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Soret and Hall Current Effects on Heat and Mass Transfer in MHD Flow of a Viscous Fluid through Porous Medium with Variable Suction

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An unsteady free convective flow through porous media of a viscous, incompressible, electrically conducting fluid past an infinite porous, non-conducting, vertical plate is analyzed. A strong uniform magnetic field is applied perpendicular to the plane of the plate which is subjected to a variable suction velocity. It is assumed that the free stream, plate temperature, species concentration oscillates about a non-zero mean and the concentration level of the diffusing species is small compared with other concentrations. Taking Hall current into account, equations of motion, energy and concentration are solved by perturbation technique. The effects of the various parameters, entering into the problem, on the primary and secondary velocity are shown graphically while the skin friction coefficient, Nusselt number and Sherwood number obtained in terms of amplitudes and phase angles are listed in tabular form and discussed.

Key Words: Hall Current; Mass Transfer; Heat Source; Soret Number; Porous Medium

Introduction

MHD free convection flow is encountered in cooling of nuclear reactors, cosmic fluid dynamics, solar physics, motion of earth's core, aeronautics, chemical engineering, meteorology, planetary magnetospheres, etc. In practice cooling of porous structure is achieved by forcing the liquid or gas through capillaries of solid. Porous media are considered useful in diminishing the natural free convection which would otherwise occur intensely on a vertical heated surface. Along with the free convection currents caused by the temperature difference, the flow is also affected by the difference in concentration on material constitution. Gebhart and Pera [1-4] made extensive studies on combined heat and mass transfer flow. The effect of suction parameter on free convection and mass transfer MHD flow being an effective method of controlling the boundary layer has been studied by several authors [5-8]. Acharya *et al*. [9] have studied magnetic field effects on the free convection and mass transfer flow through porous medium with constant suction and heat flux. Singh and Singh [10] have extended the above problem by taking variable suction at the plate. Eckert and Drake [11] have pointed out that in a convective fluid when the flow of mass is caused by a temperature difference one cannot neglect the diffusion effect (Soret effect) due to its importance in engineering science. Jha and Singh [12] and Jha [13] presented study of a free convection and mass transfer flow past an infinite vertical plate moving impulsively in its own plane taking into account Soret effect. Kafoussias [14] extended this work to MHD free convective flows. Sattar and Alam [15] have studied Soret effects on MHD free convection and mass transfer flow past an impulsively started vertical porous plate in a rotating fluid without taking into account the Hall current effect.

When the strength of the magnetic field is strong one cannot neglect the effects of Hall currents. It is of considerable importance and interest to study how the results of the hydrodynamical problems get modified by the effects of Hall current. Gupta [16] have discussed the effects of Hall currents on the oscillatory MHD flow past a flat plate. Singh [6] have analyzed the effects of Hall currents on the free convection flow past an infinite vertical porous plate. Hence our aim is to study Soret effects as well as heat source effects on MHD free convection and mass transfer flow of a viscous fluid past an infinite porous non-conducting vertical plate with variable suction through porous medium taking Hall currents into account.

Formulation of the Problem

We consider the flow of a viscous incompressible electrically conducting fluid through porous medium bounded by a vertical porous plate. The x^*-axis is assumed to be oriented vertically upward along the plate and the y^*-axis normal to the plane of the plate. The porous plate is subjected to a variable suction. A strong uniform magnetic field H_0 is taken to be acting along the y^*-axis and the induced magnetic field is assumed

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to be negligible so that $\vec{H} = (0, H_0, 0)$. The plate temperature and concentration oscillate about a constant mean and the species concentration is assumed at low level. The plate temperature and concentration are instantaneously raised or lowered to $T_w^*(> T_\infty^*)$ and $C_w^*(> C_\infty^*)$, where T_∞^* and C_∞^* are temperature and concentration of the ambient flow. The permeability K_p^* of the porous medium is considered to be constant and the variable suction velocity is assumed to be of the form $v^*(t^*) = -v_0^*(1 + \in Ae^{in^*t^*})$, where $v_0^* > 0, \in << 1$, n^* is the frequency of oscillations, t^* is the time and A is a positive constant. The schematic presentation of the physical problem is shown in Fig. 1.

The equation of conservation of electric charge is $\nabla . \vec{J} = 0$ which gives $J_y^* =$ constant, where $\vec{J} = (J_x^*, J_y^*, J_z^*)$. Since the plate is electrically non-conducting, $J_y^* = 0$ and is zero everywhere in the flow. When the magnetic field is large, the generalized Ohm's law, in absence of electric field, neglecting the ion slips and thermo electric effect (Meyer [17]) yields:

$$J_x^* - \omega_e \tau_e J_z^* = -\sigma \mu_e H_0 w^* \quad \dots (1)$$

$$J_z^* + \omega_e \tau_e J_x^* = \sigma \mu_e H_0 u^* \quad \dots (2)$$

The solutions of equation (1) and (2) are

$$J_x^* = \frac{\sigma \mu_e H_0}{1 + m^2}(m u^* - w^*) \quad \dots (3)$$

$$J_z^* = \frac{\sigma \mu_e H_0}{1 + m^2}(u^* + m w^*) \quad \dots (4)$$

where $m(= \omega_e \tau_e)$ is Hall parameter, ω_e: the cyclotron frequency, τ_e: the electron collision time, μ_e: the magnetic permeability, σ: the electrical conductivity of the fluid. Within the frame work of such assumptions, neglecting the Joule heating and viscous dissipation terms and taking the usual Boussinesq's approximation into account, the governing equations for momentum, energy and concentration in non-dimensional form are:

$$\frac{1}{4}\frac{\partial u}{\partial t} - (1 + \in Ae^{int})\frac{\partial u}{\partial y} = \frac{1}{4}\frac{\partial U}{\partial t} + \frac{\partial^2 u}{\partial y^2} - \frac{M}{1 + m^2}(u - U + mw) + Gr\,T + Gm\,\theta - \frac{1}{K_p}(u - U), \quad \dots (5)$$

$$\frac{1}{4}\frac{\partial w}{\partial t} - (1 + \in Ae^{int})\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} + \frac{M}{1 + m^2}[m(u - U) - w] - \frac{1}{K_P} w, \quad \dots (6)$$

$$\frac{1}{4}\frac{\partial \theta}{\partial t} - (1 + \in Ae^{int})\frac{\partial \theta}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2} + S\theta, \quad \dots (7)$$

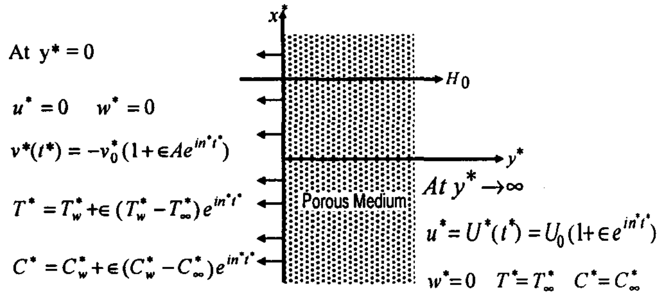

Fig. 1: Schematic presentation of the physical problem

$$\frac{1}{4}\frac{\partial C}{\partial t} - (1 + \in Ae^{int})\frac{\partial C}{\partial y} = \frac{1}{Sc}\frac{\partial^2 C}{\partial y^2} + S_0\frac{\partial^2 \theta}{\partial y^2} \quad \dots (8)$$

The non-dimensional boundary conditions are:

$$\left.\begin{array}{llll} u = w = 0, & \theta = C = 1 + \in e^{int} & at & y = 0 \\ u = 1 + \in e^{int}, & w \to 0, \quad \theta = C = 0 & as & y \to \infty \end{array}\right\} \quad \dots (9)$$

The non-dimensional quantities introduced in the above equations are defined as:

$$u = \frac{u^*}{U_0}, \quad w = \frac{w^*}{U_0}, \quad y = \frac{y^* v_0^*}{\upsilon}, \quad t = \frac{t^* v_0^{*2}}{4\upsilon},$$

$$U = \frac{U^*}{U_0}, \quad n = \frac{4\upsilon n^*}{v_0^{*2}}, \quad K_P = \frac{v_0^{*2} K_p^*}{\upsilon^2}$$

$\theta = \dfrac{T^* - T_\infty^*}{T_w^* - T_\infty^*}$ (dimensionless temperature),

$C = \dfrac{C^* - C_\infty^*}{C_w^* - C_\infty^*}$ (dimensionless concentration),

$Gm = \dfrac{g\beta^* \upsilon (C_w^* - C_\infty^*)}{U_0 v_0^{*2}}$ (modified Grashoff number),

$Gr = \dfrac{g\beta \upsilon (T_w^* - T_\infty^*)}{U_0 v_0^{*2}}$ (Grashoff number),

$M = \dfrac{\mu_e^2 H_0^2 \sigma \upsilon}{\rho v_0^{*2}}$ (Hartmann number),

$Sc = \dfrac{\upsilon}{D}$ (Schmidt number), $S = \dfrac{Q^* \upsilon}{v_0^{*2}}$ (heat source parameter), $S_0 = \dfrac{D_1 (T_w^* - T_\infty^*)}{\upsilon (C_w^* - C_\infty^*)}$ (Soret number) and

$Pr = \dfrac{\mu C_p}{k}$ (Prandtl number),

where β: volumetric coefficient of thermal expansion, β^*: volumetric coefficient of thermal expansion with concentration, $U^*(t^*)$: free stream oscillatory velocity,

U_0: mean free stream velocity, C_p: specific heat at constant pressure, k: thermal conductivity, K_p: constant permeability of the porous medium, D: chemical molecular diffusivity, D_1: thermal diffusivity, Q^*: heat source parameter and all the other quantities have their usual meanings. The " $*$" represents the dimensional quantities.

Introducing $q = u + iw$, the equations (5) and (6) transform to

$$\frac{1}{4}\frac{\partial q}{\partial t} - (1 + \in A e^{int})\frac{\partial q}{\partial y} =$$
$$\frac{1}{4}\frac{\partial U}{\partial t} + \frac{\partial^2 q}{\partial y^2} - \frac{M(1-im)}{(1+m^2)}(q-U)$$
$$+ Gr\,\theta + Gm\,C - \frac{1}{K_p}(q-U). \qquad \dots (10)$$

The transformed boundary conditions are

$$\left.\begin{array}{lll} q = 0, & \theta = C = 1 + \in e^{int} \ at & y = 0 \\ q = 1 + \in e^{int}, & \theta = C = 0 \quad as & y \to \infty \end{array}\right\} \dots (11)$$

Solution of the Problem

Since the perturbation parameter $\in$ (<<1) is a positive quantity, thus, we assume the solution of the system of equations (7), (8) and (10) of the following form:

$$\left.\begin{array}{l} q(y,t) = (1-q_0) + \in (1-q_1)\, e^{int} + O(\in^2) \\ \theta(y,t) = \theta_0(y) + \in \theta_1(y)\, e^{int} + O(\in^2) \\ C(y,t) = C_0(y) + \in C_1(y)\, e^{int} + O(\in^2) \end{array}\right\}. \qquad \dots (12)$$

Substituting (12) in (7), (8), (10) and equating harmonic and non-harmonic terms neglecting those containing $\in^2, \in^3 \dots$ etc., we get six equations. The solutions of these equations under the boundary conditions (11) give $q_0, q_1, \theta_0, \theta_1, C_0$ and C_1. Using these in (12), we get

$$q(y,t) = (M_r - i M_i) + \in (L_r - i L_i)\, e^{int}, \dots (13)$$

$$\theta(y,t) = e^{-c_0 y} + \in [S_r - i S_i]\, e^{int}, \qquad \dots (14)$$

$$C(y,t) = (1 + d_1 S_0 Sc)\, e^{-Sc\,y} -$$
$$d_1 S_0 Sc\, e^{-c_0 y} + \in (N_r - i N_i)\, e^{int}, \qquad \dots (15)$$

where

$$M_r = 1 - D_3 - a_4(Gr - Gm\, d_1 S_0\, Sc)\, e^{-c_0 y} - a_6 Gm\, e^{-Sc\,y},$$
$$M_i = F_3 - b_4(Gr - Gm\, d_1 S_0 Sc)\, e^{-c_0 y} - b_6 Gm\, e^{-Sc\,y},$$

$$L_r = 1 - D_4 - (Gr\, H_4 - Gm\, S_0 Sc\, H_8)\, X_1 - (Gr\, H_5 - Gm\, S_0 Sc\, H_9)\, Y_1$$
$$- Gm(H_6 X_2 + H_7 Y_2) - A(H_{10} X_3 + H_{11} Y_3)$$
$$- \left[Gr(E_3 + A c_0\, E_9) - Gm\, S_0 Sc\, c_0 (A E_{11} + E_5)\right] e^{-c_0 y}$$
$$- A Sc\, Gm \left[\frac{4}{n}(1 + d_1 S_0 Sc)\, E_7 + E_{13}\right] e^{-Sc\,y},$$

$$L_i = F_4 + (Gr\, H_5 - Gm\, S_0 Sc\, H_9)\, X_1$$
$$- (Gr\, H_4 - Gm\, S_0 Sc\, H_9)\, Y_1$$
$$+ Gm(H_7 X_2 - H_6 Y_2) + A(H_{11} X_3 - H_{10} Y_3)$$
$$- \left[Gr(E_4 + A c_0 E_{10}) - Gm\, S_0 Sc\, c_0 (A E_{12} - E_6)\right] e^{-c_0 y}$$
$$+ A Sc\, Gm \left[\frac{4}{n}(1 + d_1 S_0 Sc)\, E_8 - E_{14}\right] e^{-Sc\,y},$$

$$S_r = (1 - a_2)\, X_1 + b_2 Y_1 + a_2 e^{-c_0 y},$$
$$S_i = (1 - a_2)\, Y_1 - b_2 X_1 + b_2 e^{-c_0 y},$$
$$N_r = D_2 - S_0 Sc(e_3 X_1 - e_4 Y_1) - S_0 Sc\, C_0 e_5 e^{-c_0 y}$$
$$N_i = S_0 Sc(e_4 X_1 - e_3 Y_1) + S_0 Sc\, c_0 e_6 e^{-c_0 y}$$
$$- \frac{4 A Sc}{n}(1 + d_1 S_0 Sc)\, e^{-Sc\,y} - F_2.$$

The shear stress, Nusselt number and Sherwood number can now be obtained easily from equations (13), (14) and (15).

Skin friction coefficient (τ) at the plate in terms of amplitude and phase is:

$$\tau = \left(\frac{\partial q}{\partial y}\right)_{y=0} = \tau_0 + \in |D| \cos(nt + \alpha). \qquad \dots (16)$$

Heat transfer coefficient (Nu) at the plate in terms of amplitude and phase is:

$$Nu = \left(\frac{\partial \theta}{\partial y}\right)_{y=0} = c_0 + \in |R| \cos(nt + \beta). \qquad \dots (17)$$

Mass transfer coefficient (Sh) at the plate in terms of amplitude and phase is:

$$Sh = \left(\frac{\partial C}{\partial y}\right)_{y=0} = Sc[1 + d_1 S_0 (1 - c_0)] +$$
$$+ \in |G| \cos(nt + \gamma) \qquad \dots (18)$$

The various constants appearing in the text are listed in the appendix.

It is pertinent to point out that for A=0 and $K_p \to \infty$, the problem reduces to the one obtained by Singh [18] for the case of ordinary medium with constant suction velocity at the plate. By this limiting case the validity of the results of the present problem are tested.

Results and Discussion

In order to understand the problem physically, numerical calculations are carried out for the primary velocity field (u) and secondary velocity field (w). To be realistic, the values of the Prandtl number (Pr) are chosen to be 0.71 and 7.00 which correspond to air and water respectively. The values of Schmidt number (Sc) are chosen to represent hydrogen (Sc=0.22), air (Sc=0.60) and oxygen (Sc=0.75) respectively. The values of Grashoff number (Gr) and modified Grashoff number (Gm) are taken for the cooling of the plate. The values of Soret number (S_0), Magnetic parameter (M), Hall parameter (m), heat source parameter (S) and frequency parameter (n) are chosen arbitrarily.

The variations of primary velocity (u) and secondary velocity field (w) are shown in Figs. 2 and 3 respectively. A deep study of these figures reveal that an increase in M and Sc give rise to decrease in primary and secondary velocities, however, the primary and secondary velocities increase with the increase of S, S_0, m and K_p.

Table 1 presents the numerical values of skin-friction Coefficient (τ) in terms of amplitude $|D|$, phase angle $tan\alpha$, the steady part of skin-friction (τ_0) and the non-steady part (τ) of skin-friction for $\in=0.1$, ${}_{\pi}A=0.5$, n=10, Pr=0.71 and $nt=\frac{\pi}{2}$ to show the effects of various parameters. It is observed that an increase in m, S_0, Gr, Gm and K_p lead to an increase in the value of $|D|$, while an increase in M, S and Sc lead to a decrease in $|D|$. The value of $tan\alpha$ decreases due to the increase in m, S_0, Gr, Gm and K_p while increases due to increase in M, S and Sc. The skin-friction coefficients τ_0 for steady and τ for unsteady parts decrease due to increase of M and S but increase due to the increase in m, S_0, Gr, Gm and K_p. It is

Table 1: Values of skin-friction τ_0 and τ, amplitude |D| and phase $tan\alpha$ at $\in$=0.1, A = 0.5, n = 10, Pr = 0.71, nt = π/2

m	M	S	Gr	Gm	S_0	Sc	K_P	$\|D\|$	$tan\alpha$	τ_0	τ
0.5	5	0.1	5	4	1	0.60	0.5	6.7301	-0.3779	11.162	11.673
0.5	5	0.1	5	4	1	0.60	1.0	7.1223	-0.5225	11.616	11.946
1.0	5	0.1	5	4	1	0.60	0.5	8.1535	-0.8324	12.116	12.638
0.5	10	0.1	5	4	1	0.60	0.5	6.3106	-0.1915	11.083	11.202
0.5	5	0.15	5	4	1	0.60	0.5	6.3684	-0.1815	10.639	10.753
0.5	5	0.1	10	4	1	0.60	0.5	8.3054	-0.3798	20.736	21.031
0.5	5	0.1	-5	4	1	0.60	0.5	3.5802	-0.3694	0.1259	0.2499
0.5	5	0.1	5	8	1	0.60	0.5	8.8726	-0.5366	13.955	14.374
0.5	5	0.1	5	4	2	0.60	0.5	7.5992	-0.5724	11.638	12.016
0.5	5	0.1	5	4	1	0.75	0.5	6.3054	-0.1846	11.409	11.523

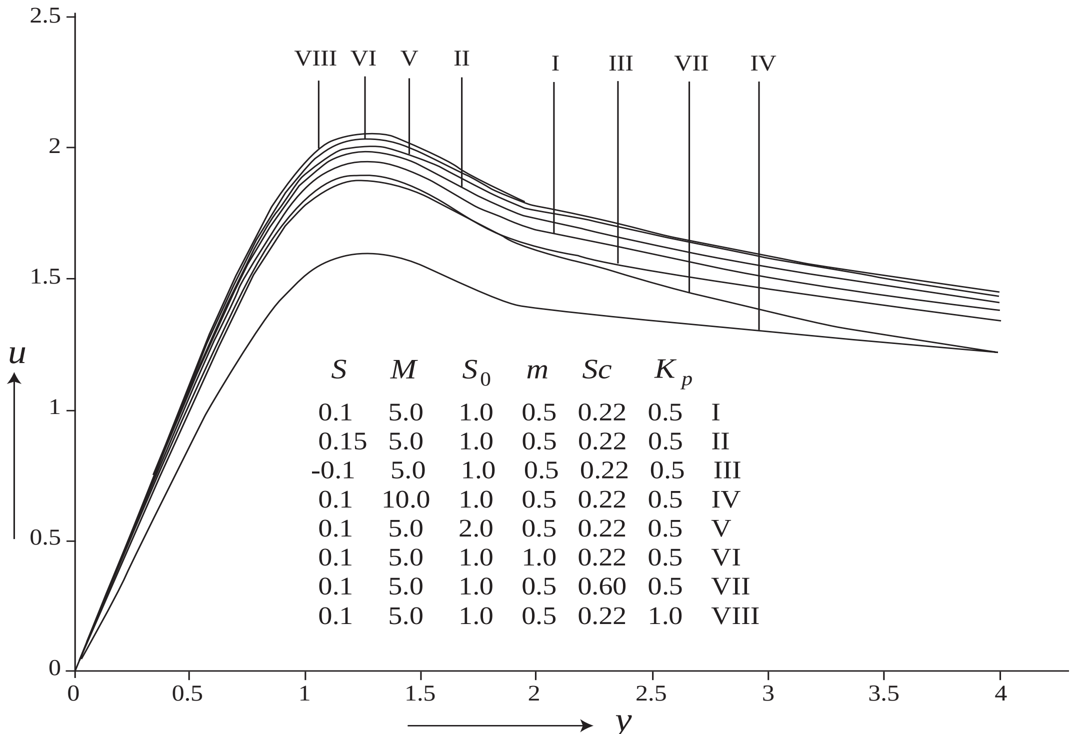

Fig. 2: Primary velocity profile
(Pr = 0.71, Gr = 5.0, Gm = 4.0, n = 10, A = 0.5, $\in$=0.1, nt=π/2)

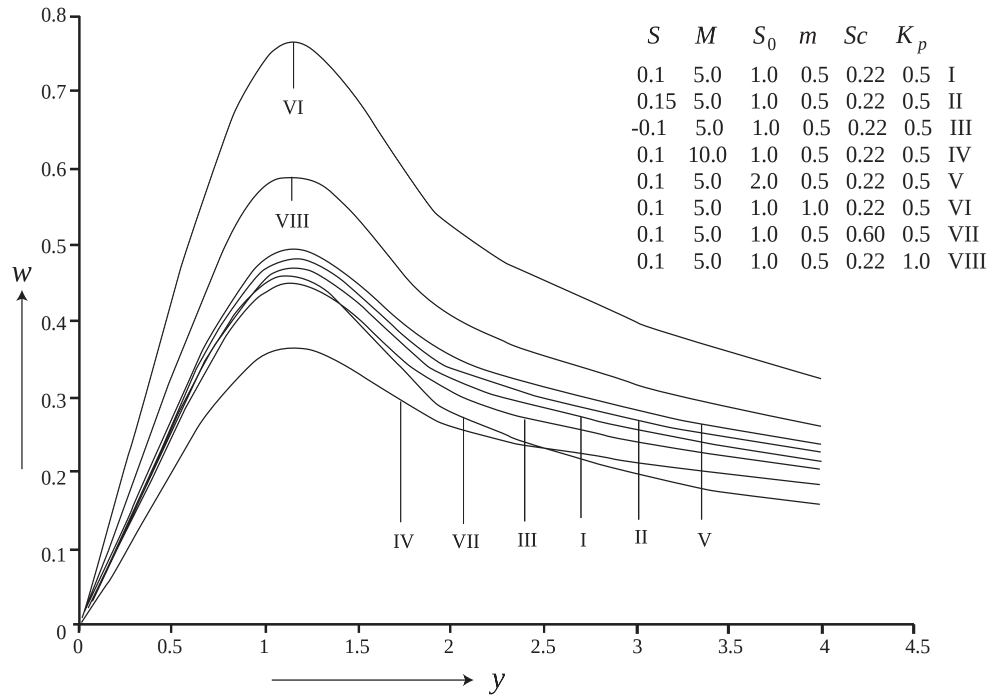

Fig. 3: Secondary velocity profile
(*Pr* = 0.71, *Gr* = 5.0, *Gm* = 4.0, *n* = 10, *A* = 0.5, ∈=0.1, *nt*=π/2)

Table 2: Amplitude (R), phase tanβ and rate of heat transfer Nu at ∈=0.1, A = 0.5, nt = π/2

S	*Pr*	*n*	|*R*|	*tanβ*	*Nu*
0.10	0.71	5	1.3104	0.4515	0.5357
0.15	0.71	5	1.2842	0.4776	0.4394
0.10	7.00	5	10.473	0.0606	6.8352
0.10	0.71	10	1.6521	0.5929	0.5053
0.10	0.71	15	1.9329	0.6638	0.4827

interesting to note that τ_0 increases and τ decreases with the increase of *Sc*.

The Nusselt number *Nu* along with its amplitude |*R*| and phase angle *tanβ* are shown in Table 2 for ∈=0.1, *A*=0.5 and $nt=\frac{\pi}{2}$. It is noticed from this Table that with the increase in the strength of the heat source, the phase angle *tanβ* increases. However the strength of the heat source have opposite effect on the amplitude |*R*| and the rate of heat transfer *Nu* i.e. amplitude |*R*| and the rate of heat transfer *Nu* decrease with the increasing heat source strength. The values in the Table also clearly show that |*R*|, *tanβ*, increase and *Nu* decreases with the increase of the frequency of oscillations (*n*). The amplitude |*R*| and rate of heat transfer *Nu* are much higher but the phase angle *tanβ* is very much less in the case of water (*Pr*=7.00) than that in the case of air (*Pr*=0.71).

The numerical values of the Sherwood number *Sh*, its amplitude |*G*| and the phase angle *tanγ* are listed in Table 3 for various values S_0, *Sc* and *S* at ∈=0.1, *A*=0.5, *n*=5, *Pr*=0.71 and $nt=\frac{\pi}{2}$. It is observed that |*G*| and *tanγ*

Table 3: Amplitude |G| , phase tanγ and rate of mass transfer Sh at ∈=0.1, A = 0.5, n=5, nt = π/2, Pr=0.71

S_0	*Sc*	*S*	|*G*|	*tanγ*	*Sh*
0	0.60	0.10	1.1949	2.2092	0.4911
1	0.60	0.10	3.1626	8.2525	-13.639
2	0.60	0.10	5.1976	19.359	-27.770
0	0.75	0.10	1.4057	2.4754	0.6197
1	0.75	0.10	3.7131	9.1783	-0.7507
2	0.75	0.10	6.085	21.891	-2.1204
0	0.60	0.15	0.4907	1.4866	0.3796

increase with the increase of *Sc* and of S_0. However the Sherwood number *Sh* increases with *Sc* but decreases with S_0. It is interesting to note that |*G*| , *tanγ* and *Sh* all decrease with the increase of the heat source strength *S*.

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Appendix

$$\alpha_1 = \frac{Pr}{2} + \frac{1}{2\sqrt{2}}\left[\sqrt{A_0^2 + B_0^2} + A_0\right]^{1/2}, \quad \beta_1 = \frac{1}{2\sqrt{2}}\left[\sqrt{A_0^2 + B_0^2} - A_0\right]^{1/2},$$

$$\alpha_2 = \frac{Sc}{2} + \frac{1}{2}\sqrt{\frac{Sc}{2}}\left[\sqrt{Sc^2 + n^2} + Sc\right]^{1/2}, \quad \beta_2 = \frac{1}{2}\sqrt{\frac{Sc}{2}}\left[\sqrt{Sc^2 + n^2} - Sc\right]^{1/2},$$

$$\alpha_3 = \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[\sqrt{a_0^2 + 16b_3^2} + a_0\right]^{1/2}, \quad \beta_3 = \frac{1}{2\sqrt{2}}\left[\sqrt{a_0^2 + 16b_3^2} - a_0\right]^{1/2},$$

$$\alpha_4 = \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[\sqrt{a_0^2 + d_0^2} + a_0\right]^{1/2}, \quad \beta_4 = \frac{1}{2\sqrt{2}}\left[\sqrt{a_0^2 + d_0^2} - a_0\right]^{1/2},$$

$$a_0 = 1 + \frac{4M}{1+m^2} + \frac{4}{K_p}, \quad b_0 = \frac{M}{1+m^2}, \quad d_0 = n - 4b_3, \quad A_0 = Pr(Pr - 4S),$$

$$B_0 = -n\,Pr, \quad c_0 = \frac{Pr + \sqrt{A_0}}{2}, \quad a_1 = c_0^2 - Pr\,c_0 + Pr\,S, \quad b_1 = c_0^2 - Sc\,c_0, \quad d_1 = \frac{c_0}{c_0 - Sc},$$

$$a_2 = \frac{Pr\,A\,c_0\,a_1}{a_1^2 + B_0^2/16}, \quad b_2 = \frac{Pr\,A\,c_0\,B_0/4}{a_1^2 + B_0^2/16}, \quad a_3 = c_0^2 - c_0 - b_0 - \frac{1}{K_p}, \quad b_3 = mb_0, \quad a_4 = \frac{a_3}{a_3^2 + b_3^2},$$

$$b_4 = \frac{b_3}{a_3^2 + b_3^2}, \quad a_5 = d_1 a_4, \quad b_5 = d_1 b_4, \quad a_6 = \frac{(1 + d_1 S_0 Sc)E_1}{E_1^2 + b_3^2},$$

$$b_6 = \frac{(1 + d_1 S_0 Sc)b_3}{E_1^2 + b_3^2}, \quad E_1 = Sc^2 - Sc - b_0 - \frac{1}{K_p}, \quad E_2 = b_3 - \frac{n}{4}, \quad E_3 = \frac{a_3 a_2 - b_2 E_2}{a_3^2 + E_2^2}, \quad E_4 = \frac{a_2 E_2 + a_3 b_2}{a_3^2 + E_2^2},$$

$$E_5 = \frac{a_3 e_5 + E_2 e_6}{a_3^2 + E_2^2}, \quad E_6 = \frac{a_3 e_6 - E_2 e_5}{a_3^2 + E_2^2}, \quad E_7 = \frac{E_2}{E_1^2 + E_2^2}, \quad E_8 = \frac{E_1}{E_1^2 + E_2^2}, \quad E_9 = \frac{a_3 a_4 - b_4 E_2}{a_3^2 + E_2^2},$$

$$E_{10} = \frac{a_4 E_2 + a_3 b_4}{a_3^2 + E_2^2}, \quad E_{11} = \frac{a_5 a_3 - b_5 E_2}{a_3^2 + E_2^2}, \quad E_{12} = \frac{a_5 E_2 + a_3 b_5}{a_3^2 + E_2^2}, \quad E_{13} = \frac{a_6 E_1 - b_6 E_2}{E_1^2 + E_2^2}, \quad E_{14} = \frac{a_6 E_2 + b_6 E_1}{E_1^2 + E_2^2},$$

$$H_4 = \frac{H_1(1 - a_2) + K_1 b_2}{H_1^2 + K_1^2}, \quad H_5 = \frac{H_1 b_2 - K_1(1 - a_2)}{H_1^2 + K_1^2},$$

$$H_6 = \frac{A_2 H_2 + B_2 K_2}{H_2^2 + K_2^2}, \quad H_7 = \frac{H_2 B_2 + K_2 A_2}{H_2^2 + K_2^2}, \quad H_8 = \frac{H_1 e_3 + K_1 e_4}{H_1^2 + K_1^2}, \quad H_9 = \frac{H_1 e_4 - K_1 e_3}{H_1^2 + K_1^2},$$

$$H_{10} = \frac{H_3(A_3\alpha_3 - B_3\beta_3) + K_3(A_3\beta_3 + B_3\alpha_3)}{H_3^2 + K_3^2},\ H_{11} = \frac{H_3(A_3\beta_3 + B_3\alpha_3) - K_3(A_3\alpha_3 - B_3\beta_3)}{H_3^2 + K_3^2},$$

$$e_0 = \frac{nSc}{4},\ e_1 = \alpha_1^2 - \beta_1^2 - Sc\,\alpha_1,\ e_2 = 2\alpha_1\beta_1 - \frac{nSc}{4},$$

$$e_3 = \frac{e_1\{(1-a_2)(\alpha_1^2 - \beta_1^2) - 2b_2\alpha_1\beta_1\}}{e_1^2 + e_2^2} + \frac{e_2\{b_2(\alpha_1^2\beta_1^2) + (1-a_2)\alpha_1\beta_1\}}{e_1^2 + e_2^2},$$

$$e_4 = \frac{e_1\{b_2(\alpha_1^2 - \beta_1^2) + 2(1-a_2)\alpha_1\beta_1\}}{e_1^2 + e_2^2} - \frac{e_2\{(1-a_2)(\alpha_1^2\beta_1^2) - 2b_2\alpha_1\beta_1\}}{e_1^2 + e_2^2}$$

$$e_5 = \frac{b_1(AScd_1 + a_2c_0) + b_2c_0e_0}{b_1^2 + e_0^2},\ e_6 = \frac{e_0(AScd_1 + a_2c_0) - b_1b_2c_0}{b_1^2 + e_0^2},$$

$$A_2 = 1 + S_0Sc(e_3 + c_0e_5),\ B_2 = S_0Sc(e_4 + c_0e_6) - \frac{4ASc}{n}(1 + d_1S_0Sc),$$

$$A_3 = 1 - Gr\,a_4 + Gm(S_0Sc\,a_5 - a_6),\ B_3 = Gr\,b_4 - Gm(S_0Sc\,b_5 - b_6),$$

$$A_4 = 1 - Gr(H_4 + E_3 + Ac_0E_9) - AH_{10} - Gm\Big[H_6 - S_0Sc(H_8 + c_0E_5 + Ac_0E_{11}) + \frac{4ASc}{n}(1 + d_1S_0Sc)E_7 + ASc\,E_{13}\Big],$$

$$B_4 = Gr(E_4 - H_5 + Ac_0E_{10}) - AH_{11} - Gm\Big[H_7 - S_0Sc(H_9 + c_0E_6 - Ac_0E_{12}) + \frac{4ASc}{n}(1 + d_1S_0Sc)E_8 - ASc\,E_{14}\Big],$$

$$(X_n, Y_n) = (\cos\beta_n y,\quad \sin\beta_n y)e^{-\alpha_n y},\quad n = 1,2$$

$$D_n = (A_n\cos\beta_n y + B_n\sin\beta_n y)e^{-\alpha_n y},\quad n = 2,3,4$$

$$F_n = (B_n\cos\beta_n y - A_n\sin\beta_n y)e^{-\alpha_n y},\quad n = 2,3,4$$

$$H_n = \alpha_n^2 - \beta_n^2 - \alpha_n - b_0 - \frac{1}{K_p},\quad n = 1,2,3\quad K_n = 2\alpha_n\beta_n - \beta_n + E_2,\quad n = 1,2,3$$

$$\tau_0 = \alpha_3A_3 - \beta_3B_3 + Gr\,A_4c_0 - Gm\,Sc(a_5S_0c_0 - a_6),\ D = D_r + iD_i,$$

$$tan\,\alpha = \frac{D_i}{D_r},\ R = R_r + iR_i,\ tan\,\beta = \frac{R_i}{R_r},\ G = G_r + iG_i,\ tan\,\gamma = \frac{G_i}{G_r},$$

$$D_r = (\alpha_4A_4 - \beta_4B_4) + \alpha_1(Gr\,H_4 - Gm\,S_0Sc\,H_8) - \beta_1(Gr\,H_5 - Gm\,S_0Sc\,H_9) + Gm(\alpha_2H_6 - \beta_2H_7) + A(\alpha_3H_{10} - \beta_3H_{11}) + c_0\{Gr(E_3 + Ac_0E_9) - Gm\,S_0Sc\,c_0(E_5 + AE_{11})\} + AGm\,Sc^2\left\{\frac{4}{n}(1 + d_1S_0Sc\,E_7 + E_{13})\right\},$$

$$D_i = (\alpha_4B_4 + \beta_4A_4) + \alpha_1(Gr\,H_5 - Gm\,S_0Sc\,H_9) + \beta_1(Gr\,H_4 - Gm\,S_0Sc\,H_8) + Gm(\alpha_2H_7 + \beta_2H_6) + A(\alpha_3H_{11} + \beta_3H_{10})$$

$$- c_0 \{Gr(E_4 + A c_0 E_{10}) - Gm\, S_0 Sc\; c_0 (AE_{12} - E_6)\}$$

$$+ AGm\, Sc^2 \left\{\frac{4}{n}(1 + d_1 S_0 Sc)\, E_8 - E_{14})\right\},$$

$$R_r = \alpha_1(1 - a_2) - \beta_1 b_2 + a_2\, c_0\,,\; R_i = \alpha_1 b_2 + \beta_1(1 - a_2) - b_2\, c\,,$$

$$G_r = (\alpha_2 A_2 - \beta_2 B_2) - S_0 Sc(\alpha_1 e_3 + \beta_1 e_4 + c_0^2 e_5\,,$$

$$G_i = (\alpha_2 B_2 + \beta_2 A_2) - S_0 Sc(\alpha_1 e_4 + \beta_1 e_3 + c_0^2 e_6) + \frac{4ASc^2}{n}(1 + d_1 S_0 Sc)\,.$$