ldots (8)$$
$$v=R(z)\,e^{i\xi(x-ct)},$$

for the medium M_1 and for M_2 we consider P', Q', R' in place of P, Q, R. This will give us a particular solution corresponding to a group of simple harmonic waves of wave length $2\pi/\xi$, travelling with velocity c.

For the medium M_1 we have the following equations by substituting the first two expressions of (8) into the first two equations of (7):

$$\frac{d^2P}{dz^2}+\xi^2 v_1^2\,P-\frac{g}{c_1^2}\,i\xi\,Q=0,$$
$$\frac{d^2Q}{dz^2}+\xi^2 v_2^2\,Q+\frac{g}{c_2^2}\,i\xi\,P=0, \quad \ldots (9)$$

where, $v_1^2=\frac{c^2}{c_1^2}-1, \quad v_2^2=\frac{c^2}{c_2^2}-1.$ $\ldots (10)$

From eq. (9) we obtain the equations which determine P or Q as

$$\left[\left(\frac{d^2}{dz^2}+\xi^2\lambda_1^2\right)\left(\frac{d^2}{dz^2}+\xi^2\lambda_2^2\right)\right](P,Q)=0, \quad \ldots (11)$$

where, $\lambda_1^2+\lambda_2^2=v_1^2+v_2^2,$

$$\lambda_1^2\lambda_2^2=v_1^2\,v_2^2-\frac{v_0^2}{\xi^2},$$

$$v_0^2=\frac{g^2}{c_1^2\,c_2^2} \quad \ldots (12)$$

Then the solution of equation (11) will be

$$P(Z)=A_1e^{-i\xi\lambda_1 Z}+B_1e^{i\xi\lambda_1 Z}+C_1e^{-i\xi\lambda_2 Z}+D_1e^{i\xi\lambda_2 Z} \quad \ldots (13)$$

where A_1, B_1, C_1, D_1, are constants. As the effect is essentially a surface one, $P(z)$ must vanish with increasing distance from the boundary. This requirement will be satisfied if $P(z)$ contains an exponential factor in which the exponent is real and always greater than zero. So λ_1, λ_2 in M_1 and λ'_1, λ'_2 in M_2 are considered imaginary and constants corresponding to B_1, D_1 in M_1 and A_1, C_1 in M_2 should vanish. Thus, the solution of first and third equation of (7) will be

$$\phi=[Ae^{-i\xi\lambda_1 Z}+Be^{-i\xi\lambda_2 Z}]\,e^{i\xi(x-ct)},$$
$$\psi=[A'e^{-i\xi\lambda_1 Z}+B'e^{-i\xi\lambda_2 Z}]\,e^{i\xi(x-ct)}, \quad \ldots (14)$$

where, the constants A', B' are related respectively with A, B by means of either of two equations of (9). Now equating the coefficient of $e^{-i\xi\lambda_1 Z}$ and $e^{-i\xi\lambda_2 Z}$ to zero, we obtain from the first equation of (9)

$$A' = i\xi\, m_1 A,\quad B' = i\xi\, m_2 B$$

where,

$$m_j = c_1^2(\lambda_j^2 - v_1^2)/g,\quad j = 1, 2 \qquad \ldots (15)$$

Introducing the last expression of (8) into the last equation of (7) we obtain for M_1

$$v = Ce^{i\xi(v_2 z + x - ct)} \qquad \ldots (16)$$

So, the solution for M_1 are

$$\phi = [Ae^{-i\xi\lambda_1 z} + Be^{-i\xi\lambda_2 z}]\ e^{i\xi(x-ct)},$$
$$\psi = i\xi\,[m_1 Ae^{-i\xi\lambda_1 z} + m_2 Be^{-i\xi\lambda_2 z}]\ e^{i\xi(x-ct)},$$
$$v = Ce^{i\xi(-v_2 z + x - ct)} \qquad \ldots (17)$$

and those for M_2 are

$$\phi = [De^{i\xi\lambda'_1 z} + Ee^{i\xi\lambda'_2 z}]\ e^{i\xi(x-ct)},$$
$$\psi = i\xi\,[m'_1 De^{i\xi\lambda'_1 z} + m'_2 Ee^{i\xi\lambda'_2 z}]\ e^{i\xi(x-ct)},$$
$$v = Fe^{i\xi(-v'_2 z + x - ct)} \qquad \ldots (18)$$

where A, B, C, D, E, F are all constants and v_1, v_2, v'_1, v'_2, etc. are all positive imaginaries in which v'_1, v'_2 are obtained by the same fashion as are obtained v_1, v_2 by replacing ρ, λ, μ by ρ', λ', μ'.

We now apply the following boundary conditions as

1) The displacements at the boundary surface between M_1 and M_2 must be continuous at all times and places and
2) The stresses s_{31}, s_{32}, s_{33}, where

$$s_{31} = \mu\left[2\frac{\partial^2\phi}{\partial x\,dz} - \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2}\right],$$
$$s_{32} = \mu\frac{\partial v}{\partial z},$$
$$s_{33} = \lambda\nabla^2\phi + 2\mu\left[\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\psi}{\partial x\,\partial z}\right] \qquad \ldots (19)$$

for M_1 and similar expression for M_2 across the boundary surface between M_1 and M_2 must be continuous at all times and places.

Taking the help of eqs. (6), (17), (18) and the above two boundary conditions, we get six equations containing six unknown constants like A, B, C, ... etc. Eliminating these constants we obtain the following wave velocity equation in determinant form as

$$\begin{vmatrix} i\xi m_1\lambda_1 + 1 & i\xi m_2\lambda_2 + 1 & i\xi m'_1\lambda'_1 - 1 & i\xi m'_2\lambda'_2 - 1 \\ \lambda_1 - i\xi m_1 & \lambda_2 - i\xi m_2 & \lambda'_1 + i\xi m'_1 & \lambda'_2 + i\xi m'_2 \\ \mu(l_1 + n_1) & \mu(l_2 + n_2) & \mu'(l'_1 - n'_1) & \mu'(l'_2 - n'_2) \\ \rho(\xi_1 - \eta_1) & \rho(\xi_2 - \eta_2) & -\rho'(\xi'_1 + \eta'_1) & -\rho'(\xi_2 + \eta'_2) \end{vmatrix} = 0 \qquad \ldots (20)$$

where,

$$l_j = 2\lambda_j,\quad n_j = -i\xi m_j(\lambda_j^2 - 1),$$
$$\xi_j = c_1^2(\lambda_j^2 + 1) - 2c_2^2,$$
$$\eta_j = -2c_2^2\, i\xi\,\lambda_j\, m_j;\qquad j = 1, 2, \qquad \ldots (21)$$

and l'_1, l'_2 etc. are the same functions as l_1, l_2 etc. obtained by replacing $\lambda_1, \lambda_2, c_1, c_2$ etc. by λ'_1, λ'_2, c'_1, c'_2 etc. This wave velocity equation for the general elastic surface waves propagated in the interior of the earth depends on the particular value of ξ and creates a dispersion of the general wave form and on the gravity field, imposing certain changes in the wave velocity and wave form.

A Few Particular Cases

(a) *Rayleigh Waves*: The particular case of the foregoing problem is to investigate the effect of gravity on Rayleigh waves. For the propagation of Rayleigh waves, the plane boundary must be a free surface so that M_2 is replaced by a vacuum. In view of this fact we can write

$$A(l_1 + n_1) + B(l_2 + n_2) = 0,$$
$$A(\xi_1 - \eta_1) + B(\xi_2 - \eta_2) = 0. \qquad \ldots (22)$$

Eliminating the constants A and B using eqs. (10), (12), (15) and (21) we obtain

$$i\xi c_1^2\left[\sqrt{\frac{c^2}{c_1^2} - 1} + \sqrt{\frac{c^2}{c_2^2} - 1}\right]\left[4c_2^2\sqrt{\frac{c^2}{c_1^2} - 1}\sqrt{\frac{c^2}{c_2^2} - 1} + \frac{1}{c_2^2}(c^2 - 2c_2^2)^2\right]$$
$$+ 4g\left[c_1^2\sqrt{\frac{c^2}{c_1^2} - 1}\sqrt{\frac{c^2}{c_2^2} - 1} + c_2^2\right] = 0 \qquad \ldots (23)$$

in which, the higher powers of the small quantities have been neglected. The roots of this equation will determine the wave velocity of Rayleigh type in the influence of gravity.

If we neglect the gravity effect then this equation is changed to the form

$$4\sqrt{1-\frac{c^2}{c_1^2}}\sqrt{1-\frac{c^2}{c_2^2}}=\left(2-\frac{c^2}{c_2^2}\right)^2, \qquad \ldots (24)$$

which is actually the classical Rayleigh wave velocity equation, independent of ξ. On the otherhand, it is to note that due to the presence of gravity, the eq. (23) depends on the particular value of ξ and thus the wave velocity c also depends on it creating a dispersion of a general wave form. It is also to be noted that when ξ is large and as a consequence the wave length being small, the effect of gravity is sufficiently small. Again if ξ is small so that the wave length is large, the effect of gravity plays significant role on wave velocity. In this case the Rayleigh wave velocity c is slightly less than c_2. While discussing the effects of gravity we have assumed that g/c_1=0.002 and g/c_2=0.003, in conformity with actual situation of the earthquake waves of Rayleigh type.

(b) *Love Waves*: For the case of Love type of surface waves, it follows from previous discussions that the equations of motion do not contain g. So, it can be concluded that the gravity field, in any way, can not affect the propagation of Love waves.

(c) *Stoneley Waves*: According to the classical theory, Stoneley waves[4] are the generalized form of Rayeligh waves propagating along the common boundary of M_1 and M_2. Due to the influence of gravity, Stoneley wave along the common boundary of M_1 and M_2 are determined by the root of the frequency eq. (20). Now, retaining terms upto the first power of g, the eq. (2) can be approximated to the following form:-

$$\Delta'-g\sum_{j=1}^{4}\Delta_j=0 \qquad \ldots (25)$$

where

$$\Delta'=\begin{vmatrix} -1 & \nu_2 & 1 & \nu'_2 \\ \nu_1 & 1 & \nu'_1 & -1 \\ 2\mu\nu_1 & \mu\left(1-\nu_2^2\right) & 2\mu'\nu'_1 & -\mu'\left(1-\nu'^2_2\right) \\ \rho\left[c_1^2\left(\nu_1^2+1\right)-2c_2^2\right] & 2\rho c_2^2\nu_2 & -\rho'\left[c_1^2\left(\nu'^2_1+1\right)-2c_2^2\right] & 2\rho' c'^2_2\nu'_2 \end{vmatrix} \qquad \ldots (26)$$

and Δ_j (j=1, 2, 3, 4) are the value of Δ' with the first, second, third and fourth columns of it, replaced respectively by

$$\frac{c'^2_1 c_1^2}{\nu'_3 c_2^2}\begin{vmatrix} \nu_1 \\ 1 \\ -\mu\left(\nu_1^2-1\right) \\ 2\rho c_2^2\nu_1 \end{vmatrix}, \quad \frac{c'^2_1}{\nu'_3}\begin{vmatrix} -1 \\ \nu_2 \\ 2\mu\nu_2 \\ \rho\left[c_1^2\left(\nu_2^2+1\right)-2c_2^2\right] \end{vmatrix},$$

$$\frac{c'^2_1 c_1^2}{c'^2_2\nu_3}\begin{vmatrix} \nu'_1 \\ -1 \\ \mu'\left(\nu'^2_1-1\right) \\ 2\rho' c'^2_2\nu'_1 \end{vmatrix}, \quad \frac{c_1^2}{\nu_3}\begin{vmatrix} 1 \\ \nu'_2 \\ 2\mu'\nu'_2 \\ -\rho'\left[c'^2_1\left(\nu'^2_2+1\right)-2c'^2_2\right] \end{vmatrix},$$

in which,

$\xi\nu_3\left(\nu_1^2-\nu_2^2\right)=\xi\nu'_3\left(\nu'^2_1-\nu'^2_2\right)=i.$

Now, if the gravity force is neglected, eq. (25) becomes Δ'=0, which is of the following form when expanded:

$$c^4\left[(\rho-\rho')^2+(\rho\nu'_1+\rho'\nu_1)(\rho\nu'_2+\rho'\nu_2)\right.$$
$$+2Mc^2(\nu_1\nu_2\rho'-\nu'_1\nu'_2\rho-\rho+\rho')$$
$$\left.+M^2(\nu_1\nu_2+1)(\nu'_1\nu'_2+1)\right]=0 \qquad \ldots (27)$$

where, $M=2(\mu-\mu')$.

The eq. (27) represents the velocity of propagation of the classical Stoneley waves and does not depend on the particular value of ξ. But the second part of eq. (25) associated with g, represents the effect of gravity contributed by the Stoneley wave velocity c and the general wave form depends on the particular values of ξ and g and creates a dispersion of a general wave form.

Effect of Gravity on Waves in an Elastic Layer

Let us consider a homogeneous isotropic elastic layer of thickness 2h under the influence of gravity and bounded by the planes $Z=\pm h$ which are free of stress. Let, there exists a plane wave moving with constant speed c in the OX direction where, a plane longitudinal wave in the infinite space would be propagated with speed c_1 and transverse one with speed c_2. The boundary surface of the elastic space by two planes undergoes a distortion of the state of stress which influences the velocities of

propagation of the elastic waves. We now introduce a set of orthogonal cartesian axes where the origin O is at any point on the middle plane of the infinite plate held horizontally in the influence of gravity and Z axis is a line drawn into the plane vertically downward. Further, from the nature of the boundary condition, we suppose that the problem is of a plane strain one, so that, the non-zero component of displacements u and w, depending only on the co-ordinates x, z and time t.

Let g be the acceleration due to gravity. Then the components of body force are $X=0$, $Z=g$. We suppose that the initial stress due to gravity is hydrostatic and hence the state of initial stress S_{ij} are

$$S_{11}=S_{33}=S, \quad S_{13}=0, \qquad \ldots (28)$$

where, S is a function of depth. The equilibrium condition of the initial stress fields are[18]

$$\frac{\partial S}{\partial x}=0, \quad \frac{\partial S}{\partial z}+\rho g=0, \qquad \ldots (29)$$

ρ being the density of the elastic layer.

So, under the influence of initial stress, the dynamical equations of two dimensional problem are

$$\frac{\partial s_{11}}{\partial x}+\frac{\partial s_{13}}{\partial z}+\frac{\partial}{\partial x}(S_{k1}w_{1k}+S_{11}e-S_{1k}e_{1k}) + \frac{\partial}{\partial z}(S_{k3}w_{1k}+S_{13}e-S_{1k}e_{3k})=\rho\frac{\partial^2 u}{\partial t^2},$$

and

$$\frac{\partial s_{13}}{\partial x}+\frac{\partial s_{33}}{\partial z}+\frac{\partial}{\partial x}(S_{k1}w_{3k}+S_{31}e-S_{3k}e_{1k}) + \frac{\partial}{\partial z}(S_{k3}w_{3k}+S_{33}e-S_{3k}e_{3k})=\rho\frac{\partial^2 w}{\partial t^2},$$

$$; k=1,3 \qquad \ldots (30)$$

where

$$s_{ij}=2\mu e_{ij}+\lambda e\delta_{ij}, \quad e=\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z},$$

$$\omega_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right), \quad e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad \ldots (31)$$

and other notations have their usual significance[18,62].

The eqs. (3) and (4) can be written in simplified forms by introducing eqs. (1) and (2) into them as

$$\frac{\partial s_{11}}{\partial x}+\frac{\partial s_{13}}{\partial z}+\rho g(e_{31}-w_{13})=\rho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial s_{31}}{\partial x}+\frac{\partial s_{33}}{\partial z}+\rho g(e_{33}-e)=\rho\frac{\partial^2 w}{\partial t^2}. \qquad \ldots (32)$$

Now, let the displacements u and w are derivable from the displacement potentials $\phi(x,z,t)$, $\psi(x,z,t)$ by the relations[2]

$$u=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z}, \qquad w=\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial x}. \qquad \ldots (33)$$

Under the condition the wave velocity equations will be

$$\nabla^2\phi-\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}+\frac{g}{c_1^2}\frac{\partial\psi}{\partial x}=0,$$

$$\nabla^2\psi-\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}-\frac{g}{c_2^2}\frac{\partial\phi}{\partial x}=0. \qquad \ldots (34)$$

where, $c_1^2=\dfrac{\lambda+2\mu}{\rho}$, $c_2^2=\dfrac{\mu}{\rho}$,

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}. \qquad \ldots (35)$$

Now, as the planes $Z=\pm h$ are free of stress, the boundary conditions will be

$$\sigma_{33}=\sigma_{13}=0 \quad \text{on} \quad Z=\pm h, \qquad \ldots (36)$$

where,

$$\sigma_{33}=\lambda\nabla^2\phi=2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\psi}{\partial x\,\partial z}\right),$$

$$\sigma_{13}=\mu\left(2\frac{\partial^2\phi}{\partial x\,\partial z}-\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial x^2}\right). \qquad \ldots (37)$$

We have to seek the solutions of equations of (34) under the boundary conditions (36). For the solutions we assume

$$(\phi,\psi)=[\phi^*(z),\psi^*(z)]\,e^{-i(\alpha x-wt)} \qquad \ldots (38)$$

and we obtain the following equations by introducing (38) into (34) as

$$\left(\frac{d^2}{dz^2}-\nu_1^2\right)\phi^*-\frac{i\alpha g}{c_1^2}\psi^*=0,$$

$$\left(\frac{d^2}{dz^2}-\nu_2^2\right)\psi^*+\frac{i\alpha g}{c_2^2}\phi^*=0, \qquad \ldots (39)$$

where, $\nu_j=\alpha^2-w^2/c_j, \quad j=1,2. \qquad \ldots (40)$

By eliminating ϕ^* and ψ^* from equations of (39) we obtain the wave equations

$$\left[\left(\frac{d^2}{dz^2}-\lambda_1^2\right)\left(\frac{d^2}{dz^2}-\lambda_2^2\right)\right](\phi^*,\psi^*)=0, \quad \ldots (41)$$

where,

$$\lambda_1^2+\lambda_2^2=\nu_1^2+\nu_2^2,$$
$$\lambda_1^2\,\lambda_2^2=\nu_1^2\nu_2^2-\nu_3^2,$$
$$\nu_3^2=\alpha^2 g^2/c_1^2\,c_2^2. \quad \ldots (42)$$

The solution of equations (41) are

$$\phi^*=A\sinh\lambda_1 z+B\cosh\lambda_1 z+C\sinh\lambda_2 z + D\cosh\lambda_2 z,$$
$$\psi^*=A'\sinh\lambda_1 z+B'\cosh\lambda_1 z+C'\sinh\lambda_2 z + D'\cosh\lambda_2 z, \quad \ldots (43)$$

where, A', B', C', D' are related with A, B, C, D respectively by means of either of the equations (39). Now equating the co-efficient of $\sinh\lambda_1 z$, $\cosh\lambda_1 z$, $\sinh\lambda_2 z$, $\cosh\lambda_2 z$ to zero, we obtain

$$A'=im_1A,\quad B'=-im_1B,\quad C'=-im_2C,\quad D'=-im_2D$$

in which, $m_j=\frac{c_1^2}{\alpha_g}\left(\lambda_j^2-\nu_1^2\right), \quad j=1,2. \quad \ldots (44)$

Now introducing

$$\xi_j=c_1^2(\lambda_j^2-\alpha^2)+2c_2^2\alpha^2,$$
$$\eta_j=2c_2^2\alpha\lambda_j m_j,$$
$$n_j=m_j\left(\lambda_j^2+\alpha^2\right),$$
$$l_j=2\alpha\lambda_j, \qquad j=1,2 \quad \ldots (45)$$

the boundary conditions will be satisfied if

$$A\xi_1p_1-B\eta_1p_1+C\xi_2p_2-D\eta_2p_2=0,$$
$$A\eta_1q_1-B\xi_1q_1+C\eta_2q_2-D\xi_2q_2=0,$$
$$An_1p_1-Bl_1p_1+Cn_2p_2-Dl_2p_2=0,$$
$$Al_1q_1-Bn_1q_1+Cl_2q_2-Dn_2q_2=0, \quad \ldots (46)$$

where

$$p_j=\sinh\lambda_j h;\quad q_j=\cosh\lambda_j h;\quad j=1,2. \quad \ldots (47)$$

Then, the equations of (46) will be satisfied for non-zero values of A, B, C, D if

$$\begin{vmatrix} \xi_1p_1 & \eta_1p_1 & \xi_2p_2 & \eta_2p_2 \\ \eta_1q_1 & \xi_1q_1 & \eta_2q_2 & \xi_2q_2 \\ n_1p_1 & l_1p_1 & n_2p_2 & l_2p_2 \\ l_1q_1 & n_1q_1 & l_2q_2 & n_2q_2 \end{vmatrix}=0. \quad \ldots (48)$$

This equation can be written after a little simplification as

$$\frac{\tanh\lambda_1 h}{\tanh\lambda_2 h}+\frac{\tanh\lambda_2 h}{\tanh\lambda_1 h}= \frac{(\xi_1l_2-\eta_2n_1)^2+(l_1\xi_2-\eta_1n_2)^2}{(\xi_1l_1-\eta_1n_1)(\xi_2l_2-\eta_2n_2)} - \frac{-(n_1\xi_2-n_2\xi_1)^2-(l_1\eta_2-l_2\eta_1)^2}{(\xi_1l_1-\eta_1n_1)(\xi_2l_2-\eta_2n_2)} \quad \ldots (49)$$

This transcendental equation determines the wave velocity c of the waves propagated in the elastic layer and depends upon the gravity of the earth. Considering the eqs. (42), (44) and (45) it is easy to understand that the effect of gravity is very small. Again, if the effect of gravity is neglected, one can derive the gravity free wave velocity equation by passing to the limit as g tends to zero. It should be stated that, while discussing the effect of gravity on the wave propagation in the elastic layer, we have assumed g/c_1=0.002 and g/c_2=0.003 in conformity with the actual situation of the problem.

When the wave length is very small in comparison to the thickness of the layer $2h$, the quantities $\lambda_1 h, \lambda_2 h, \alpha h$ are large and hence we can assume that the ratios of the hyperbolic tangents in the transcendental eq. (49) approaches unity. Finally, we get the equation in a very simplified form which leads to calculate the velocity of propagation of Rayleigh surface waves in the influence of gravity in a homogeneous isotropic elastic solid medium as

$$(\xi_1+\eta_1)(l_2+n_2)=(\xi_2+\eta_2)(l_1+n_1) \quad \ldots (50)$$

Substituting the values of $\xi_j, \eta_j, l_j, m_j, n_j, \lambda_j$ (j=1,2) in eq. (50) and neglecting the squares and higher powers of small quantities we get the following equation

$$c_1^2(\nu_1+\nu_2)[4\nu_1\nu_2\alpha^2c_2^2+c_1^2(\alpha^2+\nu_2^2)(\alpha^2-\nu_1^2) -2c_2^2\alpha^2(\alpha^2+\nu_2^2)]+4\alpha^2g(\nu_1\nu_2c_1^2+c_2^2\alpha^2)=0. \quad \ldots (51)$$

This equation shows that the wave velocity c is highly affected by the gravity of the earth and depends on the value of α, so that under the present boundary conditions there exists a dispersion of the general wave form. It can also be noted

that when the wave length is small and as a consequence α is large, the effect of gravity will be of great significance. Again, if the gravity field is neglected, the equation (51) changes to

$$4\sqrt{1-\frac{c^2}{c_1^2}}\sqrt{1-\frac{c^2}{c_2^2}}=\left(2-\frac{c^2}{c_2^2}\right)^2,$$

which is actually the classical equation of Rayleigh surface waves[2].

Now, if the wave length is large in comparison to the thickness of the layer, the quantities $\lambda_1 h$, $\lambda_2 h, \alpha h$ may be considered as small and hyperbolic tangents can be replaced by their arguments. In that case the transcendental eq. (49) takes the form as follows:

$$\begin{aligned}&(\lambda_1-\lambda_2)^2(\xi_1 l_1-\eta_1 n_1)(\xi_2 l_2-\eta_2 n_2)\\&=\lambda_1\lambda_2[(l_2+n_2)(\xi_1+\eta_1)-(l_1+n_1)(\xi_2+\eta_2)]\\&\times[(l_2-n_2)(\xi_1-\eta_1)-(l_1-n_1)(\xi_2-\eta_2)].\end{aligned}\quad \ldots (52)$$

Effect of Gravity on Plane Lamb's Problem

Let us consider an isotropic, homogeneous, elastic semi-space under the influence of gravity and in the plane boundary $x=0$ of the semi-space a loading $Z(y,t)$ acts normal to the plane boundary. Let, g be the acceleration due to gravity and then the components of the body force are $X=0$, $Y=g$. We suppose that the initial stress due to gravity is hydrostatic. In these conditions, the state of initial stress are

$$S_{11}=S_{22}=S,\ S_2=0, \quad \ldots (53)$$

where, S is defined as the function of depth. Then the equilibrium condition of the initial stress field are[18]

$$\frac{\partial S}{\partial x}=0,\ \frac{\partial S}{\partial y}+\rho g=0, \quad \ldots (54)$$

where, ρ is the density of the Earth.

Now using eqs.(53) and (54) and introducing

$$s_{ij}=2\mu e_{ij}+\lambda e\delta_{ij},\ e=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y},$$

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \quad \ldots (55)$$

where, δ_{ij} is the Kronecker symbol and λ, μ are Lame' constants, we can write the dynamical equations of the two dimensional problem under the initial stress as

$$\begin{aligned}&\frac{\partial s_{11}}{\partial x}+\frac{\partial s_{12}}{\partial y}+\rho g\frac{\partial v}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2},\\&\frac{\partial s_{12}}{\partial x}+\frac{\partial s_{22}}{\partial y}-\rho g\frac{\partial u}{\partial x}=\rho\frac{\partial^2 v}{\partial t^2}.\end{aligned}\quad \ldots (56)$$

Now, we introduce the elastic potentials ϕ and ψ which are related with displacement u and v in the following relations

$$u=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial y},\quad v=\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial x}. \quad \ldots (57)$$

We obtain the following wave velocity equations by introducing (55) and (57) into (56):

$$\begin{aligned}&\nabla^2\phi-\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}+\frac{g}{c_1^2}\frac{\partial\psi}{\partial x}=0,\\&\nabla^2\psi-\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}-\frac{g}{c_2^2}\frac{\partial\phi}{\partial x}=0,\end{aligned}\quad \ldots (58)$$

where,

$$c_1^2=\frac{\lambda+2\mu}{\rho},\ c_2^2=\frac{\mu}{\rho},\ \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \quad \ldots (59)$$

Eliminating ϕ or ψ from equations of (58), we have the wave equations

$$\left[\left(\nabla^2-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)+n^2\frac{\partial^2}{\partial x^2}\right](\phi,\psi)=0, \quad \ldots (60)$$

where, $n^2=g^2/(c_1c_2)^2$. $\ldots (61)$

Now, we are to formulate the two dimensional Lamb's problem in the influence of gravity by finding the solutions of eq. (58) under the boundary conditions

$$\begin{aligned}&s_{11}=-Z(y,t);\ s_{12}=0 \quad \text{for } x=0\\&\text{and } \phi=\psi=0 \qquad \text{at infinity,}\end{aligned}\quad \ldots (62)$$

if the loading acts on the boundary to be bounded by a plane.

Here, the stresses S_{ij} are defined as

$$s_{11}=\lambda\nabla^2\phi+2\mu\left(\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial x\partial y}\right),$$

$$s_{22} = \lambda\nabla^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\psi}{\partial x\partial y}\right),$$

$$s_{12} = \mu\left(2\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2}\right). \quad \ldots (63)$$

Applying the double Fourier integral transform in eq. (60) defined by the formulae

$$\bar{\phi}(\chi,\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint \phi(x,y,t)\, e^{i(y\xi+\eta t)}dy\, dt,$$

$$\phi(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint \bar{\phi}(\chi,\xi,\eta)\; e^{-i(y\xi+\eta t)}d\xi\, d\eta, \quad \ldots (64)$$

we get the ordinary differential equation with independent variable x as

$$\left(\frac{d^2}{dx^2} - \lambda_1^2\right)\left(\frac{d^2}{dx^2} - \lambda_2^2\right)(\bar{\phi},\bar{\psi}) = 0, \quad \ldots (65)$$

where,

$$\lambda_1^2 + \lambda_2^2 = 2\xi^2 - K_1^2 - K_2^2 - n^2,$$
$$\lambda_1^2\lambda_2^2 = \xi^4 - \xi^2(K_1^2 + K_2^2) + K_1^2K_2^2, \quad \ldots (66)$$

and

$$K_1^2 = \frac{\eta^2}{c_1^2},\ K_2^2 = \frac{\eta^2}{c_2^2}.$$

As we know that both ϕ and ψ tend to zero at infinity, we consider the solutions of the eq. (65) in the form

$$\bar{\phi} = Ae^{-\lambda_1 x} + Be^{-\lambda_2 x},$$
$$\bar{\psi} = A'e^{-\lambda_1 x} + B'e^{-\lambda_2 x}, \quad \ldots (67)$$

where A, A' and B, B' are related by means of eq. (58).

Equating the coefficient of $e^{-\lambda_1 x}$ and $e^{-\lambda_2 x}$ to zero we get

$$A' = m_1A, B' = m_2B,$$

where,

$$m_j = \frac{c_1^2}{g\lambda_j}(\lambda_j^2 - \xi^2 + K_1^2) \quad \ldots (68)$$

Now, to find out the value of A and B we perform the double Fourier integral transform in eq. (62) and use eq. (63) and hence obtain

$$\left|2\mu\left(\frac{d^2\bar{\phi}}{dx^2} + i\xi\frac{d\bar{\psi}}{dx}\right) + \lambda\left(\frac{d^2\bar{\phi}}{dx^2} - \xi^2\bar{\phi}\right)\right|_{x=0} = -\bar{Z}(\xi,\eta),$$

$$\left|\xi^2\bar{\psi} + \frac{d^2\bar{\psi}}{dx^2} - 2i\xi\frac{d\bar{\phi}}{dx}\right|_{x=0} = 0, \quad \ldots (69)$$

where,

$$\bar{Z}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint Z(y,t)\, e^{i(y\xi+\eta t)}\, dy\, dt. \quad \ldots (70)$$

Inserting $\bar{\phi}$ and $\bar{\psi}$, already obtained, into eq. (69) we have the expression of A and B as

$$A = -q_2\bar{Z}/(p_1q_2 - p_2q_1),$$

$$B = q_1\bar{Z}/(p_1q_2 - p_2q_1), \quad \ldots (71)$$

where,

$$p_j = 2\mu\lambda_j^2 + \lambda(\lambda_j^2 - \xi^2) - i\xi\lambda_j m_j,$$
$$q_j = m_j(\xi^2 + \lambda_j^2) + 2i\xi\lambda_j,\ j = 1, 2. \quad \ldots (72)$$

Finally, we have the expression for ϕ and ψ by inverting the Fourier transform in eq. (67) as

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint (Ae^{-\lambda_1 x} + Be^{-\lambda_2 x})e^{-i(\xi y+\eta t)}\, d\xi\, d\eta,$$

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint (Am_1e^{-\lambda_1 x} + Bm_2e^{-\lambda_2 x})e^{-i(\xi y+\eta t)}\, d\xi\, d\eta. \quad \ldots (73)$$

Now, since the functions ϕ and ψ are known, the displacement and stresses can be easily calculated by using eqs. (55), (57), (71) and (73).

A Particular Case: Action of Loading Harmonic in Time

In this case, the boundary conditions will be

$$s_{11} = -Z(y)e^{-iwt}, s_{12} = 0 \text{ on } x = 0. \quad \ldots (74)$$

Taking the help of eqs. (57) and (67) we get the equations

$$u = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\iint [A(\lambda_1 - i\xi m_1)e^{-\lambda_1 x} + B(\lambda_2 - i\xi m_2)e^{-\lambda_2 x}]e^{-i(\xi y+\eta t)}\, d\xi\, d\eta,$$

$$v = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\iint [A(i\xi + m_1\lambda_1)e^{-\lambda_1 x} + B(i\xi + m_2\lambda_2)e^{-\lambda_2 x}]e^{-i(\xi y+\eta t)}\, d\xi\, d\eta, \quad \ldots (75)$$

which are the expressions for displacements u and v.

In this case, we have

$$\bar{Z}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint Z(y)e^{-i\omega t}\, e^{i(\xi y+\eta t)}\, dy\, dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\iint Z(y)e^{i\xi y}\, dy \int_{-\infty}^{\infty} e^{-it(\omega-\eta)}\, dt,$$

which can be written as

$$\bar{Z}(\xi, \eta) = \sqrt{2\pi}\, \delta(\eta - \omega)\, \bar{Z}^*(\xi) \qquad \ldots (76)$$

introducing

$$\bar{Z}^*(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z(y) e^{i\xi y}\, dy$$

and

$$\int_{-\infty}^{\infty} e^{ipt}\, dt = 2\pi\delta(p),$$

where, $\delta(p)$ is the Dirac function[30].

Finally, we have the expressions of u and v by substituting eq. (76) into (75) and doing the required integrations as

$$u = -\frac{e^{-i\omega t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{q_1(\lambda_2 - i\xi m_2) e^{-\lambda_2 x} - q_2(\lambda_1 - i\xi m_1) e^{-\lambda_1 x}}{p_1 q_2 - p_2 q_1} \right]_{\eta=\omega} \times Z^*(\xi) e^{-i\xi y}\, d\xi$$

$$v = -\frac{e^{-i\omega t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{q_1(i\xi + m_2\lambda_2) e^{-\lambda_2 x} - q_2(i\xi + m_1\lambda_1) e^{-\lambda_1 x}}{p_1 q_2 - p_2 q_1} \right]_{\eta=\omega} \times Z^*(\xi) e^{-i\xi y}\, d\xi \qquad \ldots (77a)$$

In these expressions η present in the parenthesis should be replaced by ω.

Again, if we consider another case when a horizontal concentrated force acts at the origin in OY-direction, i.e. $Z(y)=P\delta(y)$, we have $Z^*(\xi) = 1/\sqrt{2\pi}$ and then eq. (76) will be reduced to the form

$$u = -\frac{Pe^{-i\omega t}}{2\pi} \int_{-\infty}^{\infty} \left[\frac{q_1(\lambda_2 - i\xi m_2) e^{-\lambda_2 x} - q_2(\lambda_1 - i\xi m_1) e^{-\lambda_1 x}}{p_1 q_2 - p_2 q_1} \right]_{\eta=\omega} \times e^{-i\xi y}\, d\xi,$$

$$v = -\frac{Pe^{-i\omega t}}{2\pi} \int_{-\infty}^{\infty} \left[\frac{q_1(i\xi + m_2\lambda_2) e^{-\lambda_2 x} - q_2(i\xi + m_1\lambda_1) e^{-\lambda_1 x}}{p_1 q_2 - p_2 q_1} \right]_{\eta=\omega} \times e^{-i\xi y}\, d\xi. \qquad \ldots (77b)$$

From these equations, it is easy to understand that the displacement components are periodic in nature and depend on earth's gravitational field.

Effect of Gravity on Steady State Response to Moving Loads

Let a line load moves with a constant speed U over an elastic half-space for an infinitely long time so that a steady state exists in the neighbourhood of the loading as seen by an observer moving with the load. We introduce a set of orthogonal cartesian co-ordinate axes $OXYZ$ in the semi-infinite isotropic elastic medium $y \geq 0$, the origin O being at a point on the free plane boundary and the y-axis is directly vertically downward. Suppose (x, y, z) is the rectangular cartesian co-ordinate fixed in the medium which occupy half-space which is in a state of plane strain owing to the moving line load. Then the elastic displacements u, v, $(w=0)$ are derivable from the displacement potentials $\phi(x, y, t)$, $\psi(x, y, t)$ by the relations

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{dy}, \quad v = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \qquad \ldots (78)$$

If g is the acceleration due to gravity then the body force components are $X=0$, $Y=g$. Assuming that the initial stress due to gravity is hydrostatic, the state of initial stresses S_{ij} are $S_{11}=S_{22}=S$ and $S_{12}=0$, where S is the function of depth. The equilibrium conditions of the initial stress field are[18]

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} + \rho g = 0, \qquad \ldots (79)$$

ρ being the density of the medium.

Then the dynamical equations of two dimensional problem under initial stress are[18]

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho g(\omega + e_{12}) = \rho \frac{\partial^2 u}{\partial t^2},$$
$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - \rho g(e - e_{22}) = \rho \frac{\partial^2 v}{\partial t^2}, \qquad \ldots (80)$$

where, stress tensor s_{ij} is connected with strain tensor e_{ij} by the law

$$s_{ij} = 2\mu e_{ij} + \lambda e \delta_{ij} \qquad \ldots (81)$$

and

$$\omega=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y},\ e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$$

while δ_{ij} is the Kroneker symbol and λ, μ are Lame' elastic constants.

Using eq. (81), (78) and (80) we have

$$\nabla^2\phi-\frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}+\frac{g}{c_1^2}\frac{\partial\psi}{\partial x}=0,$$
$$\nabla^2\psi-\frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}-\frac{g}{c_2^2}\frac{\partial\phi}{\partial x}=0, \quad \ldots (82)$$

in which $c_1^2=\frac{\lambda+2\mu}{\rho}$, $c_2^2=\frac{\mu}{\rho}$, $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

We have assumed that the load moves in the negative direction of x-axis at a constant speed U. An observer moving with the load at the same speed would see the load as stationary.

Following Fung[31] we use Galilean transformation

$$x'=x+Ut,\ y'=y,\ t'=t \quad \ldots (83)$$

The boundary condition in the moving coordinates are

$$\sigma_{yy}=P\delta(x'),\ \sigma_{xy}=0 \text{ on } y'=0, \quad \ldots (84)$$

where $\delta'(x)$ is the Dirac-delta function and

$$\sigma_{yy}=\lambda\nabla^2\phi+2\mu\left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\psi}{\partial x\partial y}\right)$$
$$\sigma_{xy}=\mu\left(2\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial x^2}\right), \quad \ldots (85)$$

Now, as the response in the elastic half-space is in a steady state, ϕ and ψ are independent of t'. So x and t enter ϕ and ψ only in the combination of $x+Ut$. Hence, eq. (82) take the form

$$\nabla_1^2\phi-\frac{U^2}{c_1^2}\frac{\partial^2\phi}{\partial x'^2}+\frac{g}{c_1^2}\frac{\partial\psi}{\partial x'}=0,$$
$$\nabla^2\psi-\frac{U^2}{c_2^2}\frac{\partial^2\psi}{\partial x'^2}-\frac{g}{c_2^2}\frac{\partial\phi}{\partial x'}=0, \quad \ldots (86)$$

so that, $\nabla_1^2=\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}$.

Let us introduce Mach numbers M_j and the quantities α_j^2, β_j^2, $\beta_j'^2$ where,

$$M_j=U/c_j,\ \alpha_j^2=g/c_j^2,$$
$$\beta_j^2=1-M_j^2 \text{ if } M_j<1;\ j=1,2,$$
$$\beta_j'^2=M_j^2-1 \text{ if } M_j>1;\ j=1,2. \quad \ldots (87)$$

Equations (86) take the following forms when $M_j<1$ ($j=1,2$),

$$\beta_1^2\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y'^2}+\alpha_1^2\frac{\partial\psi}{\partial x'}=0,$$
$$\beta_2^2\frac{\partial^2\psi}{\partial x'^2}+\frac{\partial^2\psi}{\partial y'^2}-\alpha_2^2\frac{\partial\phi}{\partial x'}=0 \quad \ldots (88)$$

and when $M_j>1$ ($j=1,2$) the eq. (86) become

$$\beta_1'^2\frac{\partial^2\phi}{\partial x'^2}-\frac{\partial^2\phi}{\partial y'^2}-\alpha_1^2\frac{\partial\psi}{\partial x'}=0,$$
$$\beta_2'^2\frac{\partial^2\psi}{\partial x'^2}-\frac{\partial^2\psi}{\partial y'^2}+\alpha_2^2\frac{\partial\phi}{\partial x'}=0. \quad \ldots (89)$$

We obtain the equations which are satisfied by ϕ or ψ by eliminating them from eq. (88) and (89) as

$$\left[\left(\beta_1^2\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)\left(\beta_2^2\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)+\alpha_1^2\alpha_2^2\frac{\partial^2}{\partial x'^2}\right]$$

$(\phi, \psi)=0$ if $M_j<1$ $\quad \ldots (90a)$

$$\left[\left(\beta_1'^2\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}\right)\left(\beta_2'^2\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}\right)+\alpha_1^2\alpha_2^2\frac{\partial^2}{\partial x'^2}\right]$$

$(\phi, \psi)=0$ if $M_j>1$ $\quad \ldots (90b)$

$$\left[\left(\beta_1^2\frac{\partial^2}{\partial x'^2}+\frac{\partial}{\partial y'^2}\right)\left(\beta_2'^2\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}\right)-\alpha_1^2\alpha_2^2\frac{\partial^2}{\partial x'^2}\right]$$

$(\phi, \psi)=0$ if $M_1<1$ and $M_2>1$. $\quad \ldots (90c)$

Now, integrating once the changed boundary conditions obtained from eq. (84), at $y'=0$, we finally have

$$\mu\left[(M_2^2-2)\frac{\partial\phi}{\partial x'}-\alpha_2^2\psi+2\frac{\partial\psi}{\partial y'}\right]=-PH(x');$$
$$2\frac{\partial\phi}{\partial y'}-(M_2^2-2)\frac{\partial\psi}{\partial x'}-\alpha_2^2\phi=0; \quad \ldots (91)$$

where, $H(x')$ is the Heaviside unit step function.

To solve the present problem, we are to seek solutions of eqs. (90) under the condition (91). Here, we will consider three cases namely (a) supersonic case, (b) subsonic case and (c) transonic case while solving the problem.

(a) Supersonic Case: ($M_j > 1, j=1, 2$)

In this case, we have to solve the eq. (90b). We consider the solution of the form

$$\phi(x', y') = \Phi(y') e^{imx'}, \qquad \ldots (92)$$

where m is a parameter. Introducing eq. (92) into (90b), we get the following ordinary differential equation in Φ as

$$\left(\frac{d^2}{dy'^2} + m^2\gamma_1'^2\right)\left(\frac{d^2}{dy'^2} + m^2\gamma_2'^2\right)\Phi = 0 \qquad \ldots (93)$$

in which,

$$\gamma_1'^2 + \gamma_2'^2 = \beta_1'^2 + \beta_2'^2,\ \gamma_1'^2\gamma_2'^2 = \beta_1'^2\beta_2'^2 - \frac{\alpha_1^2\alpha_2^2}{m^2} \qquad \ldots (94)$$

The solution of the eq. (93) satisfying the radiation condition at infinity is

$$\Phi = Ae^{-im\gamma_1' y'} + Be^{-im\gamma_2' y'},$$

A, B being arbitary constants. Hence, the general solution of the eq. (90b) will be

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[A(m)e^{im(x'-\gamma_1' y')} + B(m)e^{im(x'-\gamma_2' y')}]dm,$$

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[A_1(m)e^{im(x'-\gamma_1' y')} + B_1(m)e^{im(x'-\gamma_2' y')}]dm. \qquad \ldots (95)$$

where, A_1 (m), $B_1(m)$ are related respectively with $A(m)$, $B(m)$ by means of eq. (89).

Now performing the necessary steps for solving A and B, we have

$$A = \frac{P(m)}{imL'}(Mm^2\nu_2' - N_2),$$
$$B = -\frac{P(m)}{imL'}(Mm^2\nu_1' - N_1), \qquad \ldots (96)$$

where $P(m)$ is the Fourier transform of $-\frac{1}{\mu}\int_0^{x'} p(x')dx'$, in which $p(x')=p(x+Ut)$ is the pressure distribution over the surface $y = 0$ and

$$\nu_j' = (\beta_1'^2 - \gamma_j'^2)/\alpha_1^2, \quad (j=1, 2);$$
$$M = M_2^2 - 2,\ N_j = \alpha_2^2 + 2i\gamma_j' m, \quad (j=1,2);$$
$$L' = (\nu_2' - \nu_1')(Mm^2 - N_1N_2) + M(N_2 - N_1)(\nu_1'\nu_2' m^2 - 1). \qquad \ldots (97)$$

Thus, finally we obtain

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(m)}{imL'}[(Mm^2\nu_2' - N_2)e^{im(x'-\gamma_1' y')} - (Mm^2\nu_1' - N_1)e^{im(x'-\gamma_2' y')}]dm,$$

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(m)}{L'}[\nu_1'(Mm^2\nu_2' - N_2)e^{im(x'-\gamma_1' y')} - \nu_2'(Mm^2\nu_1' - N_1)e^{im(x'-\gamma_2' y')}]dm. \qquad \ldots (98)$$

(b) Subsonic case: ($M_j < 1, j = 1, 2$)

Here, we will solve the eq. (90a) and for this, we consider the solution in the form

$$\phi(x', y') = \Phi(y')\, e^{imx'}. \qquad \ldots (99)$$

Utilising eqs. (99) and (90a) we obtain the ordinary differential equation in the form of Φ as

$$\left(\frac{d^2}{dy'^2} - m^2\gamma_1^2\right)\left(\frac{d^2}{dy'^2} - m^2\gamma_2^2\right)\Phi = 0 \qquad \ldots (100)$$

where,

$$\gamma_1^2 + \gamma_2^2 = \beta_1^2 + \beta_2^2,\ \gamma_1^2\gamma_2^2 = \beta_1^2\beta_2^2 - \frac{\alpha_1^2\alpha_2^2}{m^2}.$$

Obviously, the solution of the eq. (100) is

$$\Phi = Ce^{-|m|\gamma_1 y'} + De^{-|m|\gamma_2 y'}$$

and hence, the general solution of the eq. (90a) satisfying the radiation condition is

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[C(m)e^{imx'-|m|\gamma_1 y'} + D(m)e^{imx'-|m|\gamma_2 y'}]dm,$$

$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[C_1(m)e^{imx'-|m|\gamma_1 y'} + D_1(m)e^{imx'-|m|\gamma_2 y'}]dm, \qquad \ldots (101)$$

where, C_1 (m) and $D_1(m)$ are related respectively with $C(m)$ and $D(m)$ by means of eq. (88).

Now, doing the necessary calculations, finally we have the expression for C and D as

$$C = -\frac{P(m)}{imL}(K_1 + m^2 \nu_2 M)$$
$$D = \frac{P(m)}{imL}(K_1 + m^2 \nu_1 M) \quad \ldots (102)$$

where, $P(m)$ and M are already defined and

$$\nu_j = (\beta_1^2 - \gamma_j^2)/\alpha_1^2, \qquad (j = 1, 2),$$
$$K_j = 2|m|\gamma_j + \alpha_2^2, \qquad (j = 1, 2),$$
$$L = (\nu_2 - \nu_1)(K_1 K_2 - m^2 M^2) + M(K_2 - K_1)(\nu_1 \nu_2 m^2 - 1) \quad \ldots (103)$$

Hence, ultimately we have

$$\phi = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{P(m)}{imL}[(K_2 + m^2 \nu_2 M)e^{imx' - |m|\gamma_1 y'} - (K_1 + m^2 \nu_1 M)\, e^{imx' - |m|\gamma_2 y'}]\, dm,$$
$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{P(m)}{L}[\nu_1 (K_2 + m^2 \nu_2 M)e^{imx' - |m|\gamma_1 y'} - \nu_2 (K_1 + m^2 \nu_1 M)e^{imx' - |m|\gamma_2 y'}]\, dm. \quad \ldots (104)$$

(c) Transonic case: $(M_1 < 1, M_2 > 1)$

In this case, we have to solve the eq. (90c) and for that we suppose the solution of the form

$$\phi(x', y') = \Phi(y')\, e^{imx'} \quad \ldots (105)$$

Using eqs. (105) and (90c) we have the ordinary differential equation

$$\left(\frac{d^2}{dy'^2} - m^2\gamma_1^2\right)\left(\frac{d^2}{dy'^2} + m^2\gamma_2'\right)\Phi = 0 \quad \ldots (106)$$

where,

$$\gamma_2'^2 - \gamma_1^2 = \beta_2'^2 - \beta_1^2,\ \gamma_1^2 \gamma_2'^2 = \beta_1^2 \beta_2'^2 + \frac{\alpha_1^2 a_2^2}{m^2}$$

and it's solution is

$$\Phi = E\, e^{-|m|\gamma_1 y'} + F\, e^{-im\gamma_2' y'} \quad \ldots (107)$$

So, the general solution of eq. (90c) are

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[E(m)e^{imx' - |m|\gamma_1 y'} + F(m)e^{im(x' - \gamma_2' y')}]\, dm,$$
$$\psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[E_1(m)e^{imx' - |m|\gamma_1 y'} + F_1(m)e^{im(x' - \gamma_2' y')}]\, dm, \quad \ldots (108)$$

where, $E_1(m)$ and $F_1(m)$ are connected respectively with $E(m)$ and $F(m)$ by means of equations (88) and (89).

Performing all the calculations we have the expressions for E and F as

$$E = \frac{P(m)}{imL''}(m^2 \nu_3 M + N_2),$$
$$F = -\frac{P(m)}{imL''}(m^2 \nu_1 M + K_1) \quad \ldots (109)$$

where $P(m)$, M, ν_j, N_j, K_j are already defined and

$$L'' = (\nu_3 - \nu_1)(m^2 M^2 - N_2 K_1) - M(N_2 - K_1)(\nu_1 \nu_3 m^2 - 1).$$

So, finally we have

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(m)}{imL''}[(m^2 M \nu_3 + N_2)e^{imx' - |m|\gamma_1 y'} - (m^2 M \nu_1 + K_1)e^{im(x' - \gamma_2' y')}]\, dm,$$
$$\psi = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(m)}{L''}[\nu_1 (m^2 M \nu_3 + N_2)\, e^{imx' - |m|\gamma_1' y'} - (m^2 M \nu_1 + K_1)\nu_3 e^{im(x' - \gamma_2' y')}]\, dm. \quad \ldots (110)$$

So, in any of the three cases, when the functions ϕ and ψ are known, we can easily calculate the displacements and stresses using the relations (78) and (85). In all the cases, the potential functions ϕ and ψ are very much dependent on the gravity field of the Earth.

Effect of Gravity on Thermo-elastic Rayleigh Waves

The influence of gravity over thermo-elastic Rayleigh waves has been discussed in this article, where it is assumed that the gravity field produces a type of initial stress, hydrostatic in nature.

We introduce a system of orthogonal cartesian axes $OX_1X_2X_3$ where the origin O is on the free plane boundary of the semi-space $x_2 \geq 0$ and the X_2 axis is drawn vertically downward in the interior of the isotropic elastic medium having the idealised homogeneous properties. We suppose that the elastic medium is free to exchange heat with the contents of the region $x_2 < 0$ and before the disturbances, both media are at temperature T_0 everywhere. The initial stress is being produced by a slow process of creep where all shear stresses

tend to become zero after a long interval of time. So, considering the two dimensional nature of the problem, we can write the state of initial stress

S_{ij} as $S_{11}=S_{22}=S$, $S_{12}=0$, ...(111)

S being the function of depth.

The equilibrium conditions of initial stress[18] are

$$\frac{\partial s}{\partial x_1}=0,\ \frac{\partial s}{\partial x_2}+\rho g=0, \qquad \ldots(112)$$

where, ρ is the density of the material and g is the acceleration due to gravity.

So, following Biot[18] the initial stresses in the two dimensional form of dynamical equations of an elastic medium under stress can be written as

$$\frac{\partial}{\partial x_1}(s_{11}+\rho g u_2)+\frac{\partial s_{12}}{\partial x_2}=\rho\frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial}{\partial x_1}(s_{12}-\rho g u_1)+\frac{\partial s_{22}}{\partial x_2}=\rho\frac{\partial^2 u_2}{\partial t^2}, \qquad \ldots(113)$$

using eqs. (111) and (112) and considering u_i and s_{ij} $(i,j=1,2)$ being the components of displacement vectors and stress tensor respectively.

Now, to investigate the Rayleigh type surface waves in the direction of X_1 axis we consider the displacement potentials $\phi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ which are related to the displacement components u_1, u_2 by means of equations

$$u_1=\frac{\partial\phi}{\partial x_1}-\frac{\partial\psi}{\partial x_2},\ u_2=\frac{\partial\phi}{\partial x_2}+\frac{\partial\psi}{\partial x_1}, \qquad \ldots(114)$$

wherefrom we get

$$\nabla^2\phi=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2},\quad \nabla^2\psi=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2} \text{ and}$$

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$$

Then the generalised Hooke's law for an isotropic, perfectly elastic solid which conducts heat is

$$s_{ij}=\frac{E_T}{1+\nu_T}\varepsilon_{ij}+\frac{E_T\nu_T}{(1+\nu_T)(1-2\nu_T)}\varepsilon_{kk}\delta_{ij}-\frac{E_T\alpha}{3(1-2\nu_T)}(T-T_0)\delta_{ij} \qquad \ldots(115)$$

where, E_T, ν_T are the isothermal elastic constants, ε_{ij} is the strain tensor, T_0 is the initial absolute temperature and α is the co-efficient of volume expansion $\left(=\frac{\partial\varepsilon_{kk}}{\partial T}\right)$.

Now, utilising eqs. (113), (114) and (115) we have

$$\frac{\partial^2\phi}{\partial t^2}=v_T^2\nabla^2\phi-\frac{\alpha_T}{K_T\rho}+g\frac{\partial\psi}{\partial x_1},$$

$$\frac{\partial^2\psi}{\partial t^2}=v_s^2\nabla^2\psi-g\frac{\partial\phi}{\partial x_1}, \qquad \ldots(116)$$

where,

$$v_T^2=\frac{\lambda+2\mu}{\rho},\quad v_s^2=\frac{\mu}{\rho},$$

$$K_T=3\left(\frac{\partial\varepsilon_{kk}}{\partial s_{kk}}\right)_T=\frac{3(1-2\nu_T)}{E_T} \qquad \ldots(117)$$

which is termed as isothermal compressibility.

Moreover, the Fourier's law of heat conduction in an isotropic elastic medium is

$$K\nabla^2 T=\rho c_\varepsilon\frac{\partial T}{\partial t}+\frac{\alpha T_0}{K_T}\frac{\partial}{\partial t}(\nabla^2\phi), \qquad \ldots(118)$$

where, c_ε is the specific heat at constant strain and K is the thermal conductivity.

Now, the force components per unit area at the boundary $x_2=0$ are zero and the initial stress due to gravity at $x_2=0$ is also zero; hence, we have the boundary conditions

$s_{12}=s_{22}=0$ at $x_2=0$. ....(119)

Again, the thermal boundary condition will be

$$\frac{\partial T}{\partial x_2}+hT=0 \quad \text{at} \quad x_2=0. \qquad \ldots(120)$$

In setting the boundary conditions, it is considered that the maximum temperature difference across the surface $x_2=0$ is always small.

Now, we are to solve the eqs. (116) and (118) under the conditions (119) and (120) and for that we assume the solution of the form

$$[\phi,\psi,T]=[\phi^*(x_2),\ \psi^*(x_2),\ T^*(x_2)]e^{i(\eta x_1-wt)} \qquad \ldots(121)$$

Utilising the eqs. (121), (116) and (118) we have

$$\frac{d^2\phi^*}{dx_2^2}-\left(\eta^2-\frac{w^2}{v_T^2}\right)\phi^*=\frac{1}{v_T^2}\left(\frac{\alpha T^*}{K_T\rho}-ig\eta\psi^*\right),$$

$$\frac{d^2\psi^*}{dx_2^2}-\left(\eta^2-\frac{w^2}{v_s^2}\right)\psi^*=\frac{ig\eta}{v_s^2}\phi^*,$$

$$\frac{d^2T^*}{dx_2^2}-\left(\eta^2-\frac{\rho c_\varepsilon iw}{K}\right)T^*=-\frac{iw\alpha T_0}{KK_T}\left(\frac{d^2}{dx_2^2}-\eta^2\right)\phi^* \quad \ldots(122)$$

It is clear that the eq. (122) have exponential solutions and in order that the functions ϕ, ψ, and T shall describe a surface wave, they must become vanishingly small as $x_2 \to \infty$. In view of this condition, we can assume the solutions of the eq. (122) of the form

$$\phi^* = Ae^{-x_2\sqrt{\eta^2-J_1^2}}+Be^{-x_2\sqrt{\eta^2-J_2^2}}+Ce^{-x_2\sqrt{\eta^2-J_3^2}},$$

$$\psi^* = A_1e^{-x_2\sqrt{\eta^2-J_1^2}}+B_1e^{-x_2\sqrt{\eta^2-J_2^2}}+C_1e^{-x_2\sqrt{\eta^2-J_3^2}},$$

$$T^* = A_2e^{-x_2\sqrt{\eta^2-J_1^2}}+B_2e^{-x_2\sqrt{\eta^2-J_2^2}}+C_2e^{-x_2\sqrt{\eta^2-J_3^2}} \quad \ldots(123)$$

Introducing eq. (123) into eq. (122) we see that J_j^2 (j=1,2,3) are the roots of the equation

$$(w^2-v_T^2J^2)(\rho c_\varepsilon iw-KJ^2)(w^2-v_s^2J^2) - \frac{\alpha^2J^2T_0iw}{K_T^2\rho}(w^2-v_s^2J^2) - g^2\eta^2(\rho c_\varepsilon iw-KJ^2)=0 \quad \ldots(124)$$

The constants A_1 and A_2, B_1 and B_2, C_1 and C_2 are related with the constants A, B, C respectively by means of the relations

$$A_1=\alpha_1A, \quad B_1=\alpha_2B, \quad C_1=\alpha_3C,$$
$$A_2=\gamma_1A, \quad B_2=\gamma_2B, \quad C_2=\gamma_3C, \quad \ldots(125)$$

where,

$$\alpha_j=\frac{ig\eta}{w^2-J_j^2v_s^2}, \quad \gamma_j=\frac{i\alpha T_0wJ_j^2}{K_T(\rho C_\varepsilon iw-KJ_j^2)}; \quad j=1,2,3 \quad \ldots(126)$$

In evaluating the quantities $\sqrt{\eta^2-J^2}$ the root with positive real part should be taken in each case.

Now, expressing the stress s_{ij} in terms of functions ϕ, ψ, T and employing the boundary conditions (119) and (120) we get a few homogeneous equations and applying the condition of consistency between these homogeneous equations, we finally obtain

$$\begin{aligned}
&\gamma_1(h/\eta-\beta_1)[2\rho iv_T^2(\beta_2-\beta_3)(1+\beta_2\beta_3)\\
&+\alpha_2(1+\beta_2^2)\{\rho v_T^2(1-\beta_3^2)-2\rho v_s^2(1-i\alpha_3\beta_3)\}\\
&-\alpha_3(1+\beta_3^2)\{\rho v_T^2(1-\beta_2^2)-2\rho v_s^2(1-i\alpha_2\beta_2)\}\\
&+\frac{\alpha}{K_T\eta^2}\{2i(\beta_2\gamma_3-\gamma_2\beta_3)+\alpha_2\gamma_3(1+\beta_2^2)\\
&-\gamma_2\alpha_3(1+\beta_3^2)\}-4\rho v_s^2\{i(\beta_2-\beta_3)\\
&-\beta_2\beta_3(\alpha_2-\alpha_3)\}]+\gamma_2(h/\eta-\beta_2)\\
&\times[2\rho iv_T^2(\beta_3-\beta_1)(1+\beta_3\beta_1)+\alpha_3(1+\beta_3^2)\\
&\times\{\rho v_T^2(1-\beta_1^2)-2\rho v_s^2(1-i\alpha_1\beta_1)\}\\
&-\alpha_1(1+\beta_1^2)\{\rho v_T^2(1-\beta_3^2)-2\rho v_s^2(1-i\alpha_3\beta_3)\}\\
&+\frac{\alpha}{K_T\eta^2}\{2i(\beta_3\gamma_1-\gamma_3\beta_1)+\alpha_3\gamma_1(1+\beta_3^2)\\
&-\gamma_3\alpha_1(1+\beta_1^2)\}-4\rho v_s^2\{i(\beta_3-\beta_1)\\
&-\beta_3\beta_1(\alpha_3-\alpha_1)\}]+\gamma_3(h/\eta-\beta_3)[2\rho i(\beta_1-\beta_2)\\
&\times(1+\beta_1\beta_2)+\alpha_1(1+\beta_1^2)\{\rho v_T^2(1-\beta_2^2)\\
&-2\rho v_s^2(1-i\alpha_2\beta_2)\}-\alpha_2(1+\beta_2^2)\{\rho v_T^2(1-\beta^2)\\
&-2\rho v_s^2(1-i\alpha_1\beta_1)\}+\frac{\alpha}{K_T\eta^2}\{2i(\beta_1\gamma_2-\gamma_1\beta_2)\\
&+\alpha_1\gamma_2(1+\beta_1^2)-\gamma_1\alpha_2(1+\beta_2^2)\}-4\rho v_s^2\{i(\beta_1-\beta_2)\\
&-\beta_1\beta_2(\alpha_1-\alpha_2)\}]=0 \qquad \ldots(127)
\end{aligned}$$

where, $\beta_j^2=1-J_j^2/\eta^2$ (j=1,2,3). $\ldots(128)$

Eq. (127) together with eqs. (124), (126) and (128) describes the relationship between the frequency of wave (ω) and the wave number (η). If $c(=\omega/\eta)$ is the velocity of propagation of thermo-elastic Rayleigh surface wave in the influence of gravity, it can be said from eq. (127) that such waves depend on gravity field and suffer dispersion. Moreover, it is very clear that if η is large and as a consequence, the wave length is small, the effect of gravity on thermo-elastic Rayleigh wave is also very small. On the otherhand, if the value of η is small, so that the wave length is large, the effect of gravity is of great significance and plays an important role for the determination of wave velocity c. Again, if we consider that gravity field is absent, then eq. (124)

gives us the relation $J_j^2 = w^2 / v_s^2$ and the equation

$$K_T^2\rho(w^2 - v_T^2J^2)(\rho c_\varepsilon iw - KJ^2) - \alpha^2J^2T_0iw = 0 \quad \ldots(129)$$

having roots J_1^2 and J_2^2. Under this condition it is clear from equation (126) that α_1, $\alpha_2 \to 0$ and $\alpha_3 \to \infty$.

Now, in view of these conditions we observe that equation (127) takes the form of

$$\left(2 - \frac{w^2}{v_s^2\eta^2}\right)^2\left(\beta_1^2 + \beta_1\beta_2 + \beta_2^2 - 1 + \frac{w^2}{v_T^2\eta^2}\right) - 4\beta_1\beta_2\beta_3(\beta_1 + \beta_2) = \frac{h}{\eta}\left[\left(2 - \frac{w^2}{v_s^2\eta^2}\right)^2 \times (\beta_1 + \beta_2)4\beta_3\right)\left(\beta_1\beta_2 + 1 - \frac{w^2}{v_T^2\eta^2}\right)\Bigg] \quad \ldots.(130)$$

This equation determines the velocity of the thermo-elastic Rayleigh surface wave and is also subject to a dispersion of the wave form[32].

On the otherhand, due to the influence of gravity field alone, the conditions are γ_1, $\gamma_2 \to 0$ and β_3, $\gamma_3 \to \infty$ and hence the equation (127) reduces to

$$[2i\beta_1 + \alpha_1(\beta_1^2 + 1)][v_T^2(1 - \beta_2^2) - 2v_s^2(1 - i\alpha_2\beta_2)] - [2i\beta_2 + \alpha_2(\beta_2^2 + 1)][v_T^2(1 - \beta_1^2) - 2v_s^2(1 - i\alpha_1\beta_1)] = 0 \quad \ldots.(131)$$

whereas, the equation (124) takes the forms

$$(w^2 - v_T^2J^2)(w^2 - v_s^2J^2) - g^2\eta^2 = 0, \quad \ldots.(132)$$

having the roots J_1^2 and J_2^2.

So, finally neglecting squares and higher powers of small quantities and using the eqs. (126), (128) and (132), we obtain the following equations, which is nothing but the simplified form of (131):

$$v_T^2\eta(v_1 + v_2)[4v_1v_2v_s^2 - \frac{1}{v_s^2}(2v_s^2 - c^2)^2] + 4g(v_1v_2v_T^2 - v_s^2) = 0 \quad \ldots(133)$$

where,

$$v_1^2 = 1 - \frac{c^2}{v_T^2}, \quad v_2^2 = 1 - \frac{c^2}{v_s^2}. \quad \ldots(134)$$

The roots of the eq. (133) determines the velocity of the Rayleigh type surface waves in the influence of gravity and is also subject to the dispersion of wave form.

Discussions

In most of the investigations the generalised theory of internal gravity waves as proposed by Biot has been given importance. Acoustic gravity waves as a particular case of the theory of elasticity under initial stress may also be considered. In fact, the gravity field contributes an initial stress is the main theme of the problem. It was first considered by Biot[18]. Here also, following the Biot's theory and his line of development the problems have been considered. It is highly expository in nature and admits wider application. In a broader aspect the theory of acoustic gravity waves may also be considered in case of fluid motion. Gravity waves in a liquid may also be taken into account to develop the area of fluid mechanics. It is also contained in the monograph of Biot. Clearly, the gravity waves, in solids and liquids, will generate a new horizon in Mechanics of Continua. The main concept of the article is to pose a modified area in the realm of researches of the propagation of waves under the influence of gravity. This exposition and it's wider application may help the future investigator to be involved such types of works. The stability problem may also be considered.

Acknowledgement

The junior authors thank Dr D C Sanyal, Department of Mathematics, University of Kalyani for his constant inspiration and necessary help for the preparation of this review work.

References

1 A E H Love *A treatise on the Mathematical Theory of Elasticity* (1953) Cambridge University Press, Fourth Edition

2 K E Bullen *An introduction to the Theory of Seismology* (1963) Cambridge University Press, Third Edition

3 Sir Harold Jefferys *The Earth* (1959) Cambridge University Press, Fourth Edition

4 W M Ewing, W S Jardetsky and F Press (1957) *Elastic Waves in Layered Media*, McGraw Hill Book Company, New York

5 C Pekeris *Physics* **6** (1935) 133

6 K Sezawa & K Kanai *Bull Earth Res Inst Tokyo* **15** (1937) 84

7 James T Wilson *Bull Seismol Soc Amer* **32** (1942) 297

8 V T Buchwald *Quart Jour Mech Appl Math* **14** (1961) 293

9 Sir H Jefferys *Mon Nat R Astr Soc Geophys Suppl* **1** (1925) 282

10 Sir H Jefferys *Mon Nat R Astr Soc Geophys Suppl* **3** (1935) 253

11 Sir H Jefferys *Mon Nat R Astr Soc Geophys Supply* **7** (1957) 332

12 R Stoneley *Proc Roy Soc* **A 106** (1924) 416

13 R Stoneley *Mon Nat R Astr Soc Geophys Suppl* **6** (1955) 610; **7** (1955) 71

14 R Stoneley *Mon Nat R Astr Soc Geophys Suppl* **4** (1937) 43

15 Lord Rayleigh *Proc Lond Math Soc* **17** (1885) 4

16 A E H Love *Some Problems on Geodynamics* (1911) Cambridge University Press

17 T J I A Bromwich *Proc Lond Math Soc* **30** (1898) 98

18 M A Biot *Mechanics of Incremental Deformations* (1965) John Wiley and Sons Inc., New York

19 A C Eringen and E S Suhubi *Elastodynamics*, **1&2** (1974) Academic Press, New York & London

20 A C Eringen and E S Suhubi *Int Jour Engg Sci* **2** (1964) 189, 389

21 E S Suhubi *Int Jour Engg Sci* **2** (1965) 509

22 E S Suhubi *Int Jour Engg Sci* **8** (1970) 699

23 E S Suhubi *Bull Tech Univ Istanbul* **25** (1972) 116

24 E S Suhubi *Letters Appl Eng Sci* **1** (1973) 119

25 E S Suhubi *Int Jour Engg Sci* **2** (1965) 441

26 M A Biot *Proc for Fifth International Congress for Appl Mech* (1939) 117

27 M A Biot *Philosophical Magazine* **27(7)** (1939) 468

28 M A Biot *Jour Appl Phys* **10(12)** (1940) 860

29 M A Biot *Jour Appl Phys* **11(8)** (1940) 522

30 I N Snedden *Fourier Transform* (1951) McGraw Book Company Inc. New York Toronto London

31 Y C Fung *Fundamentals of Solid Mechanics* (1968) Prentice Hall of India Pvt. Ltd, New Delhi

32 P Chadwick *Progress in Solid Mechanics* **1** (1960) I N Sneddon & R Hill, North Holland Publishing Company Amsterdam

33 S N De and P R Sengupta *Pure & Appl Geophys* **3** (1973) 2242

34 S N De and P R Sengupta *Gerlands Beitr Geophys* **82(5)** (1972) 421

35 S N De and P R Sengupta *Gerlands Beitr Geophys* **85(4)** (1976) 311

36 S N De and P R Sengupta *Gerlands Beitr Geophys* **84(6)** (1975) 509

37 D Acharya and P R Sengupta *Gerlands Beitr Geophys* **85(5)** (1976) 425

38 P R Sengupta and D P Acharya *Acta Ciencia Indica* **2(4)** (1976) 406

39 P R Sengupta and S K De *Acta Geophysica Polonica* **26(4)** (1978) 291

40 D P Acharya and P R Sengupta *Bull De L'Acad Polon Des Sci* **28(8)** (1979) 419

41 R K Mal and P R Sengupta *Acta Geophysica Polonica* **31(1)** (1988) 77

42 P R Sengupta and J Chel *Acta Geophysica Polonica* **34(2)** (1986) 145

43 K C Pal and P R Sengupta *Proc Indian natn Sci Acad* **53A** (1987) 113

44 P S Das and P R Sengupta *Proc Indian natn Sci Acad* **56A** (1990) 459

45 T K Das, P K Biswas and P R Sengupta *Proc Int AMSE Conf* **3** (1990) 83

46 P K Biswas and P R Sengupta *Proc Int AMSE Conf* **3** (1990) 95

47 P S Das and P R Sengupta *Ind Jour Theo Phys* **38(1)** (1990) 59

48 A Mukherjee and P R Sengupta *Proc Int Conf Vib Prob Math Elas Phys* **1** (1990) 38

49 T K Das and P R Sengupta *Jour Nat Acad Math* **8** (1990) 38

50 A Mukherjee and P R Sengupta *Proc Nat Acad India* **61A** (1991) 411

51 K Kundu, P K Biswas and P R Sengupta *Proc Math Soc BHU* **7** (1991) 137

52 T K Das and P R Sengupta *Int Jour Math & Math Sci* **4(3)** (1991) 553

53 A Mukherjee, P R Sengupta and L Debnath *Jour Appl Math Stochastic Analysis* **4(1)** (1991) 71

54 A Mukherjee and P R Sengupta *Acta Geophysica Polonica* **39(4)** (1991) 419

55 T K Das and P R Sengupta *Jour Ind Math Soc* **58(4)** (1992) 237

56 P K Biswas and P R Sengupta *Proc Int AMSE Conf* **4** (1992) 125

57 T K Das and P R Sengupta *Sadhana* **17(2)** (1992) 315.

58 P S Das, P R Sengupta and L Debnath *Int Jour Math Sci* **15(4)** (1992) 795

59 S C Das, D P Acharya and P R Sengupta *Rev Roum Sci Tech Math Appl Tome* **37(5)** (1992) 539

60 P R Sengupta and N Das (Dey) *Proc Int AMSE Conf* **5** (1993) 161

61 S N De and P R Sengupta *Jour Accoust Soc America* **55(5)** (1973) 919

62 I S Sokolnikof *Mathematical Theory of Elasticity*, 2nd Edition (1956) McGraw Hill Book Company Inc. New York