PARTIAL AND COMPLETE SYNCHRONIZATION IN QUASIPERIODICALLY FORCED COUPLED MAPS

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(Received 03 July 2003; Accepted 28 January 2005)

We study different coupling schemes in a system of three coupled nonlinear maps in which the parameters are subject to quasiperiodic modulation. This gives rise to a phase diagram that has coexisting regimes of synchronized and desynchronized dynamics on torus and chaotic attractors, as well as on strange nonchaotic attractors (SNAs). We discuss the influence of coupling topology on the extent of partial synchronization; this suggests strategies for controlling the dynamics of assemblies of driven coupled map lattices.

Key Words: Coupled Map Lattice; Synchronization; Quasiperiodic Forcing; Strange Nonchaotic Attractor

1 Introduction

The circumstances under which coupled nonlinear systems display synchronization or desynchronization has been extensively studied in the last few years\textsuperscript{1}. The subject has a long history, dating from the observation by Huygens\textsuperscript{2} of synchronization between coupled periodic oscillators (pendula), to more recent work on the synchronization between coupled chaotic oscillators\textsuperscript{3}. Pecora and Carroll\textsuperscript{3} showed that identical (or nearly identical) nonlinear systems can be synchronized if coupled by a common drive signal even when the dynamics is chaotic, namely showing sensitivity to initial conditions. The motion of the coupled system, after transient unsynchronized dynamics, takes place on an attractor in the “synchronization manifold” which is an invariant symmetric subspace (ISS) of the phase space. The Lyapunov exponent in directions transverse to the synchronization manifold (termed transverse Lyapunov exponents\textsuperscript{3}) provide a measure of the average stability of synchronous motion. A number of studies of the dynamics of coupled chaotic systems have found different interesting cases of complete and partial\textsuperscript{6,7}, generalized\textsuperscript{8}, lag\textsuperscript{9}, and phase synchronization\textsuperscript{10,11}, along with riddled basins of attraction\textsuperscript{11}, attractor bubbling\textsuperscript{12}, and on-off intermittency\textsuperscript{13}.

Similarly, quasiperiodic modulation has also been examined in fair detail in the past few years for a variety of reasons, including the fact that with quasiperiodic driving, the dynamics can be both aperiodic and stable, on so-called strange nonchaotic attractors (SNAs)\textsuperscript{14}. These attractors have all Lyapunov exponents (LEs) nonpositive (and are hence nonchaotic) and typically are geometrically fractal (or strange).

In the present work we study a system of three coupled quasiperiodically driven nonlinear maps. Quasiperiodic forcing makes it possible to achieve synchronization\textsuperscript{15} under at least three separate scenarios since the synchronous attractors can be chaotic and strange, or nonchaotic and either strange or nonstrange. Previous studies\textsuperscript{16-18} have been restricted to two similarly coupled and driven maps and therefore here we explore the complexities introduced by different possible coupling schemes. This system of coupled logistic maps, which individually follow the period-doubling route to chaos\textsuperscript{19}, demonstrate the universal scenario of the development of periodic and chaotic coexisting regimes. Quasiperiodic forcing transforms periodic attractors to quasiperiodic ones, and often these are SNAs\textsuperscript{20}.

The enhanced stability that results from strange nonchaotic dynamics is the principal motivation for study of quasiperiodically forced coupled maps. The large number of possible parameters makes exploration of such systems difficult, and as a result we choose very specific (and restricted) parameters for coupling and forcing. It should also be pointed out that here we are interested in the detailed behaviour of small numbers of coupled maps, as opposed to studies that examine large coupled map lattices\textsuperscript{21}. The behaviour of two or three such coupled maps have been extensively studied\textsuperscript{22,23,26}. The effect of quasiperiodic forcing on
two coupled logistic map system has been studied in some detail\textsuperscript{16-18} where it has been established that due to forcing and coupling the synchronization region in parameter space shrinks. Coupled systems without external forcing have also been studied\textsuperscript{11,12} from the point of view of the so-called synchronization-desynchronization transition. In the case when particular unstable periodic orbits embedded in the synchronized chaotic attractor lose transverse stability, this gives rise to riddled basins\textsuperscript{27}.

In this paper we examine the synchronization-desynchronization phenomenon and the transitions within different regions in the phase diagram of the three coupled quasiperiodically forced maps system that support two or more coexisting attractors. The bifurcation through which partial synchronization arises and breaks down and the associated transformations of basin of attraction are followed. The coupling of the three systems may be symmetric or asymmetric, and we examine all possible schemes within the restriction of linear coupling. Since the systems have been studied previously in the absence of driving\textsuperscript{23-26} the contrasting behaviour that derives under the influence of forcing can be clearly identified. The single quasiperiodically forced logistic map has been studied extensively in the context of strange nonchaotic attractors\textsuperscript{14,28,29} , as has the system of two coupled logistic maps\textsuperscript{17,18}.

We find that the system has a region in parameter space where complete or partial synchronization takes place on torus or chaotic attractors. The transition from partial synchronization to different coexisting attractor dynamics, and to complete synchronization is studied. We examine the basins of attraction for the synchronous and partially synchronous attractors.

The coupled system we study and the various possible coupling schemes are described in the next Section. Since there are several parameters we focus attention on one for a fixed set of parameter values to explore the different possible dynamical regimes. There are a number of bifurcations, and these are discussed in relation to the Lyapunov exponents of the system in Section III. The structure of attractor basins and the phenomena of multistability are also described in Section III, and this is followed by a summary in Section IV.

2 System and Dynamics
Consider three quasiperiodically forced coupled identical logistic maps,
\[
x_{n+1} = f(x_n, \theta_n) + \beta_3 (y_n - x_n) + \beta_2 (z_n - x_n) \quad \cdots (1)
\]
\[
y_{n+1} = f(y_n, \theta_n) + \beta_1 (x_n - y_n) + \beta_2 (z_n - y_n) \quad \cdots (2)
\]
\[
z_{n+1} = f(z_n, \theta_n) + \beta_4 (x_n - z_n) + \beta_4 (y_n - z_n) \quad \cdots (3)
\]
\[
\theta_{n+1} = \theta_n + \omega \mod 1 \quad \cdots (4)
\]
where the function \( f \) is given by
\[
f(r, \theta) = \alpha (1 + \varepsilon \cos 2\pi \theta) r (1-r) \quad \cdots (5)
\]
The different control parameters are the nonlinearity \( \alpha \), the forcing strength \( \varepsilon \), and the coupling strengths \( \beta_j, j=1, \ldots, 4 \) The parameters of all three maps have been taken to be identical in this study for convenience so all nonzero \( \beta_j \)'s are set to the same value, and the variables of the three maps are denoted \( x, y \) and \( z \) for clarity. In this model, variables \( x \) and \( y \) are coupled with identical coupling strength \( \beta_3 \) and also coupled to \( z \) with identical coupling \( \beta_2 \) while the \( z \) variable is coupled with different coupling strength \( \beta_4 \) and \( \beta_4 \) with variables \( x \) and \( y \) respectively. We adopt the skew—product structure of earlier studies\textsuperscript{1} for quasiperiodically driven systems and the \( \theta \) dynamics, a rigid rotation by the irrational angle \( \omega = (\sqrt{5} - 1)/2 \), is unaffected by the variation of the other freedoms.

The phase space of the system is \( \mathbb{P} \times S^1 \), \( I \) being the unit interval. Since the dynamics of the \( \theta \) freedom is uniform, ergodic and as a consequence, trivial, it is convenient to consider the reduced phase space spanned by the variables \( x, y \) and \( z \).

The simplest form of asymptotic synchronous dynamics that can occur in the coupled map system, eq. (1) is the fully synchronized behaviour in which all three maps display the same temporal variation, \( x_n = y_n = z_n \), so that the motion is restricted to the main diagonal \( D \) in the three dimensional phase space, i.e. if
\[
|x_n - y_n|_{n \to \infty} \to 0 \quad \text{and} \quad |x_n - z_n|_{n \to \infty} \to 0 \quad \cdots (6)
\]
This property depends in general on initial conditions. The synchronization manifold is the plane \( \{ x = y = z, \theta \} \), which is an invariant symmetric subspace denoted ISS. A trajectory with initial condition in the ISS will clearly remain in the ISS since all coupling terms effectively vanish and all the maps are identically the single forced logistic map
\[
x_{n+1} = f(x_n, \theta_n) = \alpha (1 + \varepsilon \cos (2 \pi \theta_n)) x_n (1 - x_n) \quad \cdots (7)
\]

The Lyapunov exponent (LE) in the ISS is
\[
\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln |f'(x_i, \theta_i)| \quad \cdots (8)
\]
Regions in parameter space where the transverse Lyapunov exponent \( \lambda_\perp \) is negative offer the possibility
\[\text{For } \varepsilon \neq 0, \text{ it is clear that motion will remain bounded in this region so long as } \alpha (1 + \varepsilon \cos (2 \pi \theta)) [0, 4]. \text{ Thus for any } \alpha \leq 4, \text{ the largest value of } \varepsilon \text{ allowed is } 4/\alpha - 1, \text{ and the driving parameter is rescaled as } \varepsilon' = \varepsilon/(4/\alpha - 1) \text{ so that we study the system for } 0 \leq \varepsilon' \leq 1.\]
of synchronization, when the attractor in the ISS is stable (or at least weakly stable\textsuperscript{21}). Upon variation of system parameters, synchronization can be lost via a blowout bifurcation\textsuperscript{22} when the transverse LE becomes positive.

It is possible that of the three coupled systems only two synchronize: this is an example of partial synchronization\textsuperscript{23-25} (in general, when only a subset of the state variables in a dynamical system synchronize) and the motion in the subspace can be chaotic or nonchaotic. In order that in a three-dimensional system two variables undergo partial synchronization the following conditions need to be satisfied\textsuperscript{25}:

1. The corresponding plane is invariant under the action of the map.
2. The plane is stable under three-dimensional dynamics.
3. The diagonal of the plane is transversely repelling (otherwise, there would be total synchronization).

We study the case of partial synchronization restricted to the plane (x = y, z) denoted $\Pi_{xy}$. Since some of the coupling terms will vanish when

$$|x_n - y_n|_{\text{max}} \to 0$$

(and the difference between $x_n$ and $z_n$ does not go to zero), one effectively has the two-dimensional system

$$x_{n+1} = f(x_n, y_n) + \beta_2(x_n - x_0)$$
$$z_{n+1} = f(z_n, y_n) + (\beta_1 + \beta_2)(x_n - z_0)$$

The stability of spatial solution in the regime of partial synchronization will depend on the behaviour of infinitesimal perturbations transverse to the invariant submanifold $\Pi_{xy}$.

As indicated in condition (3) above, in order to avoid total synchronization, the Lyapunov exponent transverse to the diagonal $x = y$ in the plane $\Pi_{xy}$ must be positive. The stability of partial synchronization may be weak or strong, depending on whether the chaotic set embedded in the synchronization manifold $\Pi_{xy}$ is asymptotically stable (in the Lyapunov sense) or not. In the case of weak stability as defined by Milnor\textsuperscript{20}, the synchronized chaotic state is stable on the average (its transverse Lyapunov exponent is negative), but particular orbits embedded in this state are unstable in the direction perpendicular to $\Pi_{xy}$.

The coexistence of total and partial synchronization is also possible with one attractor in the ISS and a second attractor outside the diagonal but within the invariant plane $\Pi_{xy}$. Along with the possibility of unsynchronized dynamics, the different dynamical regions (depending upon the parameter values) are ($x = y$ denotes that for sufficiently large $n$, $x_n = y_n$, namely the $x$ and $y$ dynamics become identical asymptotically):

1. Complete synchronization ($x = y = z$) (CS) as discussed above.
2. Partial synchronization ($x = y$) with $z$ independent (PS).
3. Desynchronized dynamics ($x \neq y \neq z$) (DS).
4. Coexisting complete and partial synchronization (CP).
5. Coexisting complete synchronization with desynchronized dynamics (CD).
6. Coexisting partial synchronization with desynchronized dynamics (PD).
7. Coexisting complete and partial, and desynchronized dynamics (CPD).

By virtue of its higher dimensionality, the system of three coupled maps is necessarily more complex than the simple logistic equation or the two map system\textsuperscript{14,16-18}. Note that the temporal behaviour within the ISS or in the invariant plane can be quasi-periodic or chaotic depending on control parameter values, and the quasi-periodic attractors can be either simple (torus attractors) or strange.

3 Simulations and Results

In order to explore the phenomenon of partial synchronization, it is necessary to introduce particular forms of the coupling between different subsystems. It is necessary that synchronization of a part of the system does not automatically force the rest of the system to follow suit. Symmetry under permutation of different variables of the system decides whether there can be a linear invariant subspace in which partial synchronization can take place\textsuperscript{31-32}. By making the coupling asymmetric, it is possible to produce partial synchronization, and below we discuss some of the possible coupling schemes that will permit this.

(A) Coupling Schemes

Between any two nodes, say $x$ and $y$, there can either (a) be no coupling, or (b) be direct coupling through a term of the form $\beta_2(x_n - y_n)$. Depending on whether this is present in the equation for $x_{n+1}$ or in that for $y_{n+1}$, or both, we depict this by an arrow from the $x$ node to the $y$ node (or vice versa), or both. There can therefore be four types of links between a pair of sites: single headed arrows in the two possible orientations, a double headed arrow, or none. The combination of links between three nodes give rise to 60 different coupling schemes (out of the 64 possible
combinations, in 4 of the cases, not all elements are linked), and of these, only a subset are germane to our interest here, namely the phenomenon of partial synchronization in addition to complete synchronization.

We keep the nonlinearity parameters in all the maps identical in eqs. (1-4). By choosing the variables x and y for partial synchronization, in order that the system is symmetric with respect to the interchange of x and y, we can have only identical direct (β₁), or indirect (β₂) coupling. The coupling schemes depicted in Fig. 1 will allow for both partial and complete synchronization.

A) β₁,3,4 = 0 and β₂ = β. Here x and y are coupled indirectly via z.
B) β₁,4 = 0 and β₁,2 = β. This is similar to case A, and x and y have an additional direct connection.
C) β₁,4 = 0 and β₂,3 = β. Similar to case A, with a direct coupling between z and x.
D) β₁ = 0 and β₁,2,3 = β. Similar to case B, with a direct coupling between z and x.
E) β₁ = 0 and β₂,3,4 = β. Similar to case A, with a direct coupling between z and both x and y.
F) β₂ = 0 and β₁,3,4 = β. A direct coupling between x, y and z is coupled to both x and y.

The different coupling schemes give different bifurcation scenarios. Partial synchronization can take place in subspaces Πₓ, and Πₚ only in the subspace Πᵧ depending on the coupling topology. Even with the restrictions of identical nonlinearity, forcing and coupling parameters, there are a large number of possibilities, and in the present paper, we focus on the behaviours that obtain from these six different coupling schemes for a fixed value of the coupling strength β = 0.1, and ε' = 0.1.

The extent of partial synchronization is determined by Monte Carlo simulations, namely by evolving a number of random initial conditions and examining the fraction that becomes partially synchronized. This fraction, denoted fₓ, where a, b = x, y or z is plotted versus the nonlinearity parameter α at fixed β = 0.1 and ε' = 0.1 in Fig. 2 for the six different coupling schemes depicted in Fig. 1. The system is completely synchronized for fₓ = fₚ = 1, while partially synchronized in the plane Πₓ, for fₓ = 1, fₚ = 0.

Fig. 1 Coupling schemes which will allow for partial synchronization in the three coupled map system. A single headed arrow indicates a coupling term of the form (y - x) in the evolution equation for the x variable if the arrow points from x to y, and a double headed arrow indicates similar coupling for both variables.
In general we find that the region of partial synchronization increases with increasing the number of linkages between $z$ and $x$ or $y$, while direct coupling between $x$ and $y$ appears to suppress the regime of partial synchronization.

In the next subsection, we examine the coupling scheme $E$ in some detail. Since the coupling is asymmetric, it becomes possible to study different transitions between complete and partial synchronization, as well as the desynchronized state.

(B) Dynamics

Each of the different coupling schemes depicted in Fig. 1 differ in details, but result in a similar overall picture of the distinct dynamical regimes. Rather than describe each of these separately, we focus on one which has been examined in the absence of forcing\textsuperscript{23-26}, so that some comparison is possible. This is the coupling scheme $E$, namely where the $x$ and $y$ freedoms are indirectly coupled, both having a two-way coupling with the $z$ freedom.

The unforced system has a range of different coexisting torus attractors of periods $1, 2, 4, \ldots, 2^n$ and indeed of all periods. It is possible to have chaotic synchronization with sufficient large coupling\textsuperscript{23}. Quasiperiodic forcing suppresses the higher period behaviour, and as is well known, typically shifts the regimes of a given behaviour to lower values of the nonlinearity\textsuperscript{22}. Here too, we find that transitions to different coexisting attractor regions and complete synchronization are both shifted to lower values of control parameter with external forcing. Synchronized SNAs are also possible along with synchronized torus, quasiperiodic, and chaotic attractors.

Consider scheme $E$, $\beta = 0.0$ and $\beta_{z, 3, 4} = \beta$. Note that $D$ and $\Pi_y$ are invariant, unlike the planes $\Pi_x$ and $\Pi_{xy}$. A sample phase diagram of the different dynamical regimes is shown in Fig. 3 in the $\alpha, \epsilon$ parameter plane for two values of $\beta = 0.01$, and 0.1 respectively. For low values of $\beta$, one has mostly regions of complete, partial, desynchronized, coexisting attractors, as shown...
in Fig. 3(a). With increasing $\beta$, completely synchronized and desynchronized dynamics region decrease, while the region of partial synchronization increases. The regions of different coexisting attractors increases with the coupling strength, as seen in Fig. 3(b).

We discuss the motion in the $I_y$ plane for the case of coexisting complete, partial, and desynchronous dynamics at $\epsilon = 0.1$ and $\beta = 0.1$, corresponding to the phase diagram in Fig. 3(b). Since there can be coexisting attractors in the system, we compute the spectrum of LEs as follows. Starting with the fixed initial conditions for $z$ and $0$, a number of different initial $x$ and $y$ are sampled in the interval $[0,1]$. For any set of parameter values, there can either be one, two, or three (in general) unequal values of each LEs if there are one, two, or three coexisting attractors respectively. The largest two LEs for the system, eqs. (1-4) are shown in Fig. 4 as a function of the nonlinearity parameter $\alpha$. The system has both regular (torus) and chaotic dynamics in the different regions of complete, partial, desynchronous, and coexisting attractors. The partially synchronized dynamics has two attractors, one for $x = y$ and other for $x = z$. The different regions are labelled for convenience and correspond to different spatial and temporal dynamics of the system. Transitions to chaotic dynamics within the regions are marked by arrows. The largest transverse Lyapunov exponent is positive in region I and negative in all other regions in the parameter space we study here, showing that in region I, there is no complete synchronization, while in all other regions, the ISS has a measurable basin of attraction.

Region I: For $\alpha$ below 3.166, there is PS on a 2-band torus. The positive value of largest transverse Lyapunov exponent of the system in this region shows the instability of the ISS, and hence complete synchronization is not possible in this region.
Fig. 4 The two largest Lyapunov exponents \( \lambda_{1,2} \) as a function of the parameter \( \alpha \) for \( \beta = 0.1 \) and \( \epsilon = 0.1 \) corresponding to Fig. 3(b). The calculations are done with fixed initial \( z_0 = 0.5, \theta_0 = 0.25 \), and a large number of initial \( \xi_0, \eta_0 \), the distinct exponents realized for different initial conditions are shown.

Fig. 5 The three coexisting Complete, Partial, and Desynchronous attractors (CPD) (region III of Fig. 4) are shown for the parameter values, \( \alpha = 3.243, \beta = 0.1, \) and \( \epsilon = 0.1 \) for initial conditions \( z = 0.25, \theta = 0.5, \) and \( (x, y) = (0.131, 0.256), (0.653, 0.601), \) and \( (0.552, 0.741) \) respectively.
Region II: The bifurcation at $\alpha_s \approx 3.166$ results in the creation of an attractor in the ISS. We thus have the situation of coexisting PS and CS attractors. The nonsynchronized attractor in the PS region becomes chaotic at $\alpha \approx 3.194$, marked by arrow in Fig. 4. At the transition from PS to CS dynamics, there is a sudden increase in the basin size of the ISS attractor, similar to the expansion of attractors at interior crises\textsuperscript{34}; this described in more detail in Section III C below.

Region III: In this region there are three different coexisting dynamics: CS, PS and DS dynamics. The transition from CP to CPD is signaled, at $\alpha \approx 3.235$, by the appearance of a nonpositive Lyapunov exponent in addition to the existing positive and negative LEs. The three coexisting CS, PS, and DS attractors are shown in Fig. 5 for the parameter values, $\alpha = 3.243$, $\beta = 0.1$, and $\varepsilon = 0.1$ with different initial conditions. The system has 2-band torus attractor for CS and DS dynamics while 2-band chaotic attractors for PS dynamics. The basin corresponding to these three coexisting attractors is shown in Fig. 8(b).

Region IV: The PS regime disappears at $\alpha \approx 3.283$; the system has only CS and DS dynamics, and at the same time the nonsynchronized 2-band chaotic attractor becomes 1-band chaotic attractor. The second largest LE corresponding to the DS dynamics becomes positive at $\alpha \approx 3.337$.

Region V: Beyond $\alpha \approx 3.41$ the system is completely synchronized on a torus.

A detailed discussion of these transitions is not given here. The bifurcations are, by and large, well-known in a lower—dimensional context\textsuperscript{17}. The explosion of the attractor basin for complete synchronization, however, has a distinctive and unusual behaviour, and we turn to a discussion of this phenomenon next.

(C) Attractor Basins

One of the most interesting features of the driven coupled system when there are multiple coexisting attractors is the nature of the attractor basins and their morphologies. A basic feature of the present coupled system is the presence of multiple attractors; this is a feature shared by the undriven system\textsuperscript{23,25,34} which also displays multistability, namely there are two or more coexisting attractors for a fixed set of parameters.

Here we investigate mechanism of multistability and the corresponding attractor dynamics. There is typically one attractor in the ISS, and one or more attractors outside the ISS. We use a direct method—

starting with different initial conditions in the $(x, y, z)$ space and iterating for a sufficiently long duration (here taken to be $10^5$ iterations) -to determine basin volumes. The relative volume of a given attractor basin is estimated as the fraction of initial conditions that lead to it. This quantity is evaluated as a function of the control parameter $\alpha$. Shown in Fig. 6 are $f_{CS}$, IPS, and $f_{DS}$, namely the fraction of initial conditions attracted to CS, PS, and DS attractors respectively. The different coexisting dynamical regimes, namely CP, CPD, and CD and transitions within them are also indicated in Fig. 6. The system has a transition from PS region to coexisting CP dynamics at $\alpha \approx 3.166$ as $f_{CS}$ becomes nonzero.

At the transition to CP dynamics (Region I to II), the basin volume of ISS attractor has a power-law increase,

$$f_{CS} \sim (\alpha - \alpha_s)^\gamma$$

(12)

Such behaviour has been noted in the two-coupled map system\textsuperscript{18} and also appears to occur here at all transitions to the regime of coexisting attractors. From the data in Fig. 7, the measured scaling exponent is $\gamma \approx 0.325$, with $\alpha_s \approx 3.166$. A full understanding of this phenomenon is not available at present, though as mentioned, there are tantalizing similarities to interior crisis-like behaviour\textsuperscript{34}; we are currently exploring this avenue of investigation\textsuperscript{35}.

The basins of the different attractors are shown in Fig. 8. At (a) $\alpha = 3.228$, there is completely synchronized dynamics on 2-band torus with a coexisting partially synchronized dynamics on 2-band quasiperiodic attractor. As $\alpha$ increases, the volume of the attractor for partial synchronization dynamics decreases and at $\alpha = 3.235$ there is the appearance of a third asynchronous attractor, and $f_{DS}$ is nonzero. From the attractor basins shown in Fig. 8(b) it can be seen that the attractor with desynchronized dynamics is found in the region which was formerly the basin of PS attractor.

The system has three coexisting attractors in the range of parameter value (CPD region) before the PS region disappears in CD region at $\alpha \approx 3.283$, where the functions $f_{PS}$ approaches to zero. Before the transition to CD region, the basin of CS and PS attractors are locally intertwined as shown in Fig. 8(c) at $\alpha = 3.2824$.

The basin of coexisting CS and DS attractors in region IV is shown in Fig. 8(d-f) for $\alpha = 3.3, 3.36$, and 3.4 respectively. The basin of CS attractor decreases in this region with the control parameter $\alpha$ but there is
Fig. 6 The fraction \( f_{\text{CS}} \) as a function of \( \alpha \) for \( \beta = 0.1 \) and \( \varepsilon' = 0.1 \) for coupling scheme E. Different synchronization regions are labelled; see the text for discussion. The crisis-like increase in the ISS attractor-basin volume can be gauged by the fraction \( f_{\text{CS}} \) of initial conditions which lead to a completely synchronized attractor which is coexisting with a partially synchronized attractor.

Fig. 7 The scaling of \( f_{\text{CS}} \) versus \( (\alpha - \alpha_c) \) with \( \alpha_c = 3.165... \) near the transition from PS to CP dynamics as shown in Fig. 6.
a sudden increase at $\alpha = 3.41$, and system becomes completely synchronized. The basin has a riddled—like structure near the transition as can be seen in Fig. 8(f), where the system approaches complete synchronization region from one of coexisting complete synchronization and desynchronized dynamics.

For the other coupling topologies, similar attractor basin morphologies are observed. The main differences that we find are in the extent (in parameter space) of the different dynamical regimes, and as a consequence, we have chosen to present one representative case.

4 Discussion and Summary

We have studied a system of three coupled identical nonlinear maps with quasiperiodic forcing. The coupling scheme is chosen to allow both partial and complete synchronization. The system has a different linear invariant subspace depending on the permutation symmetry between any two variables, which provide a possibility of partial synchronization in that subspace.

By changing the coupling scheme, the regions of complete and partial synchronization, and their coexistence differ in extent, and also in the manner in which they form and/or disappear. The transitions from partial synchronization (PS) to coexisting partial (PS) and complete synchronization (CS), and to three coexisting regions of partially synchronized (PS), completely synchronized (CS), and desynchronized dynamics (DS), and further to coexisting regions of complete synchronization (CS) and desynchronized dynamics (DS) and finally to complete synchronization (CS), as well as the basins of different coexisting attractors have been studied in detail for one coupling scheme.

We have observed two types of transitions within the different regimes of multistability, namely the case of coexisting attractors. The transition from partial
synchronization to a regime where both the partial and complete synchronization coexist is accompanied by a power-law increase in the attractor basin volume. In another set of transitions, attractors appear or disappear discontinuously, as for example the transition from coexisting complete and partially synchronized to coexisting complete, partial, and desynchronized dynamics (CP → CPD), from coexisting complete, partial, and desynchronized dynamics to coexisting complete synchronized and desynchronized dynamics (CPD → CD), and from coexisting complete synchronized and desynchronized dynamics to complete synchronization (CD → CS). The basin structure develops an interesting "riddled-like" appearance near the discontinuous transitions. The dynamics is not truly riddled as in chaotic systems since the effect of quasiperiodic driving is to stabilize the motion.

The present results offer some clues as to how larger systems of coupled driven maps may behave. By analyzing larger systems as sets of triples, one can infer and/or design coupling schemes which achieve desired forms of partial or complete synchronization. Recently coupled maps on different networks and with variable coupling strength have been studied for formation of various types of network structures. The network structure can be either static or dynamics and have different phase ordering, depending on the synchronized or desynchronized motion of unit dynamics. As has become apparent from such studies of networks, depending on the nature of the coupling, clustering phenomena and a variety of other network structures can emerge. Similarly, we believe that in systems with larger numbers of coupled maps, the stability of different hyperplanes determines the onset of partial or clustering where the coupled system splits into clusters of identical oscillating elements.

Acknowledgements

This research was supported by the Department of Science and Technology, India. The authors thank Prof U Feudel and Dr Awadhesh Prasad for discussions, and for collaboration on related work.

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