PATTERN FORMATION IN CONVECTION AND REACTION-DIFFUSION SYSTEMS

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We discuss in simple terms, the formation of certain often observed patterns in fluids and reaction-diffusion systems.

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1 Introduction

Spatiotemporal pattern formation is a common phenomenon. The ubiquity of it goes to indicate that we basically live in a nonlinear world. Nonlinearity is the backbone of any spatiotemporal pattern. Almost every system has a homogeneous stationary state as a solution. Pattern formation has to imply an instability in this 'basic' state. Instability or exchange of stability can only happen in a nonlinear system which alone is capable of exhibiting several solutions. The instability generally sets in as some parameter (dimensionless) called the control parameter, is varied. Consequently a problem of pattern formation involves the following steps.

(a) Writing down the nonlinear equations governing the dynamics of the system.

(b) Finding the trivial state (homogeneous, stationary).

(c) Checking the stability of the 'trivial' stationary state as a control parameter is varied.

(d) When an instability occurs, checking to see whether the new state indicated on the basis what is called a linear stability analysis will be spatially inhomogeneous, temporally inhomogeneous or both.

(e) Finally one needs to check that the nonlinear system does support the state whose formation is indicated by the linear stability analysis.

We set the tone by a trivial example. Consider the evolution equation for the real variable x (step 'a'),

\[
\frac{dx}{dt} = rx - x^3
\]

The stationary state, \( x_s \) is time independent and to find it we set \( \dot{x} = 0 \). The result is \( x_s = 0 \) or \( \pm \sqrt{r} \).

Obviously, for \( r < 0 \) there is only one stationary state, while for \( r > 0 \) there are three of them (Fig. 1). The 'trivial' state is \( x_s = 0 \) (step 'b'). We now need to check the stability of \( x_s = 0 \) on the basis of a linear stability analysis as \( r \) is varied. To do so one linearizes the equation of motion around \( x_s = 0 \). We write \( x = x_s + \delta x \) and linearizing in \( \delta x \), get

\[
\frac{d\delta x}{dt} = r\delta x - 3x_s^2\delta x
\]
\[
= r\delta x
\]

Clearly for \( t \to \infty \), \( \delta x \to 0 \) if \( r < 0 \) and \( \delta x \to \infty \) if \( r > 0 \). Thus \( x_s = 0 \) is stable for \( r < 0 \) and unstable for \( r > 0 \). Thus we have taken care of step 'c'. At \( r = 0 \), an instability occurs and we now look for the other possible solutions. As we have already noted there are two other stationary solutions \( x_s = \pm \sqrt{r} \). For \( r > 0 \), these are the states that could become relevant (step 'd'). Finally, we need to check whether \( x_s = \pm \sqrt{r} \) is stable in which case, we may assert that the system supports the new stationary state \( \pm \sqrt{r} \) for \( r > 0 \). With \( x_s = \pm \sqrt{r} \), linearization leads to

\[
\frac{d\delta x}{dt} = r\delta x - 3x_s^2\delta x = -2r\delta x
\]

Clearly \( \delta x \to 0 \) for \( t \to \infty \) if \( r > 0 \). Hence the stationary states \( x_s = \pm \sqrt{r} \) are stable. This completes step 'c'. Since there are two stable states, we have to bring in the concept of the basin of attraction. Certain initial conditions (here all the positive ones) evolve to \( +\sqrt{r} \), while others to \( -\sqrt{r} \).

In the particularly straightforward example above, we did not have a steady to patterned transition. A slight generalization of eq. (1) achieves that. Instead of the real variable \( x \), we introduce a complex variable

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$z$ and make the control parameter $r$ complex as well. The evolution equation is

$$\dot{z} = rz - |z|^2z$$

with $z = x + iy$ and $r = r_1 + ir_2$. Writing this as a two dimensional dynamical system

$$\dot{x} = r_x x - r_y y - (x^2 + y^2)x$$
$$\dot{y} = r_y y + r_x x - (x^2 + y^2)y$$

This system clearly has a trivial stationary state $x = y = 0$. How about its stability? We linearize around $x = y = 0$ and find for $x = x_0 + \delta x$ and $y = y_0 + \delta y$

$$\delta \dot{x} = r_x \delta x - r_y \delta y$$
$$\delta \dot{y} = r_y \delta x + r_x \delta y$$

Being coupled linear differential equations, these will have solutions of the form $e^{\lambda t}$, with the eigenvalue $\lambda$ being determined from

$$\begin{vmatrix} \lambda - r_1 & r_2 \\ -r_2 & \lambda - r_1 \end{vmatrix} = 0$$

or,

$$(\lambda - r_1)^2 + r_2^2 = 0$$

or,

$$\lambda = r_1 \pm ir_2$$

So long as $r_1 < 0$, the fluctuations $\delta x$, $\delta y$ decay to zero in the long time limit. For $r_1 > 0$, the perturbations grow but there is an oscillatory part. Hence, we see that a temporal pattern is indicated on the basis of a linear stability analysis for $r_1 > 0$. However one needs to understand whether this state is supported by the nonlinear systems. This is best analyzed by writing $z = \rho e^{\theta t}$, whence eq. (2) becomes

$$\dot{\rho} = r_1 \rho - \rho^3$$
$$\dot{\theta} = \frac{d}{r_2}$$

For $r_1 > 0$, the stable solution is $\rho^2 = r_1$ (we have seen this from the previous example) and have $z = \sqrt{r_1} e^{i\theta t}$ for $r_1 > 0$. The transition now is from the ‘trivial’ state to a periodic (temporal pattern) state. This transition is in general called a Hopf bifurcation.

The two kinds of patterns that we will focus on in this article are those formed in hydrodynamic systems\textsuperscript{12} and patterns formed in reaction-diffusion systems. For hydrodynamic systems our concern will be with convective patterns. This is a rather common mode of pattern formation. The criss-cross pattern seen in a dried up pond, the somewhat imperfect honeycombs seen on pavements after a cooling shower on a hot day, are some of the usual examples of patterns formed by convection. The stripes seen on the back of a zebra or the bands seen on a fly are typical of patterns formed in a reaction diffusion system. We will be discussing one particular technique in the study of such systems. This is the technique of Galerkin truncation. This procedure effectively creates a finite dimensional dynamical system out of a system of partial differential equations and hence is a very useful tool. We first give a qualitative picture of convective instability in a fluid layer of thickness ‘d’. The layer is heated from below and so is top heavy. We expect this to tumble but the dynamics prevents that from happening. We take a small parcel of fluid of radius ‘r’ and displace it upwards through a distance of order ‘d’. We would like to estimate the time $\tau_1$ that this displacement takes. We assume that the displacement occurs with speed $v$, which can be found by equating the upward force of buoyancy to the viscous drag. The buoyancy force is $\frac{4}{3} \pi r^3 \Delta \rho g$ which can be written as $\frac{4}{3} \pi r^3 \rho \alpha T g$, where $\alpha$ is the expansion coefficient. The viscous drag is $6 \pi \nu v$ and dropping the numerical pre-factor $\eta v \sim r^2 \rho \alpha T g$

The time $\tau_1$ is estimated as

$$\tau_1 = \frac{d}{v} \frac{\eta d}{\rho r^2 \Delta T g} = \frac{v d}{\alpha r^2 \Delta T g}$$

The time $\tau_2$ that it takes for the heat to diffuse is

$$\tau_2 = \frac{d^2}{\lambda}$$

where $\lambda$ is the thermal diffusivity. If the instability is to occur, then $\tau_1 < \tau_2$, as otherwise the extra heat content
of the blob would be gone before it reaches the top. It would no longer be lighter than the surrounding which is necessary for the blob to keep going up and transfer the instability. Hence the condition for instability is

\[ \frac{vd}{\alpha r^2 \Delta T g} < \frac{d^2}{\lambda} \]

Taking \( r \) as \( O(d) \), then we can write the condition as

\[ \frac{\alpha(\Delta T) gd^3}{\lambda \nu} > 1 \]

The important thing is that we have found a dimensionless number

\[ R = \frac{\alpha(\Delta T) gd^3}{\lambda \nu} \]

called the Rayleigh number which is a control parameter for the system. The Rayleigh number would have to exceed a critical value for the convective instability to occur. The mathematical procedure shown in Section 2 gives a quantitative verification of these qualitative statements. Later in that section, we will introduce the Galerkin truncation which leads to the famous Lorenz model. The model, designed to understand events near the convection threshold became the first model to exhibit a strange attractor.

In Section 3, we go to a richer hydrodynamic system, the salt-water mixture heated from above. The salt diffuses very slowly in water as the widely differing diffusivities of salt and heat trigger instabilities which did not occur in the single fluid. The simple fluid goes from a homogeneous steady state to a patterned steady state. Spatial patterns are formed at the onset of the first instability because of convection but not temporal patterns. With the salt water system a spatiotemporal pattern can be formed as the system can undergo a Hopf bifurcation. The possibility of the Hopf bifurcation now produces an extra complication. The oscillatory state can be a standing wave or a travelling wave. The signature of the time dependence of the convective state produced can be found in the Nusselt number which is the amount of heat transmitted by convection from the bottom to the top plate. This is expected to be oscillatory in time when the instability occurs via Hopf bifurcation. It is the unexpected behaviour of the Nusselt number which led to a clear understanding of the oscillatory state. The competition between the standing and traveling wave patterns on the basis of a Galerkin model is the subject Section 4.

In Section 5, we turn to a different system which exhibits a wide variety of patterns. This is a system of two interacting species with widely different diffusivities. The two important features in such models of pattern formation are local self enhancement and long range inhibition. Self enhancement amplifies small local inhomogeneities. If a small increase of species A beyond its homogeneous steady state value causes a further increase of A, then it is called self enhancing. Self enhancement alone cannot produce a stable pattern. For a stable pattern, the overall increase of A due to positive feedback has got to be checked. This is done by having a fast moving antagonist B which prevents the spread of the self enhancing reaction. It is the competition between the two effects that finally leads to a stable pattern. In Section 5, we will deal with the reaction diffusion system. In this case we will study a competition between two spatial patterns which will complement the discussion of Section 4.

We end this introduction by pointing out the advantages and disadvantages of a Galerkin model. All natural systems are governed by non-linear partial differential equations. These are mathematically somewhat intractable and hence various techniques have been devised to make progress. Numerical work too, is greatly helped if there are some prior insight. It is in this context that Galerkin models are extremely useful. A Galerkin model is the reduction of the partial differential equation to a finite number of coupled ordinary differential equations. The trick is to minimize the number of differential equations and that requires a pre-knowledge of the physics that we want to describe. This in other words, is somewhat akin to saying that we need to know the pattern before we do the calculation. This is, in a way true and is the major drawback. It is a very powerful tool if one knows more or less where one stands but one can go astray if one is trying to use it without prior prejudice.

2 The Rayleigh Benard Problem

Density is dependent on temperature. Hence a fluid in a gravitational field, subject to a temperature gradient will see the lighter parts rise and the heavier fall. Temperature variation in a fluid helps to drive circulation currents, the phenomenon being called convection. Convection is the major mode of heat distribution in oceans, the atmosphere, the core of the earth and in the Sun. However, in these examples, the situation is complicated due to other factors such rotation and magnetic fields.

We shall deal with a simple system of a layer of fluid confined between two horizontal surfaces and subject to an adverse temperature gradient. This set
up is called the Rayleigh Benard system. If the temperature difference is small, the fluid remains at
rest and conduction is the only mode of heat transfer. This means that the temperature gradient has to exceed
a certain value before convection can begin. Just beyond the critical temperature gradient, convection currents assume a stationary cellular character. The stability of such a system is governed by the value of a
non-dimensional parameter, the Rayleigh number $R$, which we introduced in Section 1. In the following
discussion, this number will emerge automatically as result of mathematical manipulation.

2.1 The Basic Hydrodynamic Equations
The equation of continuity is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

...(5)

where $\rho$ and $\mathbf{v}$ denote the density and the velocity field of the fluid. In case of an incompressible fluid, this reduces to

$$\nabla \cdot \mathbf{v} = 0$$

The dynamics is governed by the Navier-Stokes equations

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g$$

...(6)

where $\mu$, $p$, and $g$ denote the coefficient of viscosity, the isotropic pressure and the acceleration due to gravity respectively. We now apply the Boussinesq approximation by which we assume that for small variations in temperature, thermodynamic properties such as viscosity and density have negligible variations and the fluid is practically incompressible. The only term in which the density variation cannot be ignored is $\rho g$. Assuming that the density is a linear function of the temperature alone, we have

$$\rho = \rho_0 (1 - \alpha \Delta T)$$

The equation of motion for a Boussinesq fluid is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + (1 + \frac{\partial \rho}{\partial \rho_0}) g$$

...(7)

where $\mathbf{v} = \mu \rho$ is the kinematic viscosity.

The equation for heat transfer is given by

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla)T = \lambda \nabla^2 T$$

...(8)

$T$ and $\lambda$ denote temperature and the coefficient of thermometric conductivity respectively.

2.2 The Perturbation Equations
Consider an infinite horizontal layer of fluid, initially at rest, with the temperature governed by conduction and the pressure in hydrostatic balance. The fluid is bounded by stationary horizontal planes at $z = 0$ and $z = d$. The basic velocity, temperature and pressure are given by

$$\mathbf{v}_b = 0,$$

$$T_b = T_1 - \frac{\Delta T z}{d},$$

$$p_b = -\rho g z + \text{constant}$$

where $\Delta T = T_1 - T_0$, $T_1$ is the temperature of the fluid at the bottom surface, $T_0$ is the temperature of the fluid at the top surface and $z$ is the vertical distance measured from the bottom plane.

We may linearize the Boussinesq eqs. (7) and (8) for small perturbations denoted by $\delta \mathbf{v}$ (with components $u$, $v$, $w$), $\delta T$ and $\delta p$ about the steady state. Let us define dimensionless variables $W = \lambda \nu / d$, $x = r / d$, $\tau = t / d^2$, and $\theta$ such that

$$T = T_1 + \delta T$$

$$= T_1 + \delta T$$

After long but straightforward algebra, we deduce

$$\nabla^2 (\nabla^2 - \frac{1}{\nu} \frac{\partial}{\partial \tau}) W = -\frac{\alpha g \Delta T d^3}{\lambda \nu} \nabla^2 \theta$$

...(9)

and

$$\nabla^2 \theta = -W$$

...(10)

with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

We will consider two possible sets of boundary conditions.

(a) At both boundaries we must have $W = 0$. For stress free condition, we have

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} = 0 \text{ on } z = 0 \text{ and } 1.$$ 

Since $W = 0$ independent of $x$ and $y$ on $z = 0$, 1, this implies

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

and hence from the incompressibility condition we have

$$\frac{\partial^2 w}{\partial z^2} = 0 \text{ on } z = 0, 1$$

(b) If both boundaries are rigid, then

$$u = v = w = 0 \text{ on } z = 0, 1.$$ 

Consequently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \text{ on } z = 0, 1$$

and hence

$$\frac{\partial w}{\partial z} = 0 \text{ on } z = 0, 1 \text{ from continuity.}$$

We assume the boundaries to be perfectly conducting in both cases so that $\theta = 0$ at $z = 0$ and 1.
2.3 Normal Mode Analysis

The disturbance has to be expanded in terms of a complete set of normal modes. In this system the infinite extension in x and y directions allow us to analyze the problem in terms of 2-dimensional periodic waves and we write

\[ W = e^{(k_1 x + k_2 y)} \psi \]

\[ \theta = e^{(k_1 x + k_2 y)} \psi \]

So the Equations (9) and (10) simplify to

\[(D^2 - k^2) (D^2 - k^2 - \frac{p}{\sigma}) f_1 = R^2 f_2 \quad \text{(11)}\]

and

\[(D^2 - k^2 - p) f_2 = -f_1 \quad \text{(12)}\]

where \( k^2 = k_1^2 + k_2^2 \).

As advertised \( R = a \Delta T g d / \lambda \nu \) has emerged as the Rayleigh number. There is another parameter \( \sigma = \nu \lambda \) called the Prandtl number. For stress free boundaries we may take

\[ f_1 = A \sin n \pi z \quad \text{and} \quad f_2 = B \sin n \pi z \]

Substitution of these gives

\[ R = \frac{(n^2 \pi^4 + k^2)^3}{k^2} \]

For a given value of \( k^2 \), the least value of \( R \) occurs at \( n = 1 \). The critical Rayleigh number for the onset of instability is determined by

\[ \frac{dR}{dk} = 0 \quad \text{or} \quad k^2 = \frac{\sigma^2}{2} \]

The critical Rayleigh number corresponding to \( k^2 = \frac{\sigma^2}{2} \) is \( R_c = 27 \pi^4 / 4 \).

This means that for \( R < R_c \), no mode is unstable. Instability sets in for \( R > R_c \) with a critical wave number given by \( k^2 = \frac{\sigma^2}{2} \). Linear stability analysis cannot give individual values for \( k_1 \) and \( k_2 \) and hence cannot predict the shape of the pattern. If \( k_1 = k_2 \) and \( k_2 = 0 \), we have a cylindrical pattern (Fig. 2), for \( k_1 = k_2 = k / \sqrt{2} \) we have a square plane forms and so on. Which plane form would be selected can be determined only from an analysis of the full nonlinear system.

2.4 Derivation of the Lorenz Equations

In this subsection our aim is to explain the basis behind Galerkin models. At the onset of instability, the disturbance is specified by a particular wave number obtained by minimizing \( R \). However the cell pattern is unspecified. We begin by choosing the simplest case of cylindrical rolls.

\[ w = a(t) \sin \pi z \cos \frac{\kappa x}{2} \]

\[ \theta = b(t) \sin \pi z \cos \frac{\kappa x}{2} + c(t) \sin 2 \pi z \]

The modes associated with the velocity and temperature must show that these fields have a circulating pattern and that a convective transfer of heat occurs from the bottom to the top plate. The modes \( \cos \kappa x \sin \pi z \) would take care of the circulating part. The convective heat current leaving the plate at \( z = 0 \) is \( \lambda \frac{d \theta}{dz} \) where the averaging is over the unit cell. This clearly means that the relevant modes which would give a non zero \( \frac{d \theta}{dz} \) are the ones with \( \cos \kappa x \) or \( \sin \kappa x \) absent. Further \( \frac{d \theta}{dz} \bigg|_{z=0} \) will be non-zero only if \( \theta \) is of the form \( \sin m \pi z \). The mode \( m = 1 \) does not couple to the other modes and is hence inconsequential. The lowest relevant mode has \( m = 2 \). Substituting these in the Boussinesq equations, equating the coefficients of the same Fourier terms on either side of the equation and by rescaling \( a, b, c \) to get \( X, Y, Z \) we arrive at

\[ \dot{X} = \sigma (-X + Y) \]

\[ \dot{Y} = -Y + rX - XZ \]

\[ \dot{Z} = -bZ + XY \]

... (14)

where

\[ r = \frac{R}{R_c}, \quad b = \frac{8}{3} \]

This completes part 'a' of the program outlined in Section 1. We now need to carry out part (b) of our program. Clearly the trivial state \( X = Y = Z = 0 \) is a stationary state of the system. Since all spatial structure is associated with the modes \( X, Y \) and \( Z \), the state \( X = Y = Z = 0 \) represents the spatially homogeneous state.

Implementing part (c) of our programme means carrying out the linear stability analysis of the state \( X = Y = Z = 0 \). Writing \( X = 0 + \delta X, Y = 0 + \delta Y \) and \( Z = 0 + \delta Z \) and linearizing in \( \delta X, \delta Y, \delta Z \) we get

\[ \delta X = \sigma (-\delta X + \delta Y) \]

\[ \delta Y = -\delta Y + r \delta X \]

\[ \delta Z = \frac{\partial \delta \theta}{\partial z} \]

---

Fig. 2 Thermal Convection Rolls
The third perturbation $\delta Z \to 0$ as $t \to \infty$. The growth rate $\lambda$ of these equations is found from the roots of

$$\det \begin{vmatrix} \lambda + \sigma & -\sigma \\ -r & \lambda + 1 \end{vmatrix} = 0$$

or,

$$\lambda^2 + \lambda(1 + \sigma) + \sigma(1 - r) = 0$$

with

$$2\lambda = -(1 + \sigma) \pm \sqrt{(1 + \sigma)^2 + 4\sigma(r - 1)}$$

Clearly one of the $\lambda$'s becomes positive for $r > 1$. Hence the trivial state becomes unstable for $r > 1$, i.e., for $R > \frac{27\pi^4}{4}$ which is exactly as it should be. Turning now to part (d) of the program, we look for other non-trivial stationary states of the system. These are found as

$$X_o = Y_o = \pm \sqrt{b(r - 1)}$$

$$Z_o = r - 1$$

...(15)

This state has the periodic spatial structure associated with convection cells.

Finally, we need to check that the system does support the stationary state with periodic structure. In other words, we need to check that the state shown in eq. (15) is stable. A linear stability analysis is now carried out about this state and we have

$$\delta X = \sigma(-\delta X + \delta Y)$$

$$\delta Y = -\delta Y + r\delta X - X_o \delta Z$$

$$\delta Z = -b\delta Z + X_o \delta Y + X_o \delta X$$

...(16)

The eigenvalues are obtained from

$$\begin{vmatrix} \lambda + \sigma & -\sigma & 0 \\ -1 & \lambda + 1 & X_o \\ -X_o & -X_o & \lambda + b \end{vmatrix} = 0$$

This leads to the cubic

$$\lambda^3 + \lambda^2(1 + \sigma + b) + \lambda(\sigma + r)b + 2\sigma b(r - 1) = 0$$

...(17)

For $r = 1 + \sigma$, it is evident that the root which is zero at $r = 1$ now becomes $\lambda = -\frac{2\sigma}{1 + \sigma}$ which shows that all three roots must be negative since the product of the
roots is negative, while the root which is \( \lambda = - (1 + \sigma) \) at \( r = 1 \) continues to be negative for \( r = 1 + \varepsilon \). Hence the non-trivial stationary state is stable, at least immediately above the threshold. From eq. (17) it can be found when the state becomes unstable. But at that large value of the Rayleigh number, this model may not be physically relevant.

3 Thermohaline Convection

We now take up the case of thermohaline convection\(^9\). Consider an infinite layer of water to which hot brine is added uniformly at the top surface. The bottom surface is maintained at a temperature \( T_b \), which is less than the temperature \( T_s \), of the top surface (Fig. 4). Hot water, being lighter, tries to stay at the top. The presence of salt introduces a top heavy component but let us assume that the overall situation (temperature and salinity) is such that we have a hydrostatically stable state. In this situation, if we displace a packet of fluid upwards, its new position the packet will have lower temperature (hence heavier) and lower salt concentration (hence lighter) than its surroundings. Now the different diffusivities of salt and heat come into play. The heat diffuses fast and thermal equilibrium is attained. But the salt diffuses slowly. Hence the packet is lighter than its surroundings and rises. A dynamical instability sets in a hydrostatically stable solution!

The basic equations are:

\[
\frac{\partial \nu}{\partial t} + (\nu \cdot \nabla) \nu = - \frac{\nabla p}{\rho} + \nu \nabla^2 \nu + g(1 + \frac{\partial \rho}{\partial \rho}) \quad \ldots \quad (18)
\]

\[
\frac{\partial T}{\partial t} + (\nu \cdot \nabla) T = \lambda \nabla^2 T \quad \ldots \quad (19)
\]

\[
\frac{\partial c}{\partial t} + (\nu \cdot \nabla) c = \lambda_s \nabla^2 c \quad \ldots \quad (20)
\]

where \( c \) is the salt concentration. The density of the fluid is now given as a linear function of both temperature and the concentration.

\[
\rho = \rho_0 (\alpha \delta T + \beta \delta c). \quad \ldots \quad (21)
\]

\( \lambda_s \) and \( \beta \) are the coefficients of diffusivity and expansion respectively corresponding to concentration gradient. The static conduction state has \( \nu = 0 \) and linear profiles for \( T(r) \) and \( c(r) \). We consider small perturbations around this steady state and denote the perturbations by \( \delta \nu \) (with components \( u, v, w \)), \( \delta T \) and \( \delta c \). We define

\[
\lambda_s = L \lambda
\]

\[
\theta = \frac{\delta T}{\Delta T}
\]

Heat diffuses cooling the blob
The cold blob, being dense, continues to sink.
Fingers of salt water develop

\[
\phi = \frac{\delta c}{\Delta c}
\]

so that

\[
T = T_s + \Delta T = T_s - \frac{\Delta T}{d} + \Delta T \theta
\]

and

\[
c = c_s + \Delta c = c_s - \frac{\Delta c}{d} + \Delta c \phi
\]

These expressions for \( \nu, T, c \) are inserted in eqs. (18) to (20) and the resulting equations are linearized in \( \delta \nu, \delta T \) and \( \delta c \). The perturbation equations in terms of the dimensionless variables \( W, \theta, \phi \) are

\[
\nabla^2 (\nabla^2 - \frac{1}{\sigma} \frac{\partial}{\partial t}) W = - \frac{\alpha g \Delta T d^3}{\lambda \nu} \nabla^2 \theta + \frac{\beta g \Delta c d^3}{\lambda \nu} \nabla^2 \phi
\]

\[
= - R_1 \nabla^2 \theta + LR_2 \nabla \phi
\]

\[
\frac{\partial \theta}{\partial t} - W = \nabla^2 \theta \quad \ldots \quad (22)
\]

\[
\frac{\partial \phi}{\partial t} - W = L \nabla^2 \phi \quad \ldots \quad (23)
\]

We now have two Rayleigh numbers

\[
R_1 = \frac{\alpha g \Delta T d^3}{\lambda \nu}
\]

\[
R_2 = \frac{\beta g \Delta c d^3}{\lambda_s \nu}
\]

Once again we analyze the perturbations in terms of normal modes. We write

\[
W = e^{(k_x + k_y) x + pr} f_x(z)
\]

\[
\theta = e^{(k_x + k_y) x + pr} f_x(z)
\]

\[
\phi = e^{(k_x + k_y) x + pr} f_x(z)
\]
The equations now simplify to
\[
(D^2 - k^2)(D^2 - k^2 - \frac{p}{\alpha}) f_1 = R_1 k^2 f_2 - LR_2 k^2 f_3
\]
\[
(D^2 - k^2 - p) f_2 = -f_1
\]
\[
(LD^2 - k^2 - p) f_3 = -f_1 \quad \ldots(26)
\]

We are interested in the marginal state when \( \Im p = 0 \). The transition from stability to instability may occur in two ways:

(a) Via a stationary state in which \( \Im p = 0 \). In this case, at the onset of instability, a stationary pattern prevails.

(b) Via an oscillatory state in which at the onset of instability oscillatory motions prevail. Here \( \Im p \neq 0 \). This also called a case of over-stability.

3.1 Normal Mode Analysis

Let us first consider the stationary state. Suppose both boundaries are free. Following the same approach that we did in the Rayleigh Benard system we take
\[
\begin{align*}
  f_1 &= A \sin \pi z \\
  f_2 &= B \sin \pi z \\
  f_3 &= C \sin \pi z
\end{align*}
\]

Substituting in eq. (27), we obtain
\[
R_1 - R_2 = \frac{(\pi^2 + k^2)^3}{k^2} \quad \ldots(28)
\]

Suppose \( R_2 \) is fixed. We find \( \frac{dR_1}{dk} = 0 \) at \( k^2 = \pi^2/2 \).

Hence
\[
r_1 - r_2 = 1
\]

where \( r_1 = R_1/R_2 \), \( r_2 = R_2/R_2 \), \( R_2 = 27\pi^4/4 \). A negative \( \Delta T \) means the temperature is lower at the bottom which is a stabilizing situation. A negative \( \Delta C \) means that the salt concentration is lower at the bottom which is a destabilizing influence. If we arrange \( \Delta T > \beta \Delta C \) i.e \( R_1 > R_2 \), then we will have a hydrostatically stable situation. The convective threshold driven by \( |R_2| \)

now occurs at \( |R_2| = 27\pi^4/4 + |R_2| \). This is possible if \( |R_2| > 27\pi^4/4 + |R_2| \).

We now consider the case of over-stability. The boundary conditions remain the same. Using the trial solutions given in eq. (28), but substituting \( p = i \omega \) in the eq. (27),
\[
\begin{align*}
  R_1 (1 + \sigma) - LR_2 (L + \sigma) \\
  = \frac{(\pi^2 + k^2)^3}{k^2} (L + \sigma) (L + \sigma + 1 + \alpha L)
\end{align*}
\]

which is the threshold for oscillatory convection with frequency
\[
\omega = \frac{(\pi^2 + k^2)^3}{2} \left( L + \alpha L + \sigma \right) + \frac{k^2 \sigma}{(\pi^2 + k^2)} (R_2 - R_1)
\]

How does one experimentally detect the onset of convection? This is done by the study of the Nusselt number \( N \), which is the ratio of the total heat transmitted from the bottom plate to the heat transmitted by conduction alone. The point at which the ratio deviates from 1 signals the onset of convection. If the onset is oscillatory, then the heat current will oscillate as a function of time after the onset of convection. For the thermohaline convection, the oscillation was not seen although the threshold Rayleigh number agreed with result shown in eq. (29). The oscillation will be absent if the bifurcation gives rise to a traveling wave pattern since averaging over the cell in the x-direction will remove the time dependence. This motivates the study of the competition between the standing and traveling waves and the extended Lorenz model that we describe now.

3.2 Extended Lorenz Model for the Thermohaline Problem

As in the case of the simple Rayleigh Benard problem, we choose the case where the quantities have no y-dependence. The disturbance may be in the form of standing or traveling waves. Since we are interested in both, we may take:

\[
\begin{align*}
  W &= \frac{1}{2} (A e^{ik} + A* e^{-ik}) \sin \pi z \\
  \theta &= \frac{1}{2} (B e^{ik} + B* e^{-ik}) \sin \pi z + C \sin 2\pi z \\
  \phi &= \frac{1}{2} (D e^{ik} + D* e^{-ik}) \sin \pi z + F \sin 2\pi z
\end{align*}
\]

Substituting these in equation, equating the coefficients of the same Fourier modes on either side of the equation as we did earlier, and by suitable rescaling \( A \to X, B \to Y, C \to Z, D \to U \) and \( F \to V \) we obtain
\[
\begin{align*}
  \dot{X} &= \sigma (-X + Y - U) \\
  \dot{Y} &= -\gamma + r_1 X - XZ \\
  \dot{Z} &= -bZ + \frac{1}{2} (X* Y + Y* X) \\
  \dot{U} &= -L U + r_2 L X - XV \\
  \dot{V} &= -b L V + \frac{1}{2} (X* U + U* X)
\end{align*}
\]

These are a set of ordinary differential equations albeit nonlinear. But they are easier to handle than the p.d.e's we started out with in the first place. There
is a trivial fixed point at origin. The linearized equations are:

\[
\begin{pmatrix}
\dot{X} \\
\dot{Y} \\
\dot{U}
\end{pmatrix} =
\begin{pmatrix}
-\sigma & \sigma & -\sigma \\
-1 & 0 & 0 \\
0 & -L & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
U
\end{pmatrix}
\]  

...(32)

The eigenvalues of the Jacobian matrix are given by the cubic

\[
\lambda^3 + \lambda^2(L + 1 + \sigma) + \lambda(\sigma L + \sigma + L) + \sigma L - \sigma r_1 L + \sigma(\lambda + 1)Lr_2 = 0
\]  

...(33)

We may have \(\lambda = 0\) (stationary state) if

\[-\sigma r_1 L + \sigma(\lambda + 1)Lr_2 = 0\]

i.e. \(r_1 - r_2 = 1\)

But this is exactly the result we had already obtained using the Boussinesq equations.

Again, if we are to have a case of over-stability, we expect the eigenvalues to be purely imaginary. Putting \(\lambda = i \omega_0\)

\[\omega^2 = \frac{\sigma L r_2}{1 + \sigma}\]

or \(r_1(\sigma + 1) - L r_2(\sigma + L) = \frac{(\sigma + 1 + \sigma + L + \sigma L) d}{\sigma} \)

...(34)

This is the same as eq. (30). Thus the extended Lorenz model has given us the same results as the Boussinesq equations.

There is another fixed point given by

\[V_o = \frac{L(X_o^2 - bL(r_1 - r_2 - 1))}{b(1 - L^2)}\]

\[Z_o = \frac{|X_o|^2 - bL(r_1 - r_2 - 1)}{b(1 - L^2)}\]

\[Y_o = X_o(r_1 - Z_o)\]

and

\[U_o = Y_o - X_o\]

With

\[|X_o|^4 + b|X_o|^3(L^2(1 + r_2) - r_2 - \delta) - b^2 L^2 \delta = 0\]

where \(\delta = r_1 - r_2 - 1\). If we put \(L^2 \ll 1\), we get

\[|X_o|^2 = \frac{bL^2}{\delta} \]

where \(r_2 = -\delta_2\).

4 The Amplitude Equations

Consider the extended Lorenz equations.

\[
\begin{align*}
\dot{X} &= \sigma(-X + Y - U) \\
\dot{Y} &= -Y + r_1 X - X Z \\
\dot{Z} &= -bZ + \frac{1}{2}(X*Y + Y*X) \\
\dot{U} &= -LU + r_1 LX - XV \\
\dot{V} &= -bLV + \frac{1}{2}(X*U + U*X)
\end{align*}
\]

Let \(r_i = r_o + \epsilon\) where \(r_o = 1\) and \(\epsilon\) is a small parameter denoting the deviation from the critical value \(r_o\). We now employ the method of 'two-timing'. Before turning to the specific problem, we would like to explain the general procedure.

Let \(t = \tau\) denote the fast time-scale of \(O(1)\) and \(T = \epsilon t\) denote the slow time scale. These can be treated as independent variables. Hence

\[\frac{d}{dt} \rightarrow \frac{\partial}{\partial \tau} + \epsilon \frac{\partial}{\partial T}\]

The given system of equations can be cast in the form

\[\hat{L} |u\rangle = |v\rangle\]

...(36)

If we expand the operator \(\hat{L}\) and the vector \(|u\rangle\) in powers of \(\epsilon\), we get

\[\{\hat{L}_n + \epsilon \hat{L}_1 |u_o\rangle + \epsilon |u_o\rangle\} = |v\rangle\]

...(37)

Now \(\hat{L}_0 = 0\).

Collecting the terms of \(O(\epsilon)\), we have

\[\hat{L}_0 |u_o\rangle + \epsilon \hat{L}_1 |u_o\rangle = |v\rangle\]

Let \(\langle a_o |\) be the left eigenvector of \(\hat{L}_o\) with eigenvalue 0 i.e.

\[\langle a_o | \hat{L}_o = 0\]

Hence contracting with \(\langle a_o |\) we get

\[\langle a_o | \hat{L}_1 |u_o\rangle + \langle a_o | \hat{L}_1 |u_o\rangle = \langle a_o | v\rangle\]

or

\[\langle a_o | \hat{L}_1 |u_o\rangle = \langle a_o | v\rangle\]

...(38)

We will see next that the procedure will yield what are called amplitude equations.

In our given problem, we have

\[\hat{L}_o = \begin{pmatrix}
\frac{\partial}{\partial r} + \sigma & -\sigma & \sigma \\
-r_o \frac{\partial}{\partial r} + 1 & 0 \\
-L r_2 & \tau 0 & \frac{\partial}{\partial r} + L
\end{pmatrix}\]

...(39)

and

\[\hat{L}_1 = \begin{pmatrix}
\frac{\partial}{\partial T} & 0 & 0 \\
-1 & \frac{\partial}{\partial r} & 0 \\
0 & 0 & \frac{\partial}{\partial T}
\end{pmatrix}\]

...(40)
Now
\[
|u_i| = \begin{pmatrix} X_o \\ Y_o \\ U_o \end{pmatrix} \quad \text{where} \quad X_o = X_1 + iX_2
\]

Now
\[
\begin{pmatrix} X_1 \\ Y_1 \\ U_1 \end{pmatrix} = A(T)e^{i\omega t} \begin{pmatrix} X_o \\ Y_o \\ U_o \end{pmatrix} + c.c
\]

and
\[
\begin{pmatrix} X_2 \\ Y_2 \\ U_2 \end{pmatrix} = B(T)e^{i\omega t} \begin{pmatrix} X_o \\ Y_o \\ U_o \end{pmatrix} + c.c
\]

We have assumed the instability to be oscillatory. Substituting these in the equation for \(Z\) in the extended Lorenz model, we have
\[
\dot{Z}_o + bZ_o = \frac{1}{2}(X'_c Y'_o + X'_o Y'_c)
\]

Solving this differential equation, we get
\[
Z_o = Z_1 + Z_2 e^{i\omega t} + Z_3 e^{2i\omega t} \quad \text{...(41)}
\]

with
\[
Z_1 = \frac{(A_1^2 + B_1^2) r_o}{(1 + i\omega)(b + 2i\omega)}
\]

\[
Z_2 = \frac{(A_2^2 + B_2^2) r_o}{(1 + i\omega)(b + 2i\omega)}
\]

\[
Z_3 = Z_3^*
\]

Similarly
\[
V_o = V'_1 + V'_2 e^{i\omega t} + V'_3 e^{2i\omega t} \quad \text{...(42)}
\]

with
\[
V'_1 = \frac{(A_1^2 + B_1^2) r_l L}{(1 + i\omega)(b + 2i\omega)}
\]

\[
V'_2 = \frac{(A_2^2 + B_2^2) r_l L}{(1 + i\omega)(b + 2i\omega)}
\]

\[
V'_3 = V'_3^*
\]

We find
\[
\langle a_o \rangle = \left(1 + \frac{\sigma}{1 + i\omega} - \frac{\sigma}{L + i\omega}\right) e^{i\omega t}
\]

Since \(\langle a_o \rangle \langle \tilde{f}_o \rangle |u_o| = \langle a_o \rangle |v|
\]

\[
|v| = \begin{pmatrix} 0 \\ -X_2 Z_o \\ -X_3 Y_o \end{pmatrix}
\]

On simplification we get the required amplitude equation
\[
\frac{dA}{dT} \left[ \frac{\sigma r_o}{(1 + i\omega)^2} - \frac{\sigma r_l L}{(L + i\omega)^2} \right] = A \sigma
\]

\[
- A \left[ A^2 + |B|^2 \right] \left( \frac{2 r_o \sigma}{b(1 + i\omega)(1 + i\omega)} \right) - \frac{2 r_l L \sigma}{(L + i\omega)^2}
\]

\[
- A^* \left[ A^2 + |B|^2 \right] \left( \frac{r_o \sigma}{(1 + i\omega)^2} \right) \left( \frac{b L + 2i\omega}{(b L + 2i\omega)} \right)
\]

...(43)

The amplitude equations can be written in the form
\[
\frac{dA}{dT} = \alpha A - \beta |A|^2 A - \delta |B|^2 A - \gamma B^2 A^* \quad \text{...(44)}
\]

and
\[
\frac{dB}{dT} = \alpha B - \beta |B|^2 B - \delta |A|^2 B - \gamma A^2 A^* \quad \text{...(45)}
\]

with \(\beta = \delta + \gamma\). Since \(A, B\) and the coefficients are complex, we have a set of four equations. There are two sets of fixed points corresponding to travelling wave and standing wave solutions. We will carry out a stability analysis of these two fixed points.

4.1 The Travelling Wave Solution

This is given by \(B_o = \pm A_o\). This leads to
\[
W \propto [A e^{i\omega t - kx}] + A^* e^{-i\omega t - kx}] \sin \pi x
\]

Let us take
\[
A = r_t e^{i\phi_t}, \quad A = r_1 + i r_2
\]

\[
B = r_t e^{i\phi_t}, \quad B = r_1 + i r_2
\]

\[
\phi = \theta_1 - \theta_2, \quad \gamma = \gamma_1 + i \gamma_2
\]

\[
\delta = \delta_1 - i \delta_2
\]

For the fixed point \(B_o = \pm A_o\), we have \(r_1 = r_2\) and \(\phi = \mp \pi/2\). Let us define
\[
r_1 - r_2 = 2R
\]

\[
r_1 + r_2 = 2S
\]

Hence \(R_o = 0\) and
\[
S_o = \frac{\alpha_1}{\beta_1 + \delta_1 - \gamma_1}
\]

We find
\[
\delta \dot{R} = -4 S_o r_1 \delta R + 2 S_o r_1 \gamma_2 \delta \phi\]

\[
\delta \dot{\phi} = -8 S_o r_1 \gamma_2 \delta R - 4 \gamma_1 S_o \delta \phi\]

Let \(\delta R \sim e^{\lambda T}\) and \(\delta \phi \sim e^{\lambda T}\). Substituting in eqs. (46) and (47) and setting the secular determinant to zero, we obtain
\[
\lambda = -4 S_o (\gamma_1 \mp i \gamma_2)
\]

Hence
\[
\eta(\lambda) = -4 S_o \gamma_1 \quad \text{...(48)}
\]
If $\chi_i$ is positive the instability will die down. If negative, it will blow up. Thus the stability of the travelling wave solution to the amplitude equations depends on the sign of $\chi_i$.

4.2 The Standing Wave Solution

The fixed point corresponding to standing wave solution is given by $B = 0$ and $A \neq 0$. We retain our previous definition of $r_i$, $r_s$ and $\phi$. Here $r_{2m} = 0$ and $\phi_0 = m\pi$ with $m = 0, 1, 2, ...$ We Obtain
\[
\delta r_i = 0
\]
\[
\delta \phi = 2r_i \gamma \delta \phi
\]
\[
\delta \dot{r}_i = -2\alpha_i \delta r_i.
\] ... (49)

Since $\alpha_i > 0$, the perturbation in $r_i$ will die out. Interestingly, we see that the stability again depends on $\chi_i$. But in this case if $\chi_i$ is positive, it will blow up and if $\chi_i < 0$, it will die out. From the amplitude equation given in eq. (43), we find $\gamma$ to be positive at the threshold frequency for $L << 1$ which means that the travelling wave solution is stable and not the standing wave (Fig. 5).

5 Reaction Diffusion Systems

As stated in the introduction, the main ingredients in the pattern formation by reaction and diffusion are the existence of two competing species A and B. The dynamics of the two species are governed by the equations
\[
\frac{\partial A}{\partial t} = D\nabla^2 A + f(A, B)
\] ... (50)
\[
\frac{\partial B}{\partial t} = \nabla^2 B + g(A, B)
\]

The nonlinear functions $f$ and $g$ represent the self enhancement, antagonistic reaction etc. We will consider specific forms of $f$ and $g$ later. The primary idea in such situations is that $A = B = 0$ is a trivial solution of the system (the forms of $f$ and $g$ are always such as to allow this) which represents the homogeneous stationary state. The first question of interest is that can this state be unstable to an inhomogeneous and/or time dependent state (spatially and/or temporally patterned) in a linear stability analysis. We linearize eq. (50) around $A = B = 0$ and write the

Fig. 5 Stability diagram for eqs. (48) and (49)
linearized system with the perturbation taken to be of wave-vector

\[ k = \frac{2\pi}{\lambda}, \quad (\nabla^2 = -k^2) \]

as

\[
\frac{dA}{dt} = -DK^2A + \alpha_{11}A + \alpha_{12}B \\
\frac{dB}{dt} = -k^2B + \alpha_{21}A + \alpha_{22}B
\]

...(51)

This set of linear equations will have solutions of the form \( e^{\lambda t} \), where \( \lambda \) can be found as the solution of the algebraic equations

\[
\begin{vmatrix}
\lambda + Dk^2 - \alpha_{11} & -\alpha_{12} \\
-\alpha_{21} & \lambda + k^2 + \alpha_{22}
\end{vmatrix} = 0
\]

or

\[
\lambda^2 + \lambda[(D + 1)k^2 - \alpha_{11} - \alpha_{22}] + Dk^4 - k^2 (\alpha_{11} + D\alpha_{22}) + \alpha_{12} - \alpha_{21} \alpha_{12} - \alpha_{12} \alpha_{21} = 0 \\
\lambda^2 + \lambda(1 + D)k^2 + Dk^4 - 4\lambda^2 - 4\Delta - 4Dk^4 + 4k^2(\alpha_{11} + D\alpha_{22}) = 0
\]

...(52)

We define \( T = \alpha_{11} + \alpha_{22} \) and \( \Delta = \alpha_{12} \alpha_{21} - \alpha_{12} \alpha_{21} \) to write

\[
\lambda^2 + \lambda[(1 + D)k^2 - T] + Dk^4 - k^2 (\alpha_{11} + D\alpha_{22}) + \Delta = 0
\]

or

\[
2\lambda = T - (1 + D)k^2 \pm \sqrt{[(1 + D)k^2 - T]^2 - 4\Delta - 4Dk^4 + 4k^2(\alpha_{11} + D\alpha_{22})} = 0
\]

...(53)

In the absence of diffusion, the state \( A = B = 0 \) is stable if \( T < 0 \) and \( \Delta > 0 \). The fixed point \( A = B = 0 \) is in that limit

- a stable node if \( T < 0 \) and \( 4\Delta < T^2 \) (\( \Delta > 0 \))
- a stable spiral node if \( T < 0 \) and \( 4\Delta > T^2 \) (\( \Delta > 0 \)).

Instability can occur as a saddle if \( \Delta < 0 \) with \( T < 0 \) and \( A = B = 0 \) is certainly unstable if \( T > 0 \). An oscillatory instability can occur if \( T \) goes through a zero with \( \Delta > 0 \).

In the presence of diffusion, we can have a patterned state with wave number \( k \), and we can define

\[ T' = T - (1 + D)k^2 \]

and \( \Delta' = \Delta + Dk^2 - k^2 (\alpha_{11} + D\alpha_{22}) \). If we had a stable state in the diffusion-less situation with \( T < 0 \), then in the present scenario, \( T < 0 \) and instability can occur if \( \Delta' < 0 \). As a function of \( k \), \( \Delta' \) will have its minimum value if \( k^2 = k_0^2 = \frac{1}{2D} (\alpha_{11} + D\alpha_{22}) \). The minimum value will be given by \( \Delta' = \Delta - \frac{(\alpha_{11} + D\alpha_{22})^2}{4D} \) and the instability will occur with a patterned state of wave-number \( k_0 \) if \( \Delta' < 0 \). This instability will always occur as a stationary instability. So the new stable state is expected to be homogeneous in time but inhomogeneous in space. A spatially homogeneous but a temporally oscillating state will result if we set \( k = 0 \), take \( \Delta > 0 \) and consider \( T \) going through 0.

We now turn to a specific model known as the Gierer Meinhardt model. In this model

\[
f(A, B) = \frac{A^2}{B} - A + \sigma
\]

...(54)

\[
g(A, B) = \mu(A^2 - B)
\]

Clearly the species \( A \) is self enhancing because of the \( A^2 \) term, but \( B \) is antagonistic to \( A \) and hence turns down the growth rate. The term \( \sigma \) is the growth rate of \( A \) even in the absence of any concentration. The species \( B \) is actually helpful in the process of its growth by \( A \) through the term \( A^2 \) in \( g(A, B) \). In this case the spacetime homogeneous term is \( B = A^2 = (1 + \sigma)^2 \). This is not qualitatively different from \( A = B = 0 \) since a shift of origin would move the solution \( A = 1 + \sigma, B = (1 + \sigma)^2 \) to the origin itself. We write

\[
A = 1 + \sigma + \delta A
\]

\[
B = (1 + \sigma)^2 + \delta B
\]

...(55)

substitute in eq. (51) and linearize in \( \delta A \) and \( \delta B \). The result is

\[
\frac{\partial \delta A}{\partial t} = D \nabla^2 \delta A + \frac{1 - \sigma}{1 + \sigma} \delta B - \frac{\delta B}{(1 + \sigma)^2}
\]

...(56)

\[
\frac{\partial \delta B}{\partial t} = \nabla^2 \delta B + 2\mu(1 + \sigma) \delta B - \mu \delta B
\]

Clearly \( \alpha_{11} = \frac{1 - \sigma}{1 + \sigma} = \frac{1}{(1 + \sigma)^2} \), \( \alpha_{21} = 2\mu(1 + \sigma) \), and \( \alpha_{22} = -\mu \). The condition for instability is

\[
\Delta' = \mu - \frac{1 - \sigma}{1 + \sigma} + 2\mu \left( \frac{1}{1 + \sigma} - \frac{\sigma}{D} \right)^2 < 0
\]

which is

\[
\left( \frac{1 - \sigma}{1 + \sigma} - \frac{\sigma}{D} \right)^2 > 4\mu D
\]

...(57)

or \( \frac{1 - \sigma}{1 + \sigma} > \mu D + 2\sqrt{\mu D} \left( \sqrt{\mu D} + 1 \right)^2 - 1 \)

which is

\[
\sqrt{\mu D} < \frac{2}{\sqrt{1 + \sigma}} - 1
\]

...(58)

In the \( D - \mu \) plane this boundary is shown in Fig. 6 as the solid curve. It separates a trivial phase from a patterned phase with wave number

\[
k_0^2 = \frac{1}{2} \left[ \frac{1 - \sigma}{1 + \sigma} - \frac{1}{D} - \mu \right]
\]

...(59)
The temporally patterned phase occurs for $T = \frac{1-\sigma}{1+\sigma} - \mu = 0$. For $\mu < \frac{1-\sigma}{1+\sigma}$, the ‘trivial’ state is unstable and we have a homogeneous state which oscillates in time with frequency

$$\omega_0^2 = \frac{2\mu}{1+\sigma} - \mu \frac{1-\sigma}{1+\sigma} = \mu$$

In the $D - \mu$ plane this boundary separating temporal oscillation from the trivial state is shown as the dashed line. The two lines meet at

$$\mu_0 = \frac{1-\sigma}{1+\sigma}, \quad D_0 = \sqrt{2} - \sqrt{1+\sigma} \sqrt{1-\sigma}$$

which is a co dimension-2 point for the system.

It is also possible for diffusion to stabilize an initially unstable homogeneous state. If $T > 0$ in eq. (53), the homogeneous state is unstable in the absence of diffusion. We can stabilize the state if

$$(1 + D)k^2 > T$$

and $\Delta > 0$ ...

This kind of behaviour is typified by the chlorite-iodide-malonic acid (CIMA) system which over the last decade has turned out to be a very useful playground for observing Turing patterns because effective diffusion coefficients can be made to vary over a wide range. This system was very simply modelled by Lengyel and Epstein who focused only on the activator ($A$) and the inhibitor ($ClO_2$) and wrote down the two variable equation in two dimensions

$$\frac{\partial A_1}{\partial t} = D\nabla^2 A_1 + g(A_1, A_2)$$

$$\frac{\partial A_2}{\partial t} = \nabla^2 A_2 + f(A_1, A_2)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and

$$g(A_1, A_2) = \sigma b \left( A_2 - \frac{A_1 A_2}{1 + A_1^2} \right)$$

and $f(A_1, A_2) = a - A_2 - \frac{4 A_1 A_2}{1 + A_1^2}$

As in the GM model, there is a “trivial” state given by

$$A_{20} = \frac{a}{b}$$

$$A_{10} = 1 + \frac{a^2}{2b^2}$$

The stability of this state is governed by (for wave number $k$, so that $\nabla^2 = -k^2 = -k_1^2 - k_2^2$

$$\frac{d \delta A_1}{dt} = -Dk^2 \delta A_1$$

$$+ \sigma b \left( \delta A_2 - \frac{A_{20}}{a + A_{20}} \delta A_2 - \frac{A_{20} \delta A_1}{1 + A_{20}^2} - \frac{2 A_{20} A_{20} \delta A_2}{(1 + A_{20}^2)^2} \right)$$

$$= -Dk^2 \delta A_1 + \sigma b \left( -\frac{A_{20}}{1 + A_{20}} \delta A_2 + \frac{2 A_{20}^2 \delta A_2}{1 + A_{20}^2} \right)$$

while

$$\frac{d \delta A_2}{dt} = -k^2 \delta A_2 - \delta A_2 - 4 \frac{A_{10}}{1 + A_{20}^2} \delta A_2$$

$$- \frac{4 A_{20}^2 \delta A_1}{1 + A_{20}^2} - \frac{8 A_{10} A_{20}^2}{(1 + A_{20}^2)^2} \delta A_2$$

$$= -k^2 \delta A_2 - 4 \frac{A_{20}^2 \delta A_1}{1 + A_{20}^2} + \left( \frac{8 A_{10}^2}{1 + A_{20}^2} - 5 \right) \delta A_2$$
Substituting for \( A_{10} \) and \( A_{20} \)
\[
\frac{d}{dt} \delta A_1 = - D \kappa^2 \delta A_1 - \frac{5 a \sigma b}{a^2 + 25} \delta A_1 + \frac{2 a^2 \sigma b}{a^2 + 25} \delta A_2 \quad \text{(67)}
\]
\[
\frac{d}{dt} \delta A_2 = - k^2 \delta A_2 - \frac{20 a}{a^2 + 25} \delta A_1 + \frac{3 a^2 - 125}{a^2 + 25} \delta A_2 \quad \text{(67)}
\]
Thus,
\[
\alpha_{11} = - \frac{5 a \sigma b}{a^2 + 25} \quad \alpha_{12} = \frac{2 a^2 \sigma b}{a^2 + 25} \quad \alpha_{21} = - \frac{20 a}{a^2 + 25} \quad \alpha_{22} = \frac{3 a^2 - 125}{a^2 + 25}
\]
Clearly
\[
T = \frac{3 a^2 - 125 - 5 a \sigma b}{a^2 + 25} \quad \text{(68)}
\]
\[
\Delta = \frac{\sigma b a}{a^2 + 25} \left[ 40 a^2 - 15 \sigma a^2 + 625 \right] \quad \text{(69)}
\]
\[
= \frac{25 \sigma b a}{a^2 + 25}
\]
It is apparent that \( T \) can be positive or negative depending on the values of \( a, b \) and \( \sigma \). If \( T \) is negative, then with \( \Delta \) always positive, the diffusionless state is stable. In this case diffusion can cause an instability if \( \Delta \) becomes negative. Now the smallest positive value of \( \Delta \) is
\[
\Delta^* = \frac{25 \sigma b a}{a^2 + 25} - \left( \frac{\alpha_{11} + D \alpha_{22}}{4 D} \right)^2 \quad \text{(70)}
\]
The condition for instability is
\[
[5 a \sigma b + D(125 - 3 a^2)] = 100 \sigma b a D(a^2 + 25) \quad \text{(71)}
\]
The diffusionless state is unstable if \( T > 0 \). To stabilize this, we need
\[
k^2 (1 + D) > T = \frac{3 a^2 - 125 - 5 a \sigma b}{25 + a^2} \quad \text{(72)}
\]
This provides a minimum value of \( k^2 \). Stability will be assured if in the range of \( k^2 \) shown, \( \Delta \) is negative. The minimum value of \( \Delta \) is as shown in eq. (70). We require
\[
\frac{100 \sigma b a D}{a^2 + 25} > \left( \frac{\alpha_{11} + D \alpha_{22}}{4 D} \right)^2 \quad \text{(73)}
\]
and
\[
\frac{\alpha_{11} + D \alpha_{22}}{2 D} (1 + D) > \frac{3 a^2 - 125 - 5 a \sigma b}{25 + a^2} \quad \text{(73)}
\]
This system is particularly interesting because it can be controlled by adding an electric field. Following Riaz \textit{et al.}, we add a constant electric field in the \( x \)-direction which makes eq. (51) become
\[
\frac{\partial A}{\partial t} = D \nabla^2 A + z_2 E D \frac{\partial A}{\partial x} + g(A, B) \quad \text{(74)}
\]
The trivial state is as before, but the patterned state has to be localized in the \( x \)-direction and we take the perturbation \( \delta A, \delta B \) about the trivial state as
\[
(\delta A, \delta B) = e^{i \omega \phi} \cos k_y y \quad (\delta_1, \delta_2)
\]
Now \( \nabla^2 = k_1^2 - k_2^2 \) and \( \frac{\partial}{\partial x} = k_x \), so that \( \lambda \) is to be determined from
\[
\begin{vmatrix}
\lambda + D(k_1^2 - k_2^2) - z_2 E D k_x - \alpha_{11} & -\alpha_{21} \\
-\alpha_{21} & \lambda + D(k_1^2 - k_2^2) - z_2 E D k_x - \alpha_{22}
\end{vmatrix} = 0
\]
or
\[
\lambda^2 - \lambda (\alpha_{11} + \alpha_{22} + z_2 E D k_x) + (1 + D)(k_1^2 - k_2^2)
+ \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} - \alpha_{11} (D(k_1^2 - k_2^2)) - \alpha_{22} (D(k_1^2 - k_2^2)) - z_2 E D k_x
\]
\[
= 0.
\]
It is now possible to adjust \( E \) and \( k_x \) such that
\[
\alpha_{11} + \alpha_{22} - (1 + D) (k_1^2 - k_2^2) + (z_2 D + z_1) E k_x > 0
\]
which will make the diffusion stabilized patterned state unstable once again. Thus we can have the following scenario:

<table>
<thead>
<tr>
<th>Unstable trivial</th>
<th>Patterned state stabilized</th>
<th>Destabilized by</th>
</tr>
</thead>
<tbody>
<tr>
<td>state</td>
<td>by diffusion</td>
<td>Electric Field</td>
</tr>
</tbody>
</table>

This is very different from the Turing scenario where the trivial state is stable and destabilized by a patterned state. We can have this scenario as well. We begin with \( \alpha_{11} + \alpha_{22} < 0 \) and this state is destabilized to a patterned state by the diffusion. The electric field can now change the nature of the patterned state. This has been studied numerically by Riaz \textit{et al.} In the zero-field situation, the usual Turing pattern was observed in the form of spots as shown in Fig. 7(a). At higher values of the electric field, the spots begin to get deformed and finally we observe a complete transition from spots to stripes in Fig. 7.

**Galerkin Model of the Reaction Diffusion System**

A linear stability analysis of the eqs. (50) and (54) gives the boundaries separating the homogeneous steady state from the stationary patterned (Turing) and the oscillatory (Hopf) states respectively. But the region enclosed by the two boundaries is inaccessible by linear stability analysis. In order to explore the possibility of a Hopf Turing mixed mode, one can employ the method of Galerkin truncation discussed earlier in the context of hydrodynamic instabilities.
We expand
\[ A(x, t) = A_0(t) + A_1(t) \cos kx + A_2(t) \cos 2kx \] ...
\[ B(x, t) = B_0(t) + B_1(t) \cos kx + B_2(t) \cos 2kx \]
Neglecting the fast decaying modes \( B_1 \) and \( B_2 \), we obtain a four mode truncation. The four coupled equations that are obtained on substitution can provide an indication of a Hopf-Turing bifurcation.

\[ \dot{A}_0 = -A_0 + \frac{A_1^2}{2B_0} + \frac{A_2^2}{2B_0} \sigma \]
\[ \dot{A}_1 = -(1 + D k^2) A_1 + \frac{2A_0 A_1}{B_0} + \frac{A_0 A_2}{B_0} \]
\[ \dot{A}_2 = -(1 + 4 D k^2) A_2 + \frac{A_1^2}{2B_0} + \frac{2A_0 A_2}{B_0} \]
\[ \dot{B}_0 = \mu \left[ A_1^2 + \frac{1}{2} (A_1^2 + A_2^2) - B_0 \right] \]

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Experimental observation of Turing patterns in chemical reactions was made possible by the use of continuously fed gel reactors. A host of spatiotemporal phenomena has been studied using the CIMA reaction mentioned earlier. The interaction of two or more stationary Turing modes or between Hopf and Turing instabilities leads to the emergence of a large variety of interesting patterns which have been the focus of numerical simulations. A popular example is the 'black eye' pattern observed in the CIMA reaction where there is a black centre (high concentration of Iodide) surrounded by a white ring (low concentration of Iodide) and these eye-like spots are arranged into a hexagonal lattice. Such a pattern is attributed to the spatial resonance of two interacting Turing modes with different wavelengths.

6 Conclusion
The demonstration in Sections 2, 3 and 4 shows how effective the Galerkin truncation can be if we know what is the phenomenon we are looking for. Once we realize that the physical situation would involve a competition between standing waves and travelling waves, it is possible to set up an appropriate truncation since we can visualize the modes that would be involved in that process. In Section 5, we have discussed the pattern formation in reaction diffusion system. This is yet another area where Galerkin truncation should play an important role in future. An important element in pattern formation that we have not touched upon is the role of defects. Defects are likely to be more amenable to perturbative techniques rather than Galerkin truncation.
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