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Relativistic Third-order Viscous Hydrodynamics

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Employing the iterative solution of Boltzmann equation in relaxation time approximation, we derive a third-order evolution equation for shear stress tensor from its kinetic definition. To this end we first obtain the non-equilibrium phase-space distribution function, $f(x,p)$, up to second-order in gradients. The expression for $\delta f(x,p)$ thus obtained does not lead to the violation of the experimentally observed $1/\sqrt{m_T}$ scaling of the longitudinal femtoscopic radii, as is the case with the widely used Grad's approximation, and hence is better suited for hydrodynamic modelling of relativistic heavy-ion collisions. Subsequently, we quantify the significance of this new derivation within one-dimensional scaling expansion and demonstrate that the results obtained using third-order viscous equations are in excellent agreement with the exact solution of Boltzmann equation as well as transport results.

Key Words : Relativistic Hydrodynamics; Kinetic Theory; Boltzmann Equation; Shear Stress Tensor

Introduction

The long-wavelength, low frequency limit of the microscopic dynamics leads to an effective hydrodynamical description of the system. The collective behaviour of the hot and dense, strongly interacting matter, created in high-energy heavy-ion collisions, has been studied quite extensively within the framework of relativistic hydrodynamics. The theory of relativistic hydrodynamics is formulated as a gradient expansion where ideal hydrodynamics is zeroth-order. However, as all fluids are inherently dissipative by virtue of uncertainty principle (Danielewicz and Gyulassy, 1985), the dissipative effects can not be ignored. Although the first-order theories, collectively known as relativistic Navier-Stokes theory (Eckart, 1940; Landau and Lifshitz, 1987) incorporate dissipation, however they suffer from acausality and numerical instability. The second-order, Israel-Stewart (IS) theory (Israel and Stewart, 1979) solves the acausality problem (Huovinen and Molnar, 2009) but may not guarantee stability.

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The IS theory has been applied quite successfully in explaining a wide range of collective phenomena observed in high-energy heavy-ion collision experiments at Relativistic Heavy-Ion Collider (RHIC) and recently at Large Hadron Collider (LHC). Despite its success, the formulation of IS theory is based on strong assumptions and approximations, namely: use of second moment of Boltzmann equation and the Grad's 14-moment approximation (Israel and Stewart, 1979). While it was shown later that the dissipative equations can be obtained directly from their definitions without resorting to the second-moment of Boltzmann equation (Denicol *et al*., 2010), it has been shown recently that both these assumptions are unnecessary and instead of 14-moment approximation, iterative solution of Boltzmann equation can be employed to derive the dissipative evolution equations (Jaiswal, 2013a).

Apart from these theoretical issues in the formulation, IS theory suffers from several other shortcomings on the phenomenological level. In the case of one-dimensional scaling expansion (Bjorken, 1983), IS theory has resulted in unphysical effects such as reheating of the expanding medium (Muronga, 2004) and negative longitudinal pressure (Martinez and Strickland, 2009). Moreover, comparison of IS equations with transport results show disagreement for $\eta/s > 0.5$ indicating the breakdown of second-order theory (Huovinen and Molnar, 2009; El *et al*., 2009). Furthermore, inclusion of dissipative corrections to the phase-spcae distribution function, $f(x, p)$, via Grad's 14-moment approximation leads to the violation of experimentally observed $1/\sqrt{m_T}$ scaling of the longitudinal Hanbury Brown-Twiss (HBT) radii.

To extend the range of applicability of the IS equations, second-order dissipative equations were derived from Boltzmann equation where the collision term was generalized to include nonlocal effects through gradients of $f(x, p)$ (Jaiswal, *et al*., 2013a). Moreover, it was shown that the inclusion of third-order corrections to the evolution equation of shear stress tensor led to an improved agreement with transport results (El *et al*., 2010; Jaiswal, 2013b). Furthermore, a general moment method was devised to improve Grad's 14-moment approximation beyond its current scope by introducing orthogonal basis in momentum expansion (Denicol *et al*., 2012). The correct and consistent formulation of relativistic viscous hydrodynamics is not yet conclusively settled and is currently a topic of intense investigation (Denicol *et al*., 2010; Jaiswal 2013a; Jaiswal *et al*., 2013b; El *et al*., 2010; Jaiswal, 2014; Jaiswal *et al*., 2014; Denicol *et al*., 2012; Jaiswal *et al*., 2013c; Bhalerao *et al*., 2013, 2014).

Relativistic Hydrodynamics

The equation of motion governing the hydrodynamic evolution of a relativistic system with no net conserved charges is obtained from the local conservation of energy and momentum, $\partial_\mu T^{\mu\nu} = 0$. In terms of single-particle phase-space distribution function, the energy-momentum tensor of a macroscopic system can be expressed as (deGroot *et al.*, 1980)

$$T^{\mu\nu} = \int dp \, p^\mu p^\nu \, f(x,p) = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{1}$$

where $dp \equiv g d\mathbf{p}/[(2\pi)^3|\mathbf{p}|]$, g being the degeneracy factor, p^μ is the particle four-momentum, and $f(x,p)$ is the phase-space distribution function. In the tensor decomposition, ϵ, P, and $\pi^{\mu\nu}$ are energy density, thermodynamic pressure, and shear stress tensor, respectively. The projection operator $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is orthogonal to the hydrodynamic four-velocity u^μ defined in the Landau frame: $T^{\mu\nu}u_\nu = \epsilon u^\mu$. The metric tensor is Minkowskian, $g^{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$. Here we have restricted ourselves to a system of massless particles (ultrarelativistic limit) for which the bulk viscosity vanishes.

The conservation of the energy-momentum tensor, when projected along and orthogonal to u^μ, leads to the evolution equations for ϵ and u^μ:

$$\dot\epsilon + (\epsilon + P)\theta - \pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0, \qquad (\epsilon + P)\dot u^\alpha - \nabla^\alpha P + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} = 0, \tag{2}$$

where we employ the standard notation $\dot A \equiv u^\mu \partial_\mu A$ for comoving derivative, $\theta \equiv \partial_\mu u^\mu$ for expansion scalar, $A^{(\alpha}B^{\beta)} \equiv (A^\alpha B^\beta + A^\beta B^\alpha)/2$ for symmetrization, and $\nabla^\alpha \equiv \Delta^{\mu\alpha}\partial_\mu$ for space-like derivatives. For the massless case, the equation of state relating energy density and pressure is $\epsilon = 3P \propto \beta^{-4}$. The Landau matching condition $\epsilon = \epsilon_0$ is employed to fix the inverse temperature, $\beta \equiv 1/T$, where ϵ_0 is the equilibrium energy density. The derivatives of β,

$$\dot\beta = \frac{\beta}{3}\theta - \frac{\beta}{12P}\pi^{\rho\gamma}\sigma_{\rho\gamma}, \qquad \nabla^\alpha \beta = -\beta \dot u^\alpha - \frac{\beta}{4P}\Delta^\alpha_\rho \partial_\gamma \pi^{\rho\gamma}, \tag{3}$$

can be obtained from Eq. (2), where $\sigma^{\rho\gamma} \equiv \nabla^{(\rho}u^{\gamma)} - (\theta/3)\Delta^{\rho\gamma}$ is the velocity stress tensor (Jaiswal, 2013b).

When the system is close to local thermodynamic equilibrium, the distribution function can be written as $f = f_0 + \delta f$, where $\delta f \ll f_0$, $f_0 = \exp(-\beta\, u \cdot p)$ is the equilibrium distribution function of Boltzmann particles at vanishing chemical potential and $u.p \equiv u_\mu p^\mu$. Projecting the traceless symmetric part of Eq. (1), we obtain an expression for the shear stress tensor and its time evolution,

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \, p^\alpha p^\beta \, \delta f, \qquad \dot\pi^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \, p^\alpha p^\beta \, \delta \dot f \tag{4}$$

where $\Delta^{\mu\nu}_{\alpha\beta} \equiv \Delta^\mu_{(\alpha}\Delta^\nu_{\beta)} - (1/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}$. In the following, we iteratively solve the Boltzmann equation to determine δf and subsequently derive evolution equation for shear stress tensor.

Viscous Evolution Equations

We start from the relativistic Boltzmann equation with the relaxation-time approximation for the collision term (Anderson and Witting, 1974),

$$p^\mu \partial_\mu f = -\,(u\!\cdot\! p)\,\frac{\delta f}{\tau_R} \quad \Rightarrow \quad f = f_0 - \frac{\tau_R}{(u\!\cdot\! p)}\,p^\mu \partial_\mu f, \tag{5}$$

where τ_R is the relaxation time. Expanding the distribution function f about its equilibrium value in powers of space-time gradients, i.e., $f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots$ and solving Eq. (5) iteratively, we obtain (Jaiswal, 2013a,b),

$$\delta f^{(1)} = -\frac{\tau_R}{u\cdot p}\,p^\mu \partial_\mu f_0, \qquad \delta f^{(2)} = \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu \Big(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\Big). \tag{6}$$

Substituting $\delta f = \delta f^{(1)}$ in the expression for $\pi^{\mu\nu}$ in Eq. (3), performing the integrations, and retaining only first-order terms, we obtain $\pi^{\mu\nu} = 2\tau_R\beta_\pi\sigma^{\mu\nu}$, where $\beta_\pi = 4P/5$ (Jaiswal, 2013a).

To obtain the second-order evolution equation for shear stress tensor, we rewrite Eq. (5) in the form $\delta\dot f = -\dot f_0 - p^\gamma\nabla_\gamma f/(u\!\cdot\! p) - \delta f/\tau_R$. Using this expression for $\delta\dot f$ in Eq. (4), we obtain

$$\dot\pi^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_R} = -\Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta \left(\dot f_0 + \frac{1}{u\!\cdot\! p} p^\gamma \nabla_\gamma f\right). \tag{7}$$

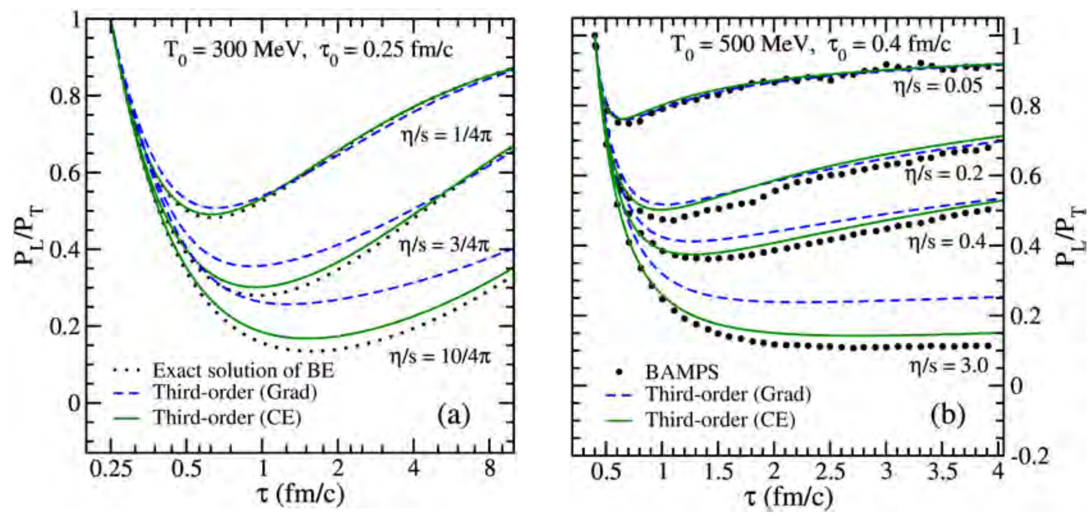
Using Eq. (6) for $\delta f^{(1)}$ and Eq. (3) for derivatives of β, and keeping terms up to quadratic order in gradients, the second-order shear evolution equation is obtained as (Jaiswal, 2013a)

$$\dot\pi^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta, \tag{8}$$

where $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$. Using Eq. (3) and (8) in Eq. (6), we arrive at (Bhalerao *et al*., 2014; Chattopadhyay *et al*., 2014),

$$\begin{aligned}
\delta f = {}& \frac{f_0\beta}{2\beta_\pi(u\!\cdot\! p)}\,p^\alpha p^\beta \pi_{\alpha\beta} - \frac{f_0\beta}{\beta_\pi}\bigg[\frac{\tau_\pi}{u\!\cdot\! p}\,p^\alpha p^\beta \pi^\gamma_\alpha\,\omega_{\beta\gamma} - \frac{5}{14\beta_\pi(u\!\cdot\! p)}\,p^\alpha p^\beta \pi^\gamma_\alpha\,\pi_{\beta\gamma} \\
& + \frac{(u\!\cdot\! p)}{70\beta_\pi}\,\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{6\tau_\pi}{5}\,p^\alpha \dot u^\beta \pi_{\alpha\beta} + \frac{\tau_\pi}{3(u\!\cdot\! p)}\,p^\alpha p^\beta \pi_{\alpha\beta}\theta + \frac{\tau_\pi}{5}\,p^\alpha\left(\nabla^\beta\pi_{\alpha\beta}\right) \\
& - \frac{3\tau_\pi}{(u\!\cdot\! p)^2}\,p^\alpha p^\beta p^\gamma \pi_{\alpha\beta}\dot u_\gamma + \frac{\tau_\pi}{2(u\!\cdot\! p)^2}\,p^\alpha p^\beta p^\gamma\left(\nabla_\gamma \pi_{\alpha\beta}\right) \\
& - \frac{\beta + (u\!\cdot\! p)^{-1}}{4(u\!\cdot\! p)^2\beta_\pi}\,\left(p^\alpha p^\beta \pi_{\alpha\beta}\right)^2\bigg] + \mathcal{O}(\delta^3),
\end{aligned} \tag{9}$$

where the first term on the right-hand side of the above equation corresponds to the first-order correction, δf_1, whereas the terms within square brackets are of second order, δf_2. It is straightforward to show that the form of δf in Eq. (9) is consistent with the definition of the shear stress tensor, Eq. (4), and satisfies the matching condition $\epsilon = \epsilon_0$ and the Landau frame definition $u_\nu T^{\mu\nu} = \epsilon u^\mu$ at each order (Bhalerao *et al.*, 2014). It is important to note that the experimentally observed $1/\sqrt{m_T}$ scaling of the longitudinal HBT radii (Adcox *et al.*, 2002; Bearden *et al.*, 2001), also predicted by ideal hydrodynamics, is violated by incorporating viscous correction through Grad's 14-moment approximation (Teaney, 2003). However the form of δf given in Eq. (9) does not lead to such unphysical effects (Bhalerao *et al.*, 2014). To obtain a third-order evolution equation for shear stress tensor, we substitute δf from Eq. (9) in Eq. (7). Keeping terms up to cubic order in derivatives, after a straightforward but tedious algebra, we finally obtain a third-order evolution equation for shear stress tensor (Jaiswal, 2013b):



A B

Fig. 1: (A): Time evolution of P_L/P_T obtained using exact solution of Boltzmann equation (dotted line), second-order equations (dashed lines), and third-order equations (solid lines). (B): Time evolution of P_L/P_T in BAMPS (dots), third-order calculation from entropy method, Eq. (11) (dashed lines), and the present work (solid lines). Both figures are for isotropic initial pressure configuration $(\pi_0 = 0)$ and various η/s

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} =& -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{25}{7\beta_\pi}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} \\ & - \frac{1}{3\beta_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta - \frac{38}{245\beta_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{22}{49\beta_\pi}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma} \\ & - \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) - \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) \\ & + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right) \\ & - \frac{2}{7}\tau_\pi\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{2}{7}\tau_\pi\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{10}{63}\tau_\pi\pi^{\mu\nu}\theta^2 + \frac{26}{21}\tau_\pi\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta. \end{aligned} \tag{10}$$

This is the main result of the present work. We compare the above equation with that obtained in Ref. (El *et al*., 2010) by invoking the second law of thermodynamics,

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau'_\pi} + 2\beta'_\pi\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{5}{36\beta'_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{16}{9\beta'_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta, \tag{11}$$

where $\beta'_\pi = 2P/3$ and $\tau'_\pi = \eta/\beta'_\pi$. We notice that the right-hand-side of the above equation contains one second-order and two third-order terms compared to three second-order and fourteen third-order terms obtained in Eq. (10). This confirms the fact that the evolution equation obtained by invoking the second law of thermodynamics is incomplete.

Numerical Results and Discussion

In the following, we consider boost-invariant Bjorken expansion of a massless Boltzmann gas (Bjorken, 1983). We have solved the evolution equations with initial temperature $T_0 = 300$ MeV at initial time $\tau_0 = 0.25$ fm/c and with $T_0 = 500$ MeV at $\tau_0 = 0.4$ fm/c, corresponding to initial conditions of RHIC and LHC, respectively.

Figs. 1(A) and (B) shows the proper time dependence of pressure anisotropy $P_L/P_T \equiv (P-\pi)/(P+\pi/2)$ where $\pi \equiv -\tau^2\pi^{\eta\eta}$. In Fig. 1 (A), we observe an improved agreement of third-order results from Chapman-Enskog method (green solid lines) with the exact solution of Boltzmann equation (black dotted lines) (Florkowski *et al*., 2013) as compared to third-order results from Grad's method (blue dashed lines). In Fig. 1(B) we notice that while the results from Grad's method (blue dashed lines) overestimate the pressure anisotropy for $\eta/s > 0.2$, those obtained in the present work (green solid lines) are in better agreement with the results of the parton cascade BAMPS (black dots) (El *et al*., 2010).

Summary

To summarize, we have derived a third-order evolution equation for the shear stress tensor from kinetic theory. We iteratively solved the Boltzmann equation in relaxation time approximation to obtain the non-equilibrium distribution function up to second-order in gradients. Using this form of the non-equilibrium distribution function, instead of Grad's 14-moment approximation, the evolution equation for shear tensor was derived directly from its definition. Within one-dimensional scaling expansion we demonstrated that the third-order viscous hydrodynamic equation derived here provides a very good approximation to the exact solution of Boltzmann equation. We also showed that our results are in better agreement with BAMPS compared to third-order viscous hydrodynamics derived using Grad's approximation.

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