NONLINEAR SCHRÖDINGER (NLS) FAMILY OF EQUATIONS: SPATIOTEMPORAL PATTERNS

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(Received 22 December 2003; Accepted 28 January 2005)

Nonlinear Schrödinger family of equations occurs in a variety of physical, chemical, engineering and biological problems of contemporary interest. These include problems ranging from the envelope dynamics of quasimonochromatic plane wave propagation in a weakly dispersive medium to Bose-Einstein condensates. It also describes the propagation of deep water waves as well as plasma waves and represents high intensity pulse propagation in optical fibers and photorefractive media. Further it possesses close connection with the dynamics of Heisenberg spin system. In the quantum mechanical context it is obtained in localizing the potential in the Hartree equation and in chemistry it appears as a model for mesoscopic molecular structures. The underlying systems possess fascinating spatiotemporal patterns starting from plane waves to solitary waves, bright and dark envelope solitons, exponentially localized dromions, algebraically decaying lumps, bound states, vortices, spiral waves, spatiotemporal chaos and so on. Different kinds of instabilities like Benjamin-Feir, Faraday, wave collapse, etc., giving rise to very many novel structures are also present. In particular, the cubic nonlinear Schrödinger equation is a completely integrable soliton possessing nonlinear dynamical system in (1+1) dimensions. However, in (2+1) dimensions the solution develops finite time singularities or blow up leading to wave collapse. Other types of nonlinearities such as power law, cubic-quintic, saturable and logarithmic are also of interest in different physical contexts. Wave collapse can be prevented by some of these modifications and also by introducing suitable confining potentials as in the case of Bose-Einstein condensates or coupling with mean fields or by introducing higher order dispersion. Effects of dissipation, diffusion and external forcing leading to dissipative Gross-Pitaevskii and Ginzburg-Landau type equations show a large number of physically interesting spatiotemporal patterns. In addition the dynamical equations of motion of Heisenberg ferromagnetic spin systems with or without damping have also close connection with nonlinear Schrödinger family of equations. In this article we give a brief review of this fascinating world of nonlinear Schrödinger family of equations.

Key Words: Spatiotemporal Patterns; Nonlinear Schrödinger Equation; Wave Collapse; Solitons; Localized Structures

1 Introduction

Nonlinear Schrödinger equation (NLS) is a ubiquitous nonlinear wave equation of dispersive type making its appearance in a wide variety of physical problems ranging from the envelope dynamics of a quasimonochromatic plane wave propagating in a weakly dispersive medium to Bose-Einstein condensates in condensed matter physics. It also describes water waves at the free surface of an ideal fluid as well as plasma waves and represents intense optical pulse propagation in optical fibers and photorefractive media (Kerr media). It also possesses close relation with the dynamics of Heisenberg ferromagnetic spin systems. In the quantum mechanical context it is obtained in localizing the potential of the Hartree equation and in chemistry it appears as a model for mesoscopic molecular structures.

In its simplest form, the NLS equation or more precisely the cubic NLS equation reads

\[ i \frac{\partial \psi}{\partial t} + \frac{1}{2} \nabla^2 \psi + \gamma + \psi^2 \psi = 0, \]  

(1)

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \nabla_\perp \) is the \( d \)-dimensional Laplacian which takes care of the group velocity dispersion and wave diffraction and \( \gamma \) is a parameter. \( \Delta_\perp \) denotes the Laplacian with respect to the transverse coordinates.

More general nonlinear forms are also of interest, namely the generalized NLS equation.

\[ i \frac{\partial \psi}{\partial t} + \nabla^2 \psi + \gamma f(|\psi|^2) \psi = 0, \]  

(2)

where \( f(u) \) is some smooth function of \( u \). Typical cases include:

\[ f(|\psi|^2) = |\psi|^{2a} \quad \text{(power law)} \]

\[ = |\psi|^2 - \alpha |\psi|^4 \quad \text{(cubic-quintic)} \]

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\[ = 1 - e^{-\alpha \psi} \quad \text{(exponential)} \quad \cdots (3) \]
\[ = \frac{\psi^2}{1 + c \psi^2} \quad \text{(saturable)} \]
\[ = \alpha \ln(1 + \psi^2) \quad \text{(logarithmic)} \]
and so on, where \( \alpha \) is a parameter. Vector generalization of nonlinear Schrödinger equations\(^6\) is also of high current interest, particularly in the context of multimode wave propagation in optical fibers/photorefractive media\(^6\) and Bose-Einstein condensates\(^3\). The most prominent of them is the so called Manakov model\(^9\)

\[ i \frac{\partial \Psi}{\partial t} + \nabla^2 \Psi + \gamma |\Psi|^2 \Psi = 0, \]
\[ \Psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_N)^T. \quad \cdots (4) \]

In the familiar form it reads,

\[ i \frac{\partial \psi_1}{\partial t} + \nabla^2 \psi_1 + \gamma \left( \sum_{i=1}^{N} |\psi_i|^2 \right) \psi_1 = 0, \]
\[ i \frac{\partial \psi_2}{\partial t} + \nabla^2 \psi_2 + \gamma \left( \sum_{i=1}^{N} |\psi_i|^2 \right) \psi_2 = 0, \]
\[ \vdots \]
\[ i \frac{\partial \psi_N}{\partial t} + \nabla^2 \psi_N + \gamma \left( \sum_{i=1}^{N} |\psi_i|^2 \right) \psi_N = 0, \quad \cdots (5) \]

In a different context, namely Bose-Einstein condensation, existence of localized structures and vortices become important, where the NLS eq. (1) in the presence of a confining external potential (for example optical lattice potential) becomes relevant,

\[ \psi_1 + \nabla^2 \psi_1 + V(r) \psi_1 + \gamma |\psi|^2 = 0. \quad \cdots (6) \]

Eq. (6) is also popularly known as the Gross-Pitaevskii equation\(^7,10\) derived as a mean field approximation to the Bose-Einstein condensates. Typical potentials include isotropic/anisotropic harmonic oscillator, periodic and other potentials\(^11,14\).

In each of the above evolution equations, the spatial dimension \( d \) plays a crucial role. Apart from plane wave solution exhibiting interesting stability property, NLS equations with one spatial dimension are dominated by solitary wave solutions and envelope solitons (bright or dark depending upon the sign of \( \gamma \)). However, dimension two becomes a critical dimension in the sense that in eq. (1), the solution blows up after a finite time or one says that wave collapse dominates the system\(^1\). This tendency to collapse can be offset by different mechanisms: (i) introduction of more general nonlinearities (as eq. (3)), (ii) increase in the number of components (vector generalization), (iii) introduction of suitable confining potentials, (iv) coupling to mean field as, for example, in the case of Davey-Stewartson equation, etc. In this article, we will review some of these possibilities in some detail.

Further, it will be of considerable physical relevance when instabilities and singularities set in and to consider the effect of dissipation/diffusion as well as external forcing to see whether interesting spatio-temporal patterns can arise. In this situation the Ginzburg-Landau (GL) equation\(^15\) is of considerable relevance. Its general form reads

\[ (a + ib) \psi_t + \alpha \psi_t + (1 + i \beta) \nabla^2 \psi_t + (c + i \delta) |\psi|^2 \psi_t = 0, \quad \cdots (7) \]

where \( a, b, c, d, \alpha \) and \( \beta \) are real parameters. The GL equation is a ubiquitous equation in condensed matter physics and numerous studies have been made on it to bring out the spatio-temporal patterns underlying the system. It also helps to understand how dissipation/diffusion affects the solution behaviour of the NLS equation. In a similar fashion, the role of dissipation/diffusion plays a very crucial role on the formation of stable vortices and localized states in Bose-Einstein condensates. It this case, the Gross-Pitaevskii equation reads as\(^16\)

\[ (\alpha - i) \psi_t + \nabla^2 \psi_t + V(r) \psi_t + \gamma |\psi|^2 = 0. \quad \cdots (8) \]

Finally, the dynamics of the Heisenberg ferromagnetic spin system in the presence of Gilbert damping\(^17\), specified by the Landau-Lifshitz equation,

\[ \vec{S}_t = \vec{S} \times \nabla \vec{S} + \mu \vec{S} \times (\vec{S} \times \nabla \vec{S}), \]
\[ \vec{S}_t = (S_x, S_y, S_z), \quad \vec{S}^2 = 1 \quad \cdots (9) \]

takes the following form under stereographic projection,

\[ i(1 + i \lambda) \omega, + \nabla^2 \omega - \frac{2 \omega' (\nabla \omega)^2}{(1 + \omega^2)} = 0, \]
\[ \omega = \frac{S_x + i S_y}{1 + S_z^2}. \quad \cdots (10) \]

where \( \lambda \) is the damping parameter, which again has close analogy with nonlinear Schrödinger equations.

It is thus clear that the world of nonlinear Schrödinger equations is quite large, fascinating, challenging and all encompassing. These equations occur in a wide spectrum of science and are of interest to mathematicians, physicists, engineers and other scientists. It is aim of this article to present an overview of the present progress and future challenges. The organization of the paper is as follows. In section 2.
we present some preliminary discussions on the occurrence of nonlinear Schrödinger equations. Then we introduce in section 3 the basic symmetries and conservation laws admitted by the NLS equation with power law nonlinearities. In particular we identify a critical dimension $d = 2$, where $\sigma$ is the power of nonlinearity and $d$ is the dimension of the space, when scale and pseudoconformal invariances exist. Consequently in section 4, we point out the existence of the fascinating phenomena of wave collapse or blow up of solution at finite time at $\sigma d = 2$. Then in section 5, we briefly touch of upon the Benjamin-Feir type modulational instability. Section 6 is devoted to the study of integrable solitary possessing $(1+1)$ dimensional NLS equations, both scalar and vector type, and spatiotemporal patterns exhibited by them in the presence of perturbing forces. In section 7 we examine the various possibilities of modifying the NLS equation to tame the tendency for wave collapse and obtaining bound, localized states in the critical dimension case. In section 8 we briefly touch upon some of the important spatiotemporal patterns admitted by the Landau-Ginzburg equation. Section 9 is devoted to point out the connection between the dynamics of the Heisenberg ferromagnetic spin system and the NLS type equations. Finally, in section 10 we present our conclusion.

2 Preliminaries

Consider a scalar nonlinear wave equation written formally as

$$L(\bar{\psi}, \bar{\nabla}) \psi + G(\psi) = 0,$$  ...(11)

where $L$ is a linear operator and $G$ is a nonlinear function of $\psi$ and its derivatives. One can look for appropriate quasimonochromatic wave of the form

$$\psi = \epsilon \psi_0 \text{e}^{i(\hat{k} \cdot \hat{r} - \omega t)}, \quad \epsilon \ll 1 \quad ...(12)$$

with a constant amplitude $\psi_0$ corresponding to a dispersion relation

$$L(\hat{k}, \omega, -i\hat{k}) = 0 \Rightarrow \omega = \omega(\hat{k}) \quad ...(13)$$

When one takes into account nonlinear effects over long time scales and large propagation distances through a regular perturbation expansion, avoiding secular terms in a multiple scale expansion wherein $\psi_0$ is assumed to be slowly varying over space and time, the NLS equation can be deduced (for spatial isotropic case)\(^4\) for the function $\psi$. The same derivation can also be made in terms of a Fourier mode coupling expansion as is generally done in the plasma physics literature (see for example ref. [1]).

Similarly considering a quasimonochromatic wavetrain, such as an intense laser beam, propagating in a weakly nonlinear centro-symmetric dielectric medium, one can show that it can be specified by the Maxwell wave equation for the electric field vector $\vec{E}(\vec{r}, t)$,

$$\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} + \nabla \left( \vec{\nabla} \cdot \vec{E} \right) = -\frac{1}{c^2 \epsilon_0} \vec{P}, \quad ...(14)$$

where $\epsilon_0$ is the free space permittivity, $c$ is the velocity of light and $\vec{P}$ is the induced polarization,

$$\vec{P} = \epsilon_0 \left[ \chi_1 \vec{E} + \chi_2 \vec{E} \cdot \vec{E} \right], \quad ...(15)$$

where $\chi'$s are the susceptibility tensors. Then with the monochromatic linearly polarized plane waves

$$\vec{E}_\psi(\vec{r}, t) = \left( \epsilon \text{e}^{i(\hat{k} \cdot \hat{r} - \omega t)} + \text{c.c.} \right) \hat{e}, \quad ...(16)$$

where $\hat{e}$ is the polarization vector, one can deduce the NLS eq. (1) in a slowly varying envelope approximation\(^4\). Similarly in many other contexts also one can deduce the nonlinear Schrödinger equation. For example, one can find several physically interesting situations such as the nonlinear spin wave propagation in the Heisenberg ferromagnetic spin system in the continuum limit\(^4\), formation of Bose-Einstein condensates in alkali atom\(^5\), wave propagation in photorefractive media\(^6\), etc., wherein one can deduce different versions of the NLS equation.

3 Basic Conservation Laws\(^4\)

Eq. (2) (and so eq. (1)) can be associated with a Lagrangian density

$$\mathcal{L} = \frac{i}{2} \left( \psi^* \psi - \psi^* \psi^* \right) - \left( \nabla \psi^* \cdot \nabla \psi \right) + F(\psi^2) \quad ...(17)$$

where

$$F(\lambda) = \int_0^\lambda \frac{d\lambda}{d \psi^2} \quad ...(18)$$

Consequently, extremizing the action functional,

$$S[\psi, \psi^*] = \frac{i}{2} \int_0^t d\tau \int d\vec{r} \mathcal{L}, \quad ...(19)$$

one is lead to eq. (2). The corresponding Hamiltonian density is

$$\tilde{H} = \nabla \psi \cdot \nabla \psi^* - F(\psi^2) \quad ...(20)$$

so that the Hamiltonian is

$$H = \int \left[ \nabla \psi \cdot \nabla \psi^* - F(\psi^2) \right] d\vec{r}. \quad ...(21)$$
Then, one can check that eq. (2) admits several conservation laws and constants of motion1.

(i) Invariance Under Gauge Transformation

Under the transformation
\[ \psi \rightarrow \psi' = e^{i\theta} \psi, \quad \theta : \text{real}, \quad \ldots(22) \]
the action in eq. (20) is invariant. The corresponding conservation law is
\[ \partial_t \|\psi\|^2 + \vec{V} \cdot \dot{\vec{V}} + i \left( \psi^* \nabla^* \psi' - \psi^* \nabla \psi' \right) = 0, \quad \ldots(23) \]
leading to the conservation of mass/wave energy/number density/wave power,
\[ N = \int_{\mathbb{R}^d} |\phi|^2 \, d\vec{r}. \quad \ldots(24) \]

(ii) Invariance Under Time Translation

Invariance of the action under infinitesimal time translation leads to the conservation law
\[ \delta \left[ \int \left( \psi^* \nabla^* \psi' + \psi^* \nabla \psi' \right) \right] \]
\[ - \nabla \cdot \left( \psi^* \nabla \psi' - \psi^* \nabla \psi \right) = 0, \quad \ldots(25) \]
so that the Hamiltonian/energy given in eq. (21) is the conserved quantity.

(iii) Invariance Under Space Translation

Again the invariance of the action under infinitesimal space translation leads to
\[ \delta \left[ \int \left( \psi^* \nabla^* \psi' - \psi^* \nabla \psi' \right) \right] \]
\[ - \nabla \cdot \left( \psi^* \nabla \psi' + \nabla \psi' \cdot \left( \psi^* \nabla \psi + \nabla \psi \cdot \nabla \psi \cdot \overline{L} \right) \right) = 0, \quad \ldots(26) \]
where \( \overline{L} \) is the Lagrangian density. Correspondingly the linear momentum
\[ \vec{P} = i \int_{\mathbb{R}^d} \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right) \, d\vec{r} \quad \ldots(27) \]
is conserved. Further, the mass centre of mass
\[ \langle \vec{r} \rangle = \frac{\vec{R}}{N} = \frac{1}{N} \int \vec{r} |\psi|^2 \, d\vec{r} \quad \ldots(28) \]
obeys the evolution equation
\[ N \frac{d\vec{R}}{dt} = \frac{1}{N} \int \vec{r} \partial_t \left( |\phi|^2 \right) \, d\vec{r} = \vec{P}. \quad \ldots(29) \]
Thus the mass centre moves at a constant speed.

(iv) Invariance Under Space Rotation

In this case one finds the conservation of angular momentum
\[ \vec{M} = i \int_{\mathbb{R}^d} \vec{r} \times \left( \psi^* \nabla^* \psi' - \psi^* \nabla \psi' \right) \, d\vec{r}. \quad \ldots(30) \]

(v) Galilean Invariance

The NLS equation is invariant under Galilean transformation
\[ \vec{r} \rightarrow \vec{r}' = \vec{r} - \vec{c}t, \quad t \rightarrow t' = t \quad \ldots(31) \]
\[ \psi(\vec{r}, t) \rightarrow \psi(\vec{r}', t') = e^{-i \frac{1}{2} \frac{\vec{r}^2}{4} - \frac{\vec{c}^2}{4} t} \psi(\vec{r} + \vec{c}t', t') \]
The consequence is that the velocity of the wavepacket of the centre of mass is uniform, see eq. (30).

(vi) Scale Invariance for Power Law Nonlinearity

NLS eq. (1) with power law nonlinearity
\[ f \left( |\psi|^2 \right) = \gamma |\psi|^{2\sigma} \quad \ldots(32) \]
is invariant under the scale transformation,
\[ \vec{r} \rightarrow \vec{r}' = \lambda^{-1} \vec{r}, \quad t \rightarrow t' = \lambda^{-2} t \quad \ldots(33) \]
\[ \psi(\vec{r}, t) \rightarrow \psi(\vec{r}', t') = \lambda^{\frac{2-\sigma}{\sigma}} \psi(\lambda \vec{r}, \lambda^2 t). \]

Then, the action scales like \( \lambda^{2-\sigma} \), so does the wave energy \( N = \int |\psi|^2 \, d\vec{r} \). Thus one finds that at the critical dimension \( \sigma d = 2 \), both the action and the equation of motion remain invariant under the scale transformation. Consequently, one can show that
\[ C_1 = \frac{i}{2} \int_{\mathbb{R}^d} \vec{r} \times \left( \psi^* \nabla \psi' - \psi^* \nabla \psi \right) \, d\vec{r} - 4 Ht \quad \ldots(34) \]
is conserved.

(vii) Pseudoconformal Invariance at Critical Dimension

For the power law given in eq. (32), eq. (1) admits another invariance at the critical dimension \( \sigma d = 2 \). Defining
\[ l(t) = \frac{t - t_0}{t_0}, \quad t < t_0, \quad \ldots(35) \]
eq (1) at the critical dimension is invariant under the pseudo-conformal or lens transformation:
\[ \vec{r} \rightarrow \vec{r}' = \frac{\vec{r}}{l}, \quad t \rightarrow t' = \frac{1}{l^2} \frac{1}{s'} ds' = \frac{t_0^2 t}{l(t_0 - t)} \]
\[ \psi \rightarrow \psi'(\vec{r}', t') = l^2 \psi(\vec{r}, t) \exp \left( -i \frac{l_0^2 |\psi|^2}{4 l} \right) \quad \ldots(36) \]
\[ = l^2 \psi(\vec{r}, t) \exp \left( i \frac{d|\psi|^2}{4 l^2} \right) \]
with
\[ a = -i \frac{dl}{dt} = \frac{1}{l} \frac{dl}{dt'} = \frac{t_s - t}{t_o^2}. \]  
...(37)

One can show that as a consequence of this symmetry the solution of the NLS eq. (1) develops a singularity at some finite time \( t \), for the critical dimension \( cd = 2 \). The symmetry given at eq. (37) also leads to the conserved quantity
\[ C_2 = \int \left( \nabla \psi + 2i \nabla \psi \right)^2 + \frac{4l^2}{\sigma} \left| \psi \right|^{2\sigma+1} |d\bar{F}|. \]  
...(38)

(viii) **Evolution of Variance**

Defining now the quantity ‘variance/moment’ of inertia
\[ V(t) = \int \frac{1}{\sigma} |\psi|^2 \, d\bar{F}, \]  
...(39)

one can show that
\[ \frac{d^2 V(t)}{dt^2} = 8H - 4 \frac{d\sigma - 2}{\sigma + 1} \int |\psi|^{2\sigma+2} \, d\bar{F}. \]  
...(40)

4 **NLS Equation and Modulational Instability**

The cubic NLS equation
\[ i\psi_t + \nabla^2 \psi + \gamma |\psi|^2 \psi = 0, \]  
...(41)

admits exact, almost trivial, solution of the form
\[ \psi = A \exp (i\gamma |A|^2 t), A \text{ : complex constant} \]  
...(42)

What can one say about the linear stability of this solution? Under infinitesimal perturbation
\[ \psi = \left[ A + \epsilon \psi + i \phi \right] e^{i |A|^2 t}, \]  
...(43)

eq. (41) becomes
\[ \chi_t + \nabla^2 \chi = 0, \]  
...(44a)

\[ \phi_t + \nabla^2 \phi - 2 \gamma |A|^2 \chi = 0, \]  
...(44b)

or equivalently
\[ \chi_t + \nabla^4 \chi + 2 \gamma |A|^2 \chi = 0 \]  
...(45)

One can look for harmonic perturbations \( \chi \) and \( \phi \) proportional to \( e^{kx} \cdot e^{ix_t} \) in eqs. (44) or (45) so as to give
\[ \lambda^2 + (k^4 - 2 \gamma |A|^2 k^2) = 0, \]  
...(46a)

or
\[ \lambda = \pm ik \sqrt{k^2 - 2 \gamma |A|^2}. \]  
...(46b)

If \( \gamma \) is negative, both the values of \( \lambda \) are imaginary and the perturbation remains bounded, while for \( 2 |A|^2 > k^2 |\gamma| \) there is instability which is the so-called Benjamin-Feir\textsuperscript{st} instability.

What is the nonlinear development of this instability? The answer is that the outcome is highly dependent on the space dimension. In \( (1 + 1) \) dimensions coherent localized structures, namely solitons can arise and in \( (2 + 1) \) dimensions wave collapse dominates. These aspects will be discussed in some detail in the next sections.

5 **Solitons in \( (1 + 1) \) Dimensions**

The family of nonlinear Schrödinger equations in \( (1 + 1) \) dimensions is a well studied topic and much progress has been made in understanding the nature of solutions. In particular, the cubic scalar nonlinear Schrödinger equation is a completely integrable infinite dimensional nonlinear dynamical system in the Liouville sense and it admits exponentially localized soliton solutions\textsuperscript{2,19}. A physically important vector generalization, namely the celebrated Manakov model, also turns out to be a completely integrable system, exhibiting novel soliton properties. In the following, we will give a summary of these results.

5.1 **The Scalar NLS Equation**

The one dimensional cubic nonlinear Schrödinger equation
\[ i\psi_t + \psi_{xx} + \gamma |\psi|^2 \psi = 0, \quad \gamma > 0 \]  
...(47)

is a completely integrable infinite dimensional nonlinear dynamical system in the Liouville sense (For \( \gamma < 0 \), one can obtain the so-called dark soliton solutions). It can be linearized: By considering the \( (2 \times 2) \) matrix linear eigenvalue problem (the so called AKNS eigenvalue problem)
\[ v_{1x} + i \xi v_1 = q v_2, \]  
...(48a)

\[ v_{2x} - i \xi v_2 = q^* v_1, \]  
...(48b)

and the time evolution equation
\[ v_t = -2i \xi^2 v_1 + (-2i \xi^2 q + i q_x) v_2, \]  
...(49a)

\[ v_{2t} = (2i \xi^2 v_1 - i q_x) v_2 + 2i \xi^2 v_2, \]  
...(49b)

one can easily check that the compatibility of eqs. (48) and (49) is the NLS equation (47). Consequently carrying out an inverse scattering transform analysis (see ref. [2, 19]), one can solve the initial value problem underlying eq. (47) for \( |\psi| \to 0 \) as \( |x| \to \infty \). The resulting solution leads to the so-called N-soliton (envelope/bright soliton) solution.
The 1- (envelope) soliton solution has the form
(with $\gamma = 2$)
\[
\psi(x,t) = p_{1r} e^{\phi_1} \text{sech} z_1 \quad \ldots (50)
\]
where
\[
\begin{align*}
z_1 &= p_{1l} x + (p_{1l}^2 - p_{1l}^2) t + \eta_{1l}^0, \quad \ldots (51a) \\
z_2 &= p_{1r} (x - 2 p_{1l}) + \eta_{1l}^0 + \Delta, \quad \Delta = \log \frac{1}{2 p_{1r}} \quad \ldots (51b)
\end{align*}
\]
where $p_l = p_{1rl} + ip_{1l}$, $\eta_{l}^0 = \eta_{1l}^0 + i \eta_{1l}^0$ are complex parameters. It is fully localized (see Fig. 1). The two soliton solution is
\[
\psi(x,t) = \frac{e^{\phi_1} + e^{\phi_2} + e^{\phi_3} + e^{\phi_4} + e^{\phi_5} + \eta_{l}^0}{1 + e^{\phi_1} + e^{\phi_2} + e^{\phi_3} + e^{\phi_4} + e^{\phi_5} + \eta_{l}^0}, \quad \ldots (52a)
\]
where
\[
\eta_j = p_{j} x + \hat{p}_{j} t + \eta_{l}^0, \quad j = 1, 2, p_{j}, \eta_{l}^0: \text{complex constants}
\]
\[
e^{\phi_1} = \frac{1}{(k_1 + k_2)^2}, \quad e^{\phi_2} = \frac{1}{(k_1 + k_2)^2}, \quad e^{\phi_3} = \frac{1}{(k_1 + k_2)^2},
\]
\[
e^{\phi_4} = \frac{(k_2 - k_1)^2}{(k_1 + k_2)^2(k_1 + k_2)^2}, \quad e^{\phi_5} = \frac{(k_2 - k_1)^2}{(k_2 + k_2)^2(k_1 + k_2)^2},
\]
\[
e^{\phi_6} = \frac{|k_1 - k_2|^2}{(k_2 + k_2)^2(k_1 + k_2)^2} \quad \ldots (52b)
\]
Similarly, proceeding further, one can write down the $N$-soliton solution. The one and two soliton solutions are plotted in Figs. 1.

Note that the above envelop (bright) solitons undergoes purely elastic collision (Fig. 1): Except for a finite phase-shift, the solitons after the collision do not undergo either change in shape or in velocity. They are quite stable entities.

Using the linear eigenvalue problem, one can show that the cubic NLS equation in $(1 + 1)$ dimensions admits infinite number of involutive integrals of motion and that a canonical transformation can be made to find infinite number of action and angle variables so that the Hamiltonian can be written purely in terms of action variables alone. Thus the $(1 + 1)$ dimensional cubic NLS equation is a completely integrable infinite degrees of freedom nonlinear dynamical system, admitting soliton solutions.

### 5.2 The Vector NLS Equation

Considering the vector NLS equation in $(1 + 1)$ dimensions
\[
i\vec{\psi}_t + \nabla^2 \vec{\psi} + \gamma \left( \vec{\psi} \cdot \vec{\psi}^* \right) \vec{\psi} = 0,
\]
\[
\vec{\psi} = \left[ \psi_1, \psi_2, \psi_3, \ldots, \psi_N \right]^T, \quad \ldots (53)
\]
or equivalently in component form
\[
i\psi_j + \nabla^2 \psi_j + \gamma \left( \sum_{\ell=1}^{N} \psi_\ell^2 \right) \psi_j = 0,
\]
\[
\begin{align*}
& j = 1, 2, \ldots, N, \quad \ldots (54)
& \end{align*}
\]
one can show that it also admits a Lax pair. For example, for $N = 2$ (Manakov model case, see for example, ref. [19]),
\[
L = \begin{pmatrix}
-i\lambda & 0 & \psi \\
0 & -i\lambda & \psi^* \\
-\psi^* & -\psi & 0
\end{pmatrix} \quad \ldots (55a)
\]
and
\[
M = -4i\lambda^2 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} + 4i\lambda \left[ \begin{pmatrix}
0 & 1 & 0 \\
-\psi & -\psi & 0 \\
\psi & \psi & -\psi
\end{pmatrix} \right] + 2i \begin{pmatrix}
|\psi|^2 & \psi & \psi^* \\
\psi & |\psi|^2 & \psi^* \\
\psi^* & \psi^* & |\psi|^2
\end{pmatrix} \quad \ldots (55b)
\]

![Fig. 1 The cubic NLS equation: one and two soliton solutions](image-url)
such that $L_i - M_i + [L, M] = 0$ is equivalent to eqs. (53) or (54) for $N = 2$. Consequently the vector NLS eq. (53) or (54) also becomes a completely integrable soliton system. However, the solutions in this kind of vector NLS equations admit a completely different, nontrivial soliton interaction: Under collision the shape of the soliton changes, giving rise to an intensity redistribution. The underlying state changes can be represented by a linear fractional transformation (LFT) which then allows one to develop logic gates, which are precursors to an all optical computer in homogeneous bulk media.

To realize the above, let us consider the one and two soliton solutions of the two component NLS equation:

(i) One Soliton Solution

\[
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} k_i \text{sech} h_i e^{i\theta_i},
\]

\[|A_1|^2 + |A_2|^2 = 1, \quad h_i = h_{_{\text{Ir}}} + i h_{_{\text{Ir}}}, \quad k_i = k_i(x + i k_i t) + h_{_{\text{Ir}}}, \quad \ldots \quad (56)\]

where $k_i$ and $h_{_{\text{Ir}}}$ are complex parameters.

(ii) Two Soliton Solution

\[
q_i = \frac{\alpha^{(1)}_{i}(x + ik_i t) + \alpha^{(2)}_{i}(x + ik_i t)}{1 + \alpha^{(1)}_{i}(x + ik_i t) + \alpha^{(2)}_{i}(x + ik_i t)}, \quad i = 1, 2
\]

\[\ldots \quad (57)\]

Here

\[\eta = k_i(x + ik_i t), \quad \eta^2 = \frac{\kappa_{11}}{k_i + k_i^*}, \quad \eta^2 = \frac{\kappa_{22}}{k_i + k_i^*}, \quad \eta^2 = \frac{\kappa_{21}}{k_i + k_i^*}, \quad \ldots \quad (58)\]

\[
\begin{align*}
\alpha^{(1)}_{1} &= \frac{k_i - k_1}{(k_i + k_i^*)(k_i + k_1^*)}(\alpha^{(1)}_{1} \kappa_{12} - \alpha^{(2)}_{1} \kappa_{22}), \\
\alpha^{(1)}_{2} &= \frac{k_i - k_2}{(k_i + k_i^*)(k_i + k_2^*)}(\alpha^{(1)}_{2} \kappa_{21} - \alpha^{(2)}_{1} \kappa_{11}), \\
\alpha^{(2)}_{1} &= \frac{k_i - k_1}{(k_i + k_i^*)(k_i + k_2^*)}(\alpha^{(2)}_{2} \kappa_{21} - \alpha^{(1)}_{2} \kappa_{12}), \\
\alpha^{(2)}_{2} &= \frac{k_i - k_2}{(k_i + k_i^*)(k_i + k_2^*)}(\alpha^{(1)}_{1} \kappa_{12} - \alpha^{(2)}_{1} \kappa_{22}), \\
\kappa_{ij} &= \frac{\alpha^{(1)}_{i} \alpha^{(1)*}_{j} + \alpha^{(2)}_{i} \alpha^{(2)*}_{j}}{(k_i + k_i^*)}, \quad i, j = 1, 2, \ldots \quad (59)\end{align*}
\]

Here $\alpha^{(1)}_{i}$'s are complex parameters.

Then an asymptotic analysis of the two soliton solution (2) shows that the amplitudes of the soliton before collision and after collision are not the same, but will vary in general. In fact denoting the state variables as

\[
\rho_{1}^{*} = \frac{q^{*}_{1}(t \rightarrow \pm \infty)}{q^{*}_{2}(t \rightarrow \pm \infty)} = A_{2}^{*}, \quad j = 1, 2, \ldots \quad (60)\]

and calling $\rho_1$ and $\rho_2$ as representing the states of the solitons $S_1$ and $S_2$, respectively, before collision, while $\rho_1$ and $\rho_2$, respectively, as the states representing the soliton $S_1$ and $S_2$ after collision, it can be shown that the soliton states before and after collision can be related through a linear fractional transformation (LFT),

\[
\rho_2 = \frac{a(\rho_1) \rho_2 + b(\rho_1)}{c(\rho_1) \rho_2 + d(\rho_1)}(ad - bc) \neq 0, \quad (60a)\]

Fig. 2 Two soliton collision in the Manakov model
and
\[ \rho_2 = \frac{a'(\rho_1)\rho_1 + b'(\rho_2)}{c'(\rho_1)\rho_1 + d'(\rho_2)}, \quad (d'd' - b'c') \neq 0, \quad \ldots (60b) \]

where
\[ a = \frac{1 - g}{\rho_1^*}, \quad b = \frac{g \rho_1}{\rho_1^*}, \quad c = g \]
\[ d = (1 - g)\rho_1 + \frac{1}{\rho_1^*}, \quad g(k_1, k_2) = \frac{k_1 + k_1^*}{k_2 + k_2^*}, \quad \ldots (61a) \]
\[ a' = \frac{1 - h'}{\rho_L}, \quad b' = \frac{h' \rho_L}{\rho_L^*}, \quad c' = h' \]
\[ d' = (1 - h')\rho_L + \frac{1}{\rho_L^*}, \quad h' = h'(k_1, k_2) = g(k_2, k_1), \quad \ldots (61b) \]

The above type of shape changing collisions of solitons is also present in the case of \( N \)-component vector NLS equations, wherein the state change can be represented by a generalized linear fractional transformation of the form\(^22\)
\[ \rho_{L,N} = \sum_{j=1}^{N} C_{j}^{(1)} \rho_{j,N} \sum_{j=1}^{N} C_{j}^{(2)} \rho_{j,N}^* = 1, \quad \ldots (62) \]

Typical cases of two soliton collisions and three soliton collisions are presented in Figs. 2 and 3.

5.3 Perturbation of (1+1) NLS Equations: Spatiotemporal Patterns

Most natural nonlinear dynamical systems are nonintegrable; however, many of these may be considered as perturbations of integrable nonlinear systems. The perturbing forces could be space-time inhomogeneities and modulations, external forces of different origins, damping and dissipation as well as diffusive forces and so on.

An analysis of these perturbing forces needs to consider the different length scales of the perturbation (both space and time) with respect to the nonlinear excitations of the unperturbed case. Depending on such scales, the original entities corresponding to unperturbed systems might survive albeit necessary deformations or may undergo chaotic or complex motions or deformations or may give rise to interesting spatio-temporal patterns. For example, the perturbed (1 + 1) dimensional cubic nonlinear Schrödinger equation\(^23\)
\[ i\psi_t + \psi_{xx} + 2|\psi|^2 \psi = \epsilon \psi \cos kx, \quad \ldots (63) \]
has been shown to give rise to complex spatio-temporal patterns for suitable choice of \( k \) (see Fig. 4).

6 (2+1) NLS Equations: Wave Collapse

In section 3, we have seen that the variance evolves as a function of time for the power law NLS equation (which includes the cubic NLS also) as given by eq. (40). Consequently, one can establish the following theorem [see for example, Sulem and Sulem\(^1\)]

**Theorem 1**

Given that \( d \sigma \geq 2 \) and an initial condition \( \varphi(\vec{r}) \in H^1 \) with the variance \( V(0) \) finite that satisfies

1. \( H(\varphi) < 0 \)
2. \( H(\varphi) < 0 \) and \( i \int \varphi^* \cdot \nabla \varphi d\vec{r} < 0 \)
3. \( H(\varphi) > 0 \) and \( i \int \varphi^* \cdot \nabla \varphi d\vec{r} \leq -4 \int |H(\varphi)| \varphi^2 d\vec{r} \),

where \( H \) is the Hamiltonian. Then, there exists a time \( t_c < \infty \) such that, as \( t \rightarrow t_c \), \( |\varphi| \) and \( \nabla |\varphi| \) tend to infinity, that is the solution blows up in finite time.

**Proof:** Since \( d \sigma \geq 2 \), from eq. (40) it follows that
\[ \frac{d^2 V(t)}{dt^2} \leq 8H, \quad V(t) = \int \varphi^2 \cdot |\psi|^2 d\vec{r}, \quad \ldots (64) \]
By integration
\[ V(t) = 4h^2 + V'(0) + V(0), \quad h = H(\phi), \quad \text{...(65)} \]
where by definition
\[ V'(0) = 4i \int \phi^* \cdot \nabla \phi \, d\vec{r} \quad \text{...(66a)} \]
and
\[ V(0) = \int \phi^2 \cdot \phi^2 \, d\vec{r}. \quad \text{...(66b)} \]
It is then straightforward to check that for any of the conditions of the above theorem, there exists a time \( t_0 \) such that the right hand side of eq. (65) vanishes. This in turn implies that there exists a \( t_0 \), such that
\[ \lim_{t \to t_0} V(t) = 0. \quad \text{...(67)} \]
Now, for any given function \( f \in H^1 \), we have
\[ \int |f|^2 \, d\vec{r} = \frac{1}{d} \int (\nabla \cdot \vec{r}) |f|^2 \, d\vec{r} \]
\[ = -\frac{1}{d} \int \vec{r} \cdot \nabla (|f|^2) \, d\vec{r}, \quad \text{...(68)} \]
where \( d \) is the spatial dimension, and so one gets the uncertainty relation
\[ |f|_{L^2}^2 \leq \frac{2}{d} \nabla (|f|_{L^2}^2) |f|_{L^2}^2. \quad \text{...(69)} \]
When \( f \) is taken as the solution \( \psi \), then from (67) and the definition of \( V(t) \), and conservation of \( |\psi|_{L^2}^2 \), it follows that there exists a time \( t_0 < t_0 \), such that
\[ \lim_{t \to t_0} |\nabla \psi|_{L^2}^2 = \infty. \quad \text{...(70)} \]
Further since the Hamiltonian \( H \) is conserved, it follows that
\[ \lim_{t \to t_0^+} |\psi|_{L^2}^{2+} = \infty. \quad \text{...(71)} \]
Since \( |\psi|_{L^2}^2 \) is conserved, it implies that
\[ \lim_{t \to t_0^+} |\psi|_{L^2}^2 = \infty. \quad \text{...(72)} \]
Thus the vanishing of the variance implies that a collapse of the wave packet or the blow up of the solution at finite time takes place. A numerical analysis of the cubic nonlinear Schrödinger eq. (1) in (2 + 1) dimensions is shown in Fig. 5 to demonstrate the blow up nature of the solution.

In particular due to the condition \( cR = 2 \) in the above, one finds that for cubic nonlinearity (\( \sigma = 1 \)), the critical dimension is \( d = 2 \). For quintic nonlinearity, \( \sigma = 2 \), this implies \( d = 1 \) itself! Thus it is clear that in (2 + 1) dimensions, the cubic nonlinear Schrödinger equation given at eq. (1) will not admit localized coherent structures like the solitons in (1 + 1) dimensions.

7 Modification to NLS Equations and Taming Wave Collapse: Localized Structures and Bound States

Considering the cubic nonlinear Schrödinger eq. (1), it is inevitable that large scale instabilities leading to wave collapse or finite time blow up occurs for spatial dimension \( d \geq 2 \). However, this does not mean that the physical system necessarily develops collapse/singularity at finite time. It is more to do with the type of approximation made in modelling the physical
Fig. 5 Wave collapse in (2 + 1) dimensional cubic NLS equation

problem to the cubic NLS equation. Modifying the
equations suitably to suit the nature of the problem,
one can identify interesting bound states or localized
solutions in higher spatial dimensions. In particular, in
(2 + 1) dimensions one can identify a number of ways
of such collapse arresting modifications. Some of the
most prominent of them are as follows:
1. Coupling to a mean field.
2. Introducing suitable confining potentials.
3. Introducing suitable additional nonlinearity such as
   saturation.
4. Increasing the number of variables, like the case
   of vector NLS equations.
5. Adding higher order dispersion terms.
   We will discuss each of these possibilities briefly
   in the following.

7.1 Coupling to Mean Field/Velocity Potential: The
Davey–Stewartson Equation and Dromions

While developing a perturbation theory for the slow
modulation of the function \( \psi(\vec{r}, t) \) of eq. (12) for
the wave eq. (11), one can in addition introduce a
mean field \( \phi(\vec{r}, \tau) \) into the expansion, which is
independent of the fast variables. In this case the field
\( \psi(\vec{r}, t) \) gets coupled to the mean field variable, which
may under suitable circumstances give rise to localized
coherent structures. In particular considering the
Kadomtsev-Petviashvile equation and carrying out a
perturbation analysis for the modulational field in the
presence of a mean field \( \phi(\vec{r}, t) \), one obtains the so
called Davey-Stewartson (DS) equation:

\[
\begin{align*}
    i\psi_t + \psi_{xx} + \alpha \psi_{yy} + \gamma |\psi|^2 \psi + \phi \psi &= 0, \quad (73a) \\
    \phi_{xx} - \alpha \phi_{yy} &= -\epsilon |\psi|^2_x, \quad (73b)
\end{align*}
\]

where \( \alpha \) and \( \epsilon \) are parameters. The Davey-Stewartson
equation I corresponds to the elliptic case when \( \alpha > 0 \)
and DS-II corresponds to the hyperbolic case when \( \alpha < 0 \). The DS-I equation in the light cone coordinates can be written as

\[
\begin{align*}
    i\psi_t + \psi_{xx} + \psi_{yy} + \gamma |\psi|^2 \psi + \phi \psi + \phi_x \psi &= 0, \quad (74a) \\
    \phi_{t} - \gamma \int_\infty d\xi |\psi|^2_x - \frac{1}{2} \gamma |\psi|^2 + u_1(\eta, t), \quad (74b) \\
    \phi_x = -\frac{1}{2} \gamma \int_\infty d\xi |\psi|^2_x - \frac{1}{2} \gamma |\psi|^2 + u_2(\xi, t), \quad (74c)
\end{align*}
\]

where \( u_1(\eta, t) \) and \( u_2(\xi, t) \) are arbitrary functions
corresponding to the boundaries. The DS-I equation is
linearizable and is a completely integrable (2 + 1)
dimensional nonlinear dynamical system. It admits line
(plane) solitons, algebraically decaying solitary waves and
exponentially localized dromions, which move along
specified tracks. For example, the basic (1, 1)
dromion solution reads

\[
\psi(\xi, \eta, t) = \frac{4P_n S_0 (j k - 1) \exp \left[ P_n \xi + S_0 \eta + \frac{1}{2} \left[ P_n \xi + S_0 \eta \right]^2 \right]}{\sqrt{1 + j \exp (2P_n \xi) + k \exp (2S_0 \eta)}}.
\]

\[
(75)
\]

where \( P_1 = P_{1, k} + i P_{1, l}, S_1 = S_{1, k} + i S_{1, l} \) and \( j, k, l \) are real and
positive constants. One can also write explicit
expressions for multifermion solutions as well and study
their nonlinear interaction properties. In general these
dromions can change shape under collisions though their
particle number remains constant. A snapshot of the
basic dromion solution is given in Fig. 6.

7.2 Gross-Pitaevskii (GP) equation: Inclusion of a
Confining Potential

At ultralow temperatures, the properties of Bose-
Einstein condensates (BECs) in dilute weakly-
interacting alkali atoms are usually described by the
time dependent nonlinear mean-field Gross-Pitaevskii
(GP) equation, which can be expressed in terms of
dimensionless quantities as

\[
\begin{align*}
    i \frac{\partial \psi}{\partial t} + \nabla^2 \psi + V(r) \psi + \gamma |\psi|^2 \psi &= 0. \quad (76)
\end{align*}
\]
Here $\psi(\mathbf{r}, t)$, $\mathbf{r} \in (x, y, z)$, represents the condensate wave function, $\mathcal{V}(\mathbf{r})$ is the trap potential and $\gamma = -8 \sqrt{2} \pi N a / l$ corresponds to the strength of interatomic interaction (attractive/repulsive) between the atoms forming the condensate with $a (< 0$ or $> 0$) being the atomic scattering length. The trap potential is normally of harmonic type and is given by $\mathcal{V}(\mathbf{r}) = -\frac{1}{4} (x^2 + \kappa y^2 + vz^2)$ and the wave function $\psi$ obeys the normalization condition:

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz |\psi|^2 = 1. \quad \ldots(77)$$

The above GP eq. (76) can be reduced to a two-dimensional model by separating the degrees of freedom of the wave function as $\psi(\mathbf{r}, t) = \phi(x, y, t) \xi(z)$. This two-dimensional approximation may be valid for the BEC in a pancake-shaped trap potential for $\kappa = 1$ and $v >> 1$. There is another situation with a cigar-shaped potential ($\kappa = 1$ and $v < 1$) for which this two-dimensional approximation is valid. However, there exist some differences in the interaction term in both the cases.

The GP equation, for the two-dimensional case\textsuperscript{16}, may be written as

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + V(x, y)\psi + \mu \psi + C |\psi|^2 \psi = 0, \quad \ldots(78)$$

where $\mu$ is the chemical potential that arises due to the two-dimensional approximation and $C$ is the modified interaction strength.

The above equation admits stationary states (bound state solutions). Fig. 7 illustrates the numerically simulated stationary wave function ($\psi$) in a quasi 2-dimensional condensate.

7.2.1 Dissipative Gross-Pitaevskii (GP) Equation

Often, from a physical point of view, it becomes necessary to include dissipation in order to simulate vortex lattice formation. In such cases, a proper dissipative term can be included in the GP equation leading to the form\textsuperscript{16}

$$(i - \alpha) \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + V(x, y)\psi + \mu \psi + C |\psi|^2 \psi + \Omega L_z \psi = 0, \quad \ldots(79)$$

where $L_z = ix \partial_y - iy \partial_x$ is the angular momentum and the dimensionless parameter $\alpha$ describes the degree of dissipation.

The above eq. (79) exhibits quantized vortex states with rotational motion of the condensate around the z-axis. Such vortex states are observed in the Bose-Einstein condensates when a steady rotation is applied along the z-axis\textsuperscript{16,25}. A typical arrangement of vortex lattices formed by the above eq. (79) is shown in Fig. 8.

7.2.2 BEC in Optical Lattice (Periodic Trap Potential)

Another important aspect which has attracted much attention in recent times is the role of optical lattices. From an experimental point of view, an optical lattice potential, for example introduced by scanning the condensate by a standing wave laser, can be included along with the harmonic trap potential. Fig. 9 depicts a numerically simulated condensate in a two dimensional optical lattice potential of the form\textsuperscript{25}

$$V_{\text{opt}}(x, y) = V_0 (\sin^2 kx + \sin^2 ky). \quad \ldots(80)$$
Fig. 7  Ground state of the two dimensional BEC: solution to eq. (78) for $C = -1$ and $\mu = 0$. The trap potential is taken as $V(x, y) = -(x^2 + y^2)$.

Fig. 8  Formation of vortex states in quasi 2-dimensional Bose-Einstein condensate.

Fig. 9  BEC in optical lattice.
7.3 Saturable Nonlinearity

It has been shown that when the power law nonlinearity in eq. (2) is replaced by certain saturable nonlinearities such as exponential or other forms of saturable nonlinearities given in eq. (3), one may be able to avoid the collapse of wave solutions under suitable circumstances and obtain bounded, localized solutions in $(2 + 1)$ dimensions (see for example, Refs. [1] and [9]). In these cases a variational analysis establishes the stability of these solutions.

For example, the $(2 + 1)$ dimensional NLS equation with a saturable nonlinearity of the form

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2} \nabla^2 \psi + \frac{|\psi|^2 \psi}{1 + |\psi|^2} = 0 \quad \cdots(81)$$

reduces to the ordinary differential equation

$$i\frac{\partial q}{\partial \rho} + \frac{1}{\rho} \frac{\partial q}{\partial \rho} + \frac{|q|^2 q}{1 + |q|^2} = \alpha_0 q, \quad \cdots(82)$$

if one looks for cylindrically symmetric stationary solution of the form

$$\psi(\vec{r}, t) = q(\rho) e^{i\omega t} \quad \cdots(83)$$

Assuming that $q$ vanishes as $\rho \to \infty$ and finite at $\rho = 0$ and that

$$\frac{dq}{dr}\bigg|_{\rho=0} = \frac{d\psi}{d\rho}\bigg|_{\rho=0} = 0, \quad \cdots(84)$$

one can show that stable localized bound state solutions exist (for details, see for example, Ref. [15]).

7.4 Localized Structures in Vector NLS Equations

Recently it has been proposed that a variety of stable localized solutions, namely the so-called soliton molecules, can be realized in vector coupled nonlinear Schrödinger equations in the presence of a potential:

$$i\frac{\partial \psi_j}{\partial t} + \nabla^2 \psi_j + V(\vec{r}) \psi_j + \sum_{k=1}^{N} \gamma_{jk} |\psi_k|^2 \psi_j = 0, \quad j = 1, 2, \ldots, N \quad \cdots(85)$$

$$V(\vec{r}) = \frac{1}{2} \vec{r}^2.$$ 

It is not only the single component case which admits bound state solutions, but even the many component cases do admit stable localized solutions in both the spatial dimensional cases $d = 2$ and $d = 3$. An example of localized structure for the $N = 2$ case for $d = 2$ is shown in Fig. 10.

7.5 Inclusion of Higher Derivative Terms

Yet another way to realize stable localized entities is to introduce higher order derivatives into the equation of motion. For example, with the introduction of a fourth order dispersion effect and a combined cubic-quintic nonlinearity, the NLS equation becomes:

$$i\psi_t = \nabla^2 \psi + PV^4 \psi + \gamma |\psi|^2 \psi + \kappa |\psi^4|^2 \psi = 0 \quad \cdots(86)$$

Such fourth order dispersion term is considered to be important for electrostatic and electromagnetic waves, when two wave polarizations are taken into account. It has been shown that when $\kappa P > 0, \gamma < 0$, prevention of collapse can occur and stable bound states in 2 spatial dimensions can be obtained. Stable vortex structures have also been identified.

8 Spatio-temporal Patterns in the Ginzburg-Landau Equation

The Ginzburg-Landau (GL) equation of the form given by eq. (7) is a well studied nonlinear equation in the physics literature. It describes a diverse variety of phenomena ranging from wave propagation in nonlinear media including dissipation to second order phase transitions and also in superconductivity, superfluidity, Bose-Einstein condensation, liquid crystals.

Fig. 10 Localized solitons in two component NLS equation in the presence of potentials.

(a) $t=0$

(b) $t=50$

(c) $t=100$
and in field theory\textsuperscript{15}. The underlying dynamics in 1, 2 and 3 dimensions are of considerable importance for condensed matter physicists. From the point of view of the present study, it is a natural generalization of the cubic nonlinear Schrödinger equation discussed in the earlier sections with the inclusion of damping/dissipation and it is important to investigate the various spatiotemporal patterns which arise therein.

The Ginzburg-Landau eq. (7) can admit simple plane wave solutions of the form

\[ \psi(r, t) = A(k) \exp[i(k \cdot r - \omega(k)t + \delta)], \quad \text{(...(87)} \]

where the form of $A(k)$ and $\omega(k)$ can be obtained by substituting eq. (87) into eq. (7) and equating the real and imaginary parts. A linear stability analysis will then show the condition under which the Benjamin-Feir instability occurs\textsuperscript{18}.

In $(1+1)$ dimensions this instability can lead to localized coherent structures such as pulses, fronts, domain boundaries and homoclinics. Other possibilities are that different kinds of spatiotemporal chaos such as phase chaos, defect chaos, intermittency and so on can arise. A typical space-time plot showing spatiotemporal chaos is presented in Fig. 11.

In $(2+1)$ dimensions the GL equation admits a variety of coherent structures and spatiotemporal patterns. In particular the GL equation possesses localized patterns in 2-dimensions known as spiral waves. A typical isolated spiral solution is of the form

\[ \psi(r, t) = F(r) \exp[i(\omega \tau + n \theta + \varphi(r))], \quad \text{(...(88)} \]

where $r$ and $\theta$ are polar coordinates. The nonzero integer $n$ is the topological charge, $\omega$ is the rotational frequency of the spiral, $F(r) > 0$ is the amplitude and $\varphi(r)$ is the phase of the spiral. Also $Q = \frac{d\varphi}{dr}$ as $r \to \infty$ is the asymptotic wave number selected by the spiral and $\omega$ is the spiral frequency. One can show that spirals emit plane waves asymptotically and so they are considered as a source of solutions. A typical spiral solution in a domain with periodic boundary condition is shown in Fig. 12.

In addition, in the $(2+1)$ dimensional GL equation, the spirals become vortices when $\omega = 0$ and $\varphi = 0$. In this case $F(r)$ is a monotonic function and $F \sim ar$ for $r \to 0$ and $F'(r) \to 1 - \frac{1}{r^2}$ for $r \to \infty$. A more general isolated moving vortex solution with wave vector $\vec{k}$ is specified by

\[ \psi(\vec{r}, t) = \chi(\vec{r} - \vec{v}t) \exp(i \vec{k} \cdot \vec{r}), \quad \text{(...(89)} \]

where $|\chi|^2 \to 1 - k^2$, for $r \to \infty$ with an appropriate phase jump $\pm 2\pi$. For details ref. [15].

9 The Spin Equation and Nonlinear Schrödinger Equations

The dynamics of the continuous Heisenberg ferromagnetic spin system in $(1+1)$ dimensions described by the evolution equation for the spins,

\[ \vec{S}_r = \vec{S} \times \vec{S}_r, \quad \vec{S} = (S_x, S_y, S_z), \quad \vec{S}^2 = 1, \quad \text{...(90)} \]

has been shown to be equivalent to the cubic nonlinear Schrödinger eq. (1) in $(1+1)$ dimensions through the mapping

\[ \psi(x, t) = \kappa(x, t)e^{i\tau(x,t)dx}, \quad \text{...(91)} \]

where

\[ \kappa(x,t) = \frac{|\vec{S}|^2}{\vec{S} \cdot \vec{S}_x} \quad \text{and} \quad \tau(x,t) = \kappa^{-2}\vec{S} \cdot (\vec{S}_x \times \vec{S}_x). \quad \text{...(92)} \]

Fig. 11 Spatiotemporal chaos in $(1+1)$ dimensional Ginzburg-Landau equation\textsuperscript{15}

Fig. 12 A spiral wave in the $(2+1)$ dimensional Ginzburg-Landau equation\textsuperscript{16}
Nonlinear Schrödinger (NLS) Family of Equations: Spatiotemporal Patterns

by Lakshmanan. Correspondingly, the spin equation also becomes a completely integrable soliton system. The corresponding equation with Gilbert damping,

\[
\ddot{S} = \vec{S} \times \dddot{S} + \lambda \vec{S} \times (\vec{S} \times \dot{S}),
\]

\[
\dddot{S} = (S_x, S_y, S_z), \quad \ddot{S}^2 = 1
\]

(93)
can be mapped onto the generalized nonlinear Schrödinger equation

\[
i\psi + \psi'' + 2|\psi|^2\psi = i\lambda \left[ \psi'' - 2\psi \int_0^1 (\psi\psi' - \psi'\psi) \, dr \right]
\]

(94)

One can develop a soliton perturbation theory for this equation and analyse the nature of the perturbed soliton solutions.

In (2 + 1) dimensions, the spin equation with circular symmetry

\[
\ddot{S} = \vec{S} \times \left( \frac{1}{r} \vec{S} + \ddot{S}_r \right)
\]

(95)
can be mapped onto the generalized cubic nonlinear Schrödinger equation in radial coordinates in the form

\[
i\psi + \psi'' + \frac{1}{r} \psi' - \left( \frac{1}{r^2} - 2|\psi|^2 \right) \psi = 4 \int_0^1 \left| \psi' \right|^2 \, dr
\]

(96)

which is also an integrable soliton system. However, more general spin equations with Gilbert damping as in eq. (9) in (2 + 1) dimensions is not known to be integrable and under stereographic projection takes the form of eq. (10) which can be considered as a generalized nonlinear Schrödinger equation with damping. The study of spatiotemporal patterns underlying the above equation is of considerable physical relevance in the context of developing the field of magnetoelcronics or spintronics. For further details one may refer to refs. [31 and [32].

10 Conclusion

In this rather brief review we have tried to present a unified point of view of the fascinating world of nonlinear Schrödinger equations in (1 + 1) and (2 + 1) dimensions. Depending on the type of nonlinearity, the space dimension, presence of damping and other external forces, the NLS equation may admit uniform or steady states, plane wave solutions, periodic solutions, localized coherent structures including solitons, plane solitons, lump solitons, exponentially localized dromions, bound states and wave collapse, spirals and vortices as well as spatiotemporal chaos, to name a few. Our description is in no way either complete or exhaustive. We have only endeavored to present a flavour of the type of fascinating forms and structures these equations possess. Study of these equations is mathematically challenging and physically very relevant and one can expect to see much progress in the future in understanding these systems and their application in varied physical situations.

Acknowledgement

This work forms part of the Department of Science and Technology, Govt. of India sponsored research project.

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