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Research Paper

Analysis of Chirp Signal with Fractional Fourier Transform

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Fractional Fourier transform (FrFT) is well suited for processing of non-stationary signals especially linear frequency modulated signals and chirped signals. It is a parameterized transform which provides extremely compact representations. In this paper a closed-form expression of the fractional Fourier transform of chirp signal is derived. The FrFT of chirp is used in calculating the spread of the signal representation for linear chirp in any fractional domain. The derived expression establishes the dependence of chirp parameters on optimum FrFT angle and FrFT peak delta location. Based on this derivation, a fractional Fourier transform based filter for chirp noise removal is designed. The results derived show that the simulation results and analytical results closely match.

Key Words: Chirp Parameters; Fractional Fourier Transform; Chirped Noise; Discrete Fractional Fourier Transform

1. Introduction

A chirp signal is a typical non-stationary signal, whose instantaneous frequency changes either linearly or nonlinearly with time. It is used in sonar, radar, acoustic communication, mobile communication and biomedical applications. This type of signal may be considered as noise in optical system, microwave system, radar system, and acoustics which needs to be eliminated efficiently. The analysis of non-stationary chirp signals is required in sonar systems to detect the parameters of the chirps transmitted by other emitters. Active sonar involves the transmission of an acoustic signal which when reflected from a target provides the sonar receiver with a basis for detection and estimation of its range and radial velocity. The knowledge of transmitted waveform is advantageous for detection of chirp waveforms. In radar applications, the analysis of chirp signals is used to detect the velocity and acceleration of the remote moving target. A fixed frequency time signal is having narrowest representation in frequency domain. However, chirp signal containing a range of frequencies, produces a greater spread in the frequency domain. Time-frequency representations provide better detection and estimation capabilities. In the time-frequency plane, a chirp signal is projected onto a rotated frequency axis u with a proper angle α, that is Fractional Fourier Transform (FrFT). The energy distribution of the chirp can be more highly concentrated in the fractional Fourier domain and can be easily processed for several applications. The FrFT is ideally suited to the analysis and synthesis of linear chirp signals. The chirp rate and initial frequency of the received echo from the target in radar and sonar is estimated using Fractional Fourier Transform (FrFT). It is used to extract individual chirp from the mixture of chirp signals and estimate its parameters. Chirps produce impulses at its optimum FrFT. The optimum fractional order for a known linear chirp

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signal is defined. On the other hand, it is obtained using a search algorithm for a chirp like signal whose characteristics are not known earlier.

In real situations, chirp overlaps with the desired signal in time domain as well as frequency domain. It can not be separated either in time domain or frequency domain using conventional multiplicative filter. Thus, we need to analyze and synthesize chirp signals in a rotated co-ordinate system. FrFT is used for filtering of additive chirp from the desired signal. A linear chirp signal x(t) is defined mathematically as:

$$\text{Chirp signal } x(t) \Rightarrow A\,cos\,(ct^2 + \grave{\imath}\; t + k) \qquad (1)$$

with its phase is given by $ct^2 + \mu t + k$ and instantaneous frequency (derivative of phase) is given by $2ct + \mu$, where c is the chirp rate or slope which indicates the rate of change of instantaneous frequency and μ is initial frequency defined by instantaneous frequency at t = 0. Chirp with positive 'c' (frequency increases with time) is considered as up-chirp, otherwise it is down-chirp with negative 'c' (frequency decreases with time). 'A' denotes the amplitude.

The paper presents an analytic relationship between the chirp parameters (chirp rate and initial frequency of chirp) and optimum fractional angle α. The derived support of chirp signal in fractional domains is used in developing the performance comparison between FrFT based filtering of additive chirp noise and filtering method based on ordinary Fourier transform.

2. Fractional Fourier Transform

Fractional Fourier Transform [8 2 1] is a generalization of the conventional Fourier transform (FT), which is richer in theory, more flexible in applications, and implementation cost is same as that of computing FT. The FrFT of the α order can be interpreted as a rotation in the time-frequency plane with an angle α, and the time domain and frequency domain are the special cases of the FrFT domain with α being $2n\pi$ and $2n\pi + \pi/2$, respectively, where n is an integer. It can also be viewed as a fractional power of the Fourier operator. It reflects the characteristics of a signal in the time-frequency domain with a single variable and does not suffer from cross-terms. One extra degree of freedom (angle α) helps FrFT to do better in chirp-like signals processing. Chirp signal is the basis function of FrFT. Nowadays, signal communication system based on chirp has received a great deal of attention.

The continuous Fractional Fourier Transform (CFrFT) of a signal x(t) represented along time axis denoted by t, with angular parameter α is computed as :

$$F^{\alpha}[x(t)] = X_{\alpha}(u) = \qquad (2)$$

where the transform kernel $K_\alpha(t, u)$ of the FrFT is given by

$$K_{\alpha}(t,u) = \begin{cases} C_{\alpha} e^{jat^2 + jau^2 - jbut}, & \text{if } \alpha \text{ is not a multiple of } \pi \\ \delta(t-u), & \text{if } \alpha \text{ is a multiple of } 2\pi \\ \delta(t+u), & \text{if } \alpha + \pi \text{ is a multiple of } 2\pi \end{cases} \qquad (3a)$$

with

$$C_{\alpha} = \sqrt{\frac{1 - j\cot(\alpha)}{2\pi}},\ a = \frac{1}{2}\cot\alpha \ \text{ and } \ b = \csc\alpha \qquad (3b)$$

Here n is a positive or negative integer and $\delta(t)$ denotes Dirac-delta function. The Fourier transform is a special case for .

3. Derivation of FrFT of Chirp Signal

The continuous FrFT of a real chirp signal x(t) = is

$$X_{\alpha}(u) = C_{\alpha} \exp(jau^2) \int_{-\infty}^{\infty} \cos(\beta t^2) \exp(jat^2 - jbut)dt \quad (4)$$

Rewriting $\cos(\beta t^2)$ in Euler's form, one gets

$$\cos(\beta t^2) = \frac{\exp(j\beta t^2) + \exp(-j\beta t^2)}{2} \quad (5)$$

Substituting (5) in (4), gives

$\cos(\beta t^2)$ $\beta = \pm a$ $a = \pm jb$ $\cot\alpha$

$$I_1 = \int_{-\infty}^{\infty} \exp[j(\beta + a)t^2 - jbut]dt$$

$$\exp\left[\frac{-jb^2u^2}{4(\beta + a)}\right] \int_{-\infty}^{\infty} \exp\left[j(\sqrt{(\beta + a)}t - \frac{bu}{2\sqrt{(\beta + a)}})^2\right]dt$$

$$X_{\alpha}(u) = \frac{C_{\alpha}}{2} \exp(jau^2) \int_{-\infty}^{\infty} \left[\exp(j\beta t^2) + \exp(-j\beta t^2)\right] \exp(jat^2 - jbut)dt \quad (6)$$

$$\Rightarrow \frac{C_{\alpha}}{2} \exp(jau^2) \left[\int_{-\infty}^{\infty} \exp[j\{(\beta + a)t^2 - jbut\}]dt + \int_{-\infty}^{\infty} \exp[j\{(-\beta + a)t^2 - jbut\}]dt \right] \quad (7)$$

$$\Rightarrow \frac{1}{2} C_{\alpha} \exp(jau^2)$$

$$[I_1 + I_2 = \frac{1}{2}[Y_{\alpha}(u) + Z_{\alpha}(u)] \quad (8)$$

Where is the FrFT of positive frequency complex chirp and $Z_{\alpha}(u)$ is the FrFT of negative frequency complex chirp.

And

and

$$I_2 = \int_{-\infty}^{\infty} \exp[j(-\beta + a)t^2 - jbut]dt \quad (9)$$

We consider two cases when $\beta \neq \pm a$ and separately:

Case 1: For $\beta \neq \pm a$ we solve two integrals I_1 and I_2 separately. Solving for I_1 :

On completing the square integral I_1 becomes:

(10a)

And I_2 becomes:

$$\exp\left[\frac{-jb^2u^2}{4(-\beta + a)}\right] \int_{-\infty}^{\infty} \exp\left[j(\sqrt{(-\beta + a)}t - \frac{bu}{2\sqrt{(-\beta + a)}})^2\right]dt \quad (10b)$$

Now, Substituting $Z = \sqrt{(\beta + a)}t - \frac{bu}{2\sqrt{(\beta + a)}}$ in (10a) gives:

$$\text{or } dt = \frac{dz}{\sqrt{(\beta + a)}} \qquad (11)$$

So that

(12)

or

$$I_1 = \frac{2}{\sqrt{(\beta + a)}} \exp\left[\frac{-jb^2u^2}{4(\beta + a)}\right] \int_0^\infty \exp(jz^2)dz \qquad (13)$$

Replacing $\sqrt{-j}z = y, dz = \frac{dy}{\sqrt{-j}}$ one gets

(14)

We know that $\int_0^\infty \exp(-y^2)dy = \frac{\sqrt{\pi}}{2}$

Therefore,

$$I_1 = \frac{2}{\sqrt{(\beta + a)}\sqrt{-j}} \exp\left[\frac{-jb^2u^2}{4(\beta + a)}\right] \frac{\sqrt{\pi}}{2} \qquad (15)$$

Similarly, replacing β by $-\beta$ in (15), we get:

(16)

Now substituting values of integrals I_1 and I_2 in (8) we get

$$\left\{ \frac{\sqrt{j}}{\sqrt{(\beta + a)}} \sqrt{\pi} \exp\left[\frac{-jb^2u^2}{4(\beta + a)}\right] \right. \qquad (17)$$

$$\left\{ \frac{1}{\sqrt{(\beta + a)}} \exp\left[\frac{-jb^2u^2}{4(\beta + a)}\right] \right. \qquad (18)$$

Substituting values of α, β and C_α given by expression (3b) in (18), and simplifying, we get for

(19)

Thus, the FrFT of chirp is directly dependent on the FrFT angle α. The closed-form expression (19) can be used to describe the spread of the real chirp signal in the fractional domains.

Case 2: For $(\cot\alpha = 2\beta)$

In Equation (9), we get

$$I_2 = \int_{-\infty}^{\infty} \exp(-jbut)dt \Rightarrow \sqrt{2\pi}\delta(u) \quad (20)$$

$$\beta = a = \frac{1}{2}\cot\alpha$$

$$X_\alpha(u) = \frac{\sqrt{1+j\tan\alpha}}{\sqrt{8}} \exp(jau^2)\left[\int_{-\infty}^{\infty} \exp[j\{(c+a)t^2 - jb(u-\frac{\mu}{b})t\}]dt\right]$$

and

$$Z_\alpha(u) = \sqrt{1-2j\beta}\,\exp(j\beta u^2)\left\{\frac{\exp[\frac{ju^2(2\beta-\tan\alpha)}{2(2\beta\tan\alpha+1)}]}{\sqrt{(\beta\tan\alpha+\frac{1}{2})}} + \frac{\exp[\frac{-ju^2(2\beta+\tan\alpha)}{2(-2\beta\tan\alpha+1)}]}{\sqrt{(-\beta\tan\alpha+\frac{1}{2})}}\right\}\delta(u)$$

$$(\text{for } \beta = a = \frac{1}{2}\cot\alpha) \quad (21)$$

Thus with the FrFT tool, a chirp and a delta function can be transformed one into the other and the FrFT which produces the most compact support i.e. impulse for a given linear chirp signal is called as the optimal fractional Fourier transform for that signal [9, 6]. Thus if we know the chirp rate 2β, also called the rate of change of frequency, we can calculate the optimal fractional domain. On the other hand, if we want to estimate the chirp rate, we can use an iterative method and find the optimal α [5] by performing FrFT on the chirp for different values of α in the range from –1 to 1 and selecting the one which gives the maximum peak value. The initial frequency is zero for this chirp.

Consider an analytic chirp signal which consist of only negative frequency components and is given by x(t) = $\exp(-j*3*t^2)$. Given the chirp rate 2β = 6, the CFrFT $X_\alpha(u)$ of this chirp using the derived expression (19) is shown in Fig. 1(a) for α = 0 (time domain), (b) for α = ð/4 (fractional domain), (c) for = ð/2, (Fourier domain) and (d) for optimal domain .

We take another chirp signal x(t) = exp(j(ct^2 + µt)), with chirp rate c and initial frequency µ and compute its CFrFT. Equation (4) reduces to:

(22)

For the case $c = a = \frac{1}{2}\cot\alpha$, we get

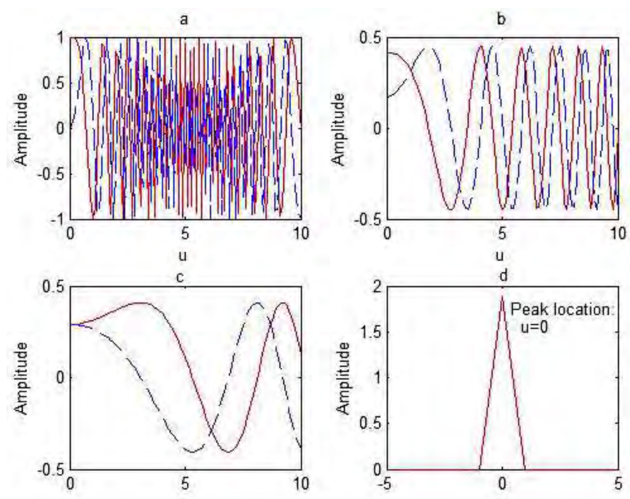

Fig. 1: Continuous Fractional Fourier transforms of x(t). The solid line corresponds to real part and dashed line corresponds to imaginary part

$$X_{\alpha}(u) = \frac{C_{\alpha}}{1} \exp(jau^2)$$

$$[\int_{-\infty}^{\infty} \exp(jb\frac{\mu}{b}t)\exp(-jbut)dt \qquad (23)$$

$$\Rightarrow \frac{C_{\alpha}}{1} \exp(jau^2)\sqrt{2\pi}\delta(u - \frac{\mu}{b}) \qquad (24)$$

(using frequency shifting property of FT).

Substituting values of C_{α}, a and b, we get

(25)

The FrFT output in equation (25) clearly shows that the position of the maximum peak within the optimal fractional Fourier domain is related to chirp signal's initial frequency. Thus, from the analytic (25), it can be written:

FrFT peak delta location $u = \mu \sin\alpha$ (26)

Thus, chirp parameters can be estimated by computing optimum angle α at which chirps will produce impulses. If the chirp signal is not known previously, the optimum α cannot be found analytically. A one-dimensional search is necessary to find it.

4. Chirped Noise Extraction

In signal processing, transform domain approach is useful to bring out the hidden information which may not be explicitly available when the signal is in original domain. If a desired signal and unwanted noise overlap in time domain as well as frequency domain, the additive noise is filtered in a rotated co-ordinate system. The filtering in fractional domain procedure consists in projecting the signal on the fractional domain, applying a classical stationary filtering procedure (e.g. FIR) and, using IFrFT, returning to the time domain.

Consider chirp as a model of additive noise. The desired signal is mixed with unwanted chirp. In order to filter chirp from the desired signal, first an optimal domain where the spectrum of chirp transforms to impulse is found. If the characteristics of chirp signal are not known, an iterative method is used to determine '*a*'. A narrow band-stop mask is applied to the FrFT of desired signal plus chirp noise in that optimal domain so that all the values of chirp component become zero. An inverse FrFT of equivalent order is applied to the resultant in order to get the reconstructed desired signal in time domain. The narrower the band-stop mask is, the less distortion the non-chirp part has. Fig. 2 shows the multiplicative filtering in a^{th} order fractional Fourier transform domain.

We perform computer simulations for a discrete Gaussian shaped desired input signal given by

, which is corrupted by an additive real chirp noise $\cos(0.004\pi n^2 + 0.2\pi n)$ so that

$$x(n) = e^{-0.01(n-30)^2/20}$$

$$+\cos(0.004\pi n^2 + 0.2\pi n), 1 \le n \le 62 \qquad (27)$$

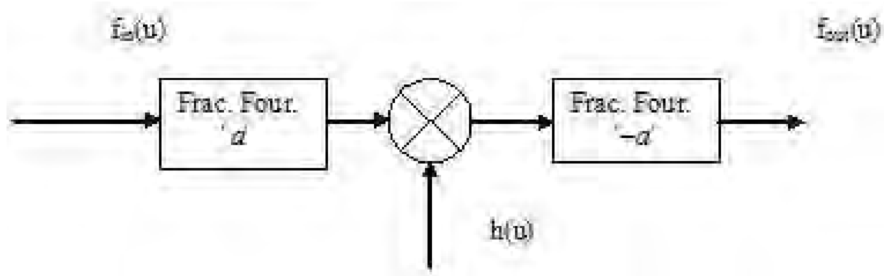

Fig. 2: Multiplicative filtering in order fractional Fourier transform domain

The signal $x(n)$ which consist of the desired signal plus noise is corrupted with unwanted chirp both in the time and frequency domains. So, we need to filter noise in the fractional Fourier domain. The chirped noise filtering method comprises of following steps:

1. Compute the DFrFT $x_1(n)$ of signal $x(n)$ for fractional angle.
2. Multiply the transform result $x_1(n)$ by the bandstop mask $m(n)$ to obtain $x_2(n) = x_1(n)\, m(n)$.
3. Perform inverse DFrFT to bring the signal back in time domain and again compute DFrFT with fractional angle-α. This is equivalent to computing the DFrFT $x_3(n)$ of the signal $x_2(n)$ for fractional angle -2α.
4. Multiply the transform result $x_3(n)$ again by the bandstop mask $m(n)$ to obtain $x_4(n) = x_3(n)\, m(n)$.
5. The last step is to compute the DFrFT $x_5(n)$ of the signal $x_4(n)$ for fractional angle α. The signal $x_5(n)$ is the desired filtered output.

$\beta \equiv \cot(\alpha/2)$ $I_1 \equiv \sqrt{2\pi}\delta(u)$

Repeat the procedure for different values of α. Here, two band-stop masks are applied because a real chirp signal is decomposed into sum of two complex chirp signals [7, 4]. Therefore, negative frequency component in real chirp will become delta function at as proved analytically and can be eliminated in fractional domain with optimal angle α. Similarly if , . Thus positive frequency component in real chirp will become delta function and this impulse of chirp can be eliminated in fractional domain with angle –α. Thus, one band-stop mask is performed in the α domain and the other is in the –α domain. The simulation results shown in Fig. 3 demonstrate the discrete chirped noise extraction based on FrFT for angle α = 1.4137. Chirp noise is removed using a band-stop filter given by

$$m(n) = \begin{cases} 0, \text{ for } 12 \le n \le 20 \\ 1, \text{ otherwise} \end{cases} \tag{28}$$

The matlab code for generating m(n) is:

m = ones(1,62);

m(12:20)=0; (29)

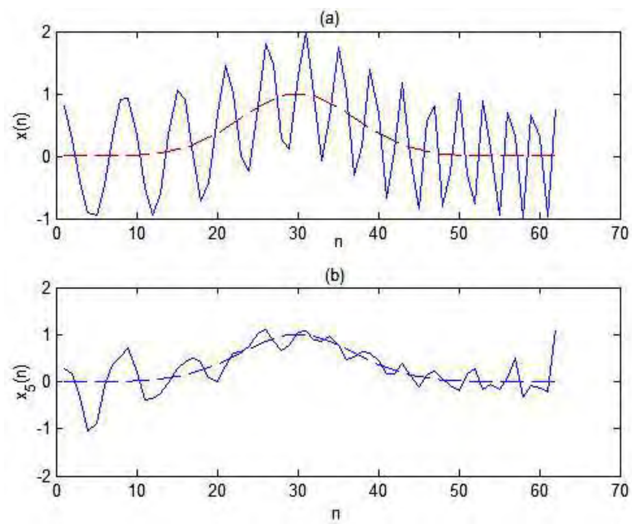

Fig. 3: (a) Gaussian plus real chirp noise $x(n)$ in time domain, (b) Signal $x_5(n)$ after filtering with fractional angle α = 1.4137. The waveform plotted by dashed line is the ideal Gaussian signal

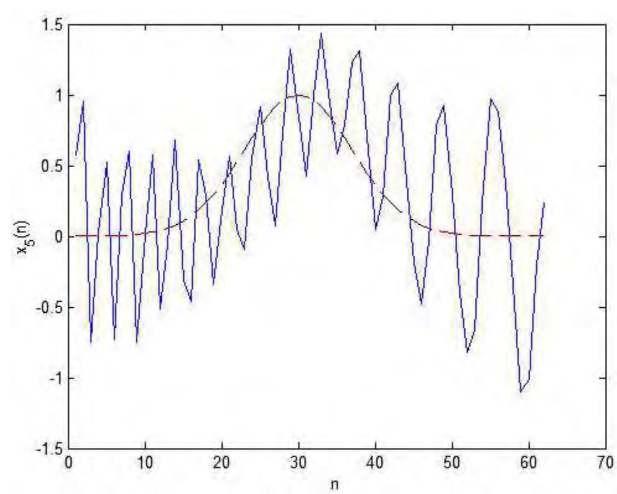

Fig. 4: The filtered output $x_5(n)$ for fractional order $\alpha = \pi / 2$ i.e. ordinary Fourier domain

In Fig. 3(a), the waveform of signal $x(n)$ is plotted by solid line. For comparison, the ideal desired Gaussian signal is plotted by dashed line in Fig. 3(a) and Fig. 3(b). Fig. 3(b) shows the filtered signals $x_5(n)$ with solid line by using the MATLAB code developed by Candan, Kutay and Ozaktas [3]. The results plotted in Fig. 4 show filtering in ordinary frequency domain.

It is clear from the results in Fig. 3(b), that chirp interference can be eliminated by filtering in optimal fractional Fourier domain with $\alpha = 1.4137$. This value is close to the analytic value 1.54 ($\beta = 0.004\pi = \cot \alpha/2$). However, the results in Fig. 4 show that unwanted chirp can not be removed from the desired signal in ordinary Fourier domain.

5. Conclusion

Based on the derived expression for fractional Fourier transform of chirp signal, it can be concluded that FrFT of chirp varies directly with the change in fractional angle α. The chirp parameters can be estimated by computing the optimal fractional angle where the spread of signal representation in time-frequency plane reduces to peak delta. The fractional filter design reveals that the chirp noise can be easily filtered out in an optimal fractional domain. Thus, it is found that for time-varying signals and systems, filtering in fractional Fourier domain can be used effectively for the separation of interference. The performance of fractional Fourier transform based filtering is compared with ordinary Fourier domain filtering of chirp noise. The simulation results reveal that for additive chirp noise removal, FrFT based filtering gives excellent results.

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